This lecture is concerned with matchings, which we explained informally at the end of the previous lecture. Here is a rigorous definition; note the difference between a perfect matching is a general graph and a perfect matching of $X$ into $Y$ in a bipartite one.

**Definition** Let $G$ be a general graph. A matching is a collection of vertex-disjoint edges. Alternatively, this is called a set of independent edges. If every vertex is covered by one of these edges, it is a perfect matching.

For a bipartite graph consisting of partition $(X, Y)$, if a matching covers all the vertices in $X$, it is a perfect matching of $X$ into $Y$.

There is an interesting and beautiful characterization for when a bipartite graph has a perfect matching of the smaller class into the larger; first, let us examine necessity.

**Lemma 1.** A necessary condition for a bipartite graph $G = (X, Y)$ to have a perfect matching of $X$ into $Y$ is that for all $T \subset X$, $|N(T)| \geq |T|$, where $N(T) \subseteq Y$ is the set of all neighbors of the subset $T$ of $X$.

**Proof.** If this were not true, there would be some subset $T$ for which $|N(T)| < |T|$. But then one could not match $T$ into $Y$, since any set of edges covering all vertices in $T$ would have to have some overlap in $N(T)$ (and in $Y$). Hence we could not possibly match $X$ into $Y$. $\Box$

The above condition turns out to also be sufficient, as we will see from the theorem below.

**Theorem 1.** (Hall’s Theorem, Hall’s “Marriage” problem) A necessary and sufficient condition for a bipartite graph $G = (X, Y)$ to have a perfect matching of $X$ into $Y$ is that for all $T \subset X$, $|N_G(T)| \geq |T|$, where $N_G(T) \subseteq Y$ is the set of all neighbors in $G$ of the subset $T$ of $X$.

**Proof.** As the Lemma above shows necessity, we only need to prove sufficiency.

We will induct on $|X|$. For $|X| = 1$, the statement is trivial. Pick now a graph $G$ satisfying the requirements, and assume that the statement is true for any bipartite graph with the smaller class $X$ having size at most $n$.

We split the problem into two cases.

a) For any $T \subset X$, $|N_G(T)| > |T|$. (Note that we exclude the possibility that $T = X$.)

b) There exists a proper subset $T \subset X$ such that $|N_G(T)| = |T|$.

These are mutually exclusive and cover all possibilities.

a) Pick any two vertices $x \in X$ and $y \in Y$ that are adjacent, match them, and eliminate them and all incident edges from the graph, creating a new graph $G'$ with a smaller set $X'$, $|X'| = |X| - 1$. The matching requirements must still be true, since they will work for $T \subset X$ with $x \notin T$ as $|N_{G'}(T)| \leq |N_G(T)| - 1 \geq |T|$. Then we apply induction.
b) There exists $T \subset X$ for which $|T| = |N_G(T)|$. We define the graphs $G_1$, with classes $(T, N_G(T))$, and all the edges in $G$ in-between, and $G_2$, with classes $(X \setminus T, Y \setminus N_G(T))$, and all the edges in $G$ in-between. Note that this ignores any edges between $X \setminus T$ and $N_G(T)$. If we can find a perfect matching of $T$ into $N_G(T)$ and a perfect matching of $X \setminus T$ into $Y \setminus N_G(T)$, we put them together and form a perfect matching for $X$ into $Y$.

So we need to show that perfect matchings of $T$ into $N_G(T)$, respectively, of $X \setminus T$ into $Y \setminus N_G(T)$ exist, which we will do by using the induction hypothesis.

We start with $G_1$. As any $R \subseteq T \subset X$ must have $N_{G_1}(R) = N_G(R) \geq |R|$ and $N_{G_1}(R) \subseteq N_G(T)$, by induction, a perfect matching of $T$ into $N_G(T)$ must exist.

We now show that we can find one in $G_2$. Suppose $R \subseteq (X \setminus T)$; then $N_{G_2}(R) \subseteq Y \setminus N_G(T)$. Note that $|N_G(R \cup T)| = |N_{G_2}(R) \cup N_G(T)| = |N_{G_2}(R)| + |N_G(T)|$, as the latter is a union of disjoint sets. But then

$$|R| + |T| = |R \cup T| \leq |N_G(R \cup T)| = |N_{G_2}(R) \cup N_G(T)| = |N_{G_2}(R)| + |N_G(T)| = |N_{G_2}(R)| + |T|,$$

so $|N_G(R)| \geq |R|$. Thus the set of inequalities is satisfied by $G_2$, and so $G_2$ must by induction also have a match of $X \setminus T$ into $Y \setminus N_G(T)$.

As we can write a matching of $G$ as a union of a match of $G_1$ and a match in $G_2$, the conclusion follows.

$\square$