In this lecture we will veer a bit from the book, since there is a lot more about the basics of posets that I would like to explore with you.

**Posets or not posets?**

- $B_n$ with the reverse-inclusion ordering; $A \leq B$ if $B \subseteq A$. Is it a poset?
  
  Yes; the relation is reflexive, transitive, and antisymmetric. The Hasse diagram of $B_3$ with the reverse-inclusion ordering is flipped.

- $\mathbb{Z}$ ordered by $x \leq y$ iff $x = y$ or $x = y - 1$. Is this a poset?
  
  No; not transitive: $1 \leq 2, 2 \leq 3$, but $1 \not\leq 3$.

**Remark 1.** We hinted at this last time. Every finite poset has at least one maximal element and one minimal element.

More on what we’ve seen last time.

**Definition** A subset of a chain is a chain, a subset of an antichain is an antichain.

The maximum size of a chain in $P$ is called the *height* of $P$ (from the Hasse diagram), sometimes denoted $\omega(P)$; the maximum size of an antichain in $P$ is denoted $\alpha(P)$ and called the *width* of $P$.

Note that for $B_3$ with inclusion, $\alpha(B_3) = 3$ and $\omega(B_3) = 4$.

**Lemma 1.** The set of all minimal (maximal) elements of a poset is an antichain.

**Proof.** If not, then there are two minimal (maximal) elements that are comparable, which is contradictory with the fact that they are minimal (maximal). □

**Theorem 1.** If $(P, \leq)$ is a finite poset, then $\alpha(P)\omega(P) \geq |P|$.

**Proof.** We will find one chain and one antichain such that their sizes multiply to more than $P$; this suffices, since their sizes are lower bounds for $\alpha(P)$, respectively $\omega(P)$.

We construct the chain and antichain by “shaving off” recurrently the antichain of minimal elements.

Let $P^{(1)}$ be the antichain of minimal elements in $P$. Then, define $P^{(2)}$ to be the antichain of minimal elements of the poset $P \setminus P^{(1)}$, $P^{(3)}$ to be the antichain of minimal elements of $P \setminus (P^{(1)} \cup P^{(2)})$, ..., $P^{(k)}$ be the antichain of minimal elements of $P \setminus (\cup_{i=1}^{k-1} P^{(i)})$, ...; this process must end since $P$ is finite. Say the last antichain thus defined is $P^{(k)}$.

Note $P = \cup_{i=1}^{k} P^{(i)}$. Thus, $|P| \leq k \max_{i} |P^{(i)}|$, and if we could construct a chain of length $k$, we would be done.

We now construct a chain of length $k$. Start with an element $x_k$ of $P^{(k)}$; since it was not minimal for $P^{(k-1)} \cup P^{(k)}$, there must be an element $x_{k-1}$ in $P^{(k-1)}$ such that $x_{k-1} < x_k$. Similarly, since $x_{k-1}$ was not a minimal element of $P^{(k-2)} \cup P^{(k-1)} \cup P^{(k)}$, there must be an element $x_{k-2}$ of $P^{(k-2)}$ such that $x_{k-2} < x_{k-1} < x_k$, and so on until we reach $x_1$. Then we will have constructed a chain of $k$ elements, and ended the proof. □
Definition A cover of a set $S$ is a collection of disjoint subsets $S_1, \ldots, S_k, S_i \cap S_j = \emptyset$ for all $i \neq j$ whose union is $S$: $\bigcup_{i=1}^{k} S_i = S$. The size of the cover is $k$.

Definition A smallest antichain cover for a poset $P$ is the smallest size of a cover of $P$ with antichains. Similarly, a smallest chain cover for a poset $P$ is the smallest size of a cover of $P$ by chains.

Remark 2. As an immediate consequence of the ideas we used in the Theorem, it follows that the size of any antichain cover of $P$ is at least $\omega(P)$. Indeed, any two elements of a chain must necessarily be in different antichains in the cover.

Remark 3. As noticed in class, the chain constructed here is in fact the longest possible chain. If there were a longer one, it would (by Pigeonhole) necessarily have to have more than one element in the same antichain in the cover constructed by recursively “shaving off” the minimal elements, and that is impossible.

But in fact we have the stronger theorem below.

Theorem 2. (Mirsky) The minimum size of an antichain cover for a poset $P$ is $\omega(P)$ (the size of the longest chain).

Proof. The $\geq$ part follows from the first remark, and in the proof of the previous theorem we constructed a cover of $P$ with antichains for which the size was no larger than the length of the longest chain (which shows $\leq$). This also shows that the antichain cover we constructed there is smallest! □