Dilworth’s Theorem has a few very notable applications. First, we have the Erdős-Szekeres Lemma.

**Lemma 1.** Let $r, s \geq 1$ be integers. A sequence of $rs + 1$ numbers must contain either an increasing subsequence of length $r + 1$ or a decreasing subsequence of length $s + 1$.

**Proof.** Let $P$ be the poset containing the $rs + 1$ numbers, call them $\{x_1, \ldots, x_{rs+1}\}$, with the following relation: we say $x_i \leq x_j$ in this ordering if $x_i \leq x_j$ as numbers and in addition $j > i$. It is not hard to verify that this is indeed a poset.

Chains in this poset are increasing subsequences. What kinds of elements are incomparable? note that if $j > i$, the only way $x_i$ and $x_j$ are incomparable are if $x_i \geq x_j$. So in other words, antichains must be composed of decreasing subsequences.

Dilworth’s theorem says that the smallest chain cover has the same size as the largest antichain. If no increasing subsequence of length $r + 1$ exists, then, by virtue of the fact that every chain has length at most $r$, the smallest chain cover must have at least $s + 1$ chains. So by Dilworth’s Theorem, the longest antichain must have length $s + 1$ or more. But an antichain is a decreasing sequence, so there is a decreasing subsequence of length at least $s + 1$. \qed

The second application is Hall’s Theorem. Recall that Hall’s Theorem says that any bipartite graph with classes $(X, Y)$ such that for any $S \subseteq X$, $|N(S)| \geq |S|$, has a matching of $X$ into $Y$. Assume wlog $|X| \leq |Y|$. Any bipartite graph can be seen as a Hasse diagram of a poset $P = X \cup Y$ with the following relationship:

- $z \leq z$, $\forall z \in P$;
- $y \leq x$ if $y \in Y$, $x \in X$, and $(x, y) \in E$.

Note that chains in $P$ have length at most 2, and that a chain cover of $P$ must have size at least $|Y|$ (since $Y$ is an antichain, and not necessarily the largest one).

**Claim 1.** We will show that the condition that $Y$ is a maximum antichain (an antichain of largest size) is equivalent to the existence of a perfect matching of $X$ into $Y$.

**Proof.** $\Rightarrow$ Suppose that $Y$ is a maximum antichain. By Dilworth’s theorem, the smallest chain cover has size $|Y|$. The chain cover consists of some number of length 2 chains, with the rest of them of length 1. The union of the length 2 chains must be a matching, since they are disjoint, and the rest of the chains are singletons.

Now assume that there exists no perfect matching of $X$ into $Y$; then the matching defined by the cover isn’t perfect. The remaining (unmatched) vertices in $X$ and $Y$ must be covered by singleton chains. In particular, every vertex in $Y$ must be covered by a distinct chain, and the unmatched vertices of $X$ must also be covered by distinct chains, and so the cover must contain strictly more than $|Y|$ chains, which is a contradiction.

Thus, there must exist a perfect matching of $X$ into $Y$. \qed
Suppose now that the Hall conditions are fulfilled. Then, there exists a chain cover of the poset that uses the chains given by the matching, plus singleton chains for the remaining vertices in $Y$, for a total of exactly $|Y|$ chains. Hence the smallest cover must be of size at most $|Y|$, and since it needs to be of size at least $|Y|$, it has size exactly $|Y|$. Hence, the largest antichain is also of size $|Y|$ by Dilworth’s theorem.

**Claim 2.** We must show that $\alpha(P) = |Y|$ iff the conditions of Hall’s Theorem are fulfilled.

*Proof.* Note that an antichain in $P$ consists of a set $A = X_A \cup Y_A$, with $X_A \subseteq X$ and $Y_A \subseteq Y$. Also note that $Y_A \subseteq Y \setminus N(X_A)$.

How large can $|A|$ be? Pick $X_A \in X$; then the largest antichain containing $X_A$ is $X_A \cup (Y \setminus N(X_A))$.

$\implies$ We prove by contrapositive. If $|N(X_A)| < |X_A|$ for some $X_A$, this means that the largest antichain containing $X_A$ has size $|X_A| + |Y| - |N(X_A)| > |Y|$. So since $\alpha(P) = |Y|$, all the conditions of Hall’s Theorem must be satisfied.

$\impliedby$ We again show this by contrapositive. $\alpha(P) > |Y|$ means that some nontrivial $X_A \subset X$ represents the intersection of $X$ with the longest antichain, and since the longest antichain containing $X_A$ has length $|X_A| + |Y| - |N(X_A)|$ and this is by assumption bigger than $|Y|$, it follows that Hall’s Theorem condition fails for $X_A$. 

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