Adjacency matrix. There are many matrices associated to a graph, the simplest and most natural of which is the adjacency one.

Given an undirected graph $G$ on set vertex $V$ with $|V| = n$, we define the $n \times n$ matrix $A$ such that $a_{i,j} = \#$ edges between $i$ and $j$. See example. This definition then easily extends to a directed graph, by simply considering the direction of the edges. Note that for a undirected graph, the adjacency matrix is symmetric, while for a directed one it is not. (A quick appeal to Math 308 material should remind you that in the former case the eigenvalues are real, while in the latter they are not necessarily so.)

Two simple lemmas below speak of how this matrix can be used to illustrate some of the properties of $G$.

**Lemma 1.** Let $G$ and $A$ be as above (directed or undirected), and let $k > 0$ an integer. Then $(A^k)_{ij}$ is the number of walks from $i$ to $j$ of length exactly $k$ (if $G$ is directed, the walks must also be).

**Proof.** The proof uses induction, and it is very easy; when $k = 1$, the definition of $A$ makes the point. Assume now that $(A^k)_{ij}$ has the desired counting property for all $i, j$, and let us show that $(A^{k+1})_{ij}$ does, also.

Matrix multiplication tells us that since $A^{k+1} = A^k A$,

$$(A^{k+1})_{ij} = \sum_{m=1}^{n} (A^{k})_{im} A_{mj}.$$  

Since $(A^k)_{im}$ counts the number of walks from $i$ to $m$ of length $k$, and $A_{mj}$ the number of edges between $m$ and $j$, the term $(A^{k})_{im} A_{mj}$ counts precisely the number of walks of length $k + 1$ between $i$ and $j$ passing through $m$ right before reaching $j$. These are all distinct. Conversely, any walk of length $k + 1$ between $i$ and $j$ passes through some $m$ right before reaching $j$, and so it is accounted for. Combined, these two observations show the lemma. \[\square\]

The above lemma provides a very simple way to test for connectivity. Recall that connectivity is defined as the property that there is a walk from any vertex $i$ to any vertex $j \neq i$ in the graph $G$ is question; notably, if there is a walk, there is a path (which is defined as a walk not repeating vertices, i.e., not including cycles). We proved this last quarter, but the reason for it is fairly simple: start with the walk between $i$ and $j$, and remove all cycles from it; the result is a path between $i$ and $j$.

Since a path can have length at most $n - 1$ (if it goes through all the vertices), it follows that if the graph $G$ is connected, given any vertices $i, j$ with $i \neq j$, there is a path/walk of length at most $n - 1$ between $i$ and $j$.

Hence the following.

**Lemma 2.** A graph $G$ with $n \times n$ adjacency matrix $A$ is connected iff the matrix $(I + A)^{n-1}$ has all positive entries (here $I$ stands in for the identity matrix having all ones on the diagonal and zeros elsewhere.)
Proof. Since

$$(I + A)^{n-1} = I + \binom{n-1}{1}A + \binom{n-1}{2}A^2 + \ldots + \binom{n-1}{n-1}A^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k}A^k.$$ 

we use the fact that for any $i \neq j$ if there is a walk there is a path of length $\leq n - 1$ to conclude that for some $k \leq n - 1$, $(A^k)_{ij} > 0$, and hence $((I + A)^{n-1})_{ij} > 0$. Since $I$ has all diagonal terms positive, these two facts put together show the lemma.