Leftover from PM: one more expectation example. Suppose \( n, k \) positive integers with \( k \leq n \). We want to compute the expected number of \( k \)-cycles in a uniformly random permutation of \([n]\).

Let \( C \) be some arbitrary \( k \)-cycle, and let \( X_C \) be its indicator variable of presence in the randomly chosen permutation \( \pi \).

Then
\[
\mathbb{E}[\text{# of } k\text{-cycles in } \pi] = \sum_C \mathbb{E}[X_C].
\]
Since the actual properties of the objects we permute do not matter, any \( k \)-cycle should be as likely as any other \( k \)-cycle to be present in \( \pi \) (FYI, this means two different \( k \)-cycles are exchangeable; they are not independent, but they are exchangeable). So the above is actually equal to
\[
\mathbb{E}[\text{# of } k\text{-cycles in } \pi] = \# \text{ } k\text{-cycles} \cdot \mathbb{E}[X_C].
\]

As \( X_C \) is an indicator variable, \( \mathbb{E}[X_C] \) is actually the probability that \( X_C \) is in \( \pi \). The latter is computed quickly as \( (n-k)!/n! \), since prescribing what \( \pi \) does on \( k \) values leaves it completely open on what it does to the remaining \( (n-k) \), and there are \( n! \) permutations overall.

On the other hand, the number of \( k \)-cycles is \( \binom{n}{k}(k-1)! \), as we have \( (k-1)! \) ways of placing a \( k \)-cycle on \( k \) numbers (we permute them in \( k! \) ways, but each cycle is obtained in this procedure in \( k \) ways, since we can start “reading” it at any of the numbers).

Putting it all together,
\[
\mathbb{E}[\text{# of } k\text{-cycles in } \pi] = \binom{n}{k}(k-1)! \frac{(n-k)!}{n!} = \frac{1}{k}.
\]

Posets. When we work with real number sets, we have the benefit of always knowing the order relationship of two numbers; \( e < \pi, -23 < 1.5 \). When the numbers are complex this stops being true; we usually do not compare complex numbers; \( i \) is neither smaller than nor larger than 1; however, the relationships between real numbers stay true even when we consider the reals as part of the complex plane.

This means that there are sets which admit only partial ordering, in that only certain elements are comparable and can be ordered.

**Example 1.** Suppose we look at the set of all subsets of \([n]\), and consider inclusion \( \subseteq \). Call this \( B_n \). For \( n = 3 \), \( B_3 = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \} \). Note that while \( \{1\} \subseteq \{1, 3\} \), there is no inclusion relationship between \( \{1, 3\} \) and \( \{2, 3\} \) or between \( \{2\} \) and \( \{1, 3\} \) ! So the latter pair of subsets are incomparable.

**Definition** Let \( P \) be a set, and \( \leq \) a relationship on the elements of this set such that

1) \( \leq \) is reflexive, i.e., \( x \leq x \) for all \( x \in P \);
2) \( \leq \) is transitive, i.e., \( x \leq y \) and \( y \leq z \) means that \( x \leq z \) for all \( x, y, z \in P \);

3) \( \leq \) is antisymmetric, i.e., \( x \leq y \) and \( y \leq x \) means \( x = y \) for all \( x, y \in P \).

We call \( \leq \) a partial ordering of \( P \), and \( P, \leq = (P, \leq) \) a partially ordered set (or a poset).

**Example 2.** [12], together with the relationship \( \leq \) defined by \( x \leq y \) if \( x \) is a divisor of \( y \), is a poset. It is not hard to verify reflexivity, transitivity, and anti-symmetry.

Oftentimes is it desirable to draw a diagram in order to understand the relationships between the elements of the set.

**Definition** The Hasse diagram of a poset \((P, \leq)\) is a graph whose vertices represent the elements of \( P \) and if \( x < y \) then \( y \) is drawn above \( x \). If, in addition, there is no \( z \in P \) such that \( x < z < y \), then we say “\( y \) covers \( x \)” and draw an edge between \( x \) and \( y \). (If we want to avoid the idea of “above” we draw every edge directed, from the smaller vertex to the larger.)

In class, we drew the Hasse diagrams for the posets \((B_3, \subseteq)\) and \(([12], |)\).

**Definition** A chain is a totally ordered subset of \( P \), formed of elements on the same directed path (or down to up path) in the Hasse diagram. Not all elements on the path have to be included in order to have a chain; for example, in the \( B_3 \) example both \( \{\emptyset, \{1\}, \{1,2,3\}\} \) and \( \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}\} \) are chains.

At the opposite end, a set of incomparable elements is called an antichain. For example, \( \{\{1\}, \{2\}, \{3\}\} \) is an antichain.

A set of disjoint antichains whose union is the entire set \( P \) is called a chain cover of \( P \). A set of disjoint antichains whose union is the entire set \( P \) is called an antichain cover of \( P \).

**Definition** Maximal/minimal elements are defined as expected, i.e., any element \( x \) such that there is no \( y \) for which \( y > x \) is called maximal, and an element \( x \) such that there is no \( y < x \) is called minimal.

We can now show the following theorem (the proof will be shown next time).

**Theorem 1.** (Dilworth’s Theorem) In a poset \( P \), the size of any maximum antichain is equal to the number of chains in any smallest chain cover.