Remark 1. Inequalities (13.1) and (13.2) in the book are missing a comma (it should be $R(k, l-1)$, not $R(kl-1)$).

Corollary 1. We showed as part of the proof of existence for the Ramsey numbers $R(k, l)$, that $R(k, l) \leq R(k-1, l) + R(k, l-1)$ for all $k, l \geq 3$.

This can be now used to construct upper bounds for $R(k, l)$; for example, $R(4, 3) \leq R(3, 3) + R(4, 2) = 6 + 4 = 10$. But as it turns out, this is not right: $R(4, 3) = 9$. Similarly, now we can bound $R(4, 4)$ from above by $2 \cdot R(4, 3) = 18$ (as clearly $R(k, l) = R(l, k)$ by symmetry); in this case equality holds. See textbook for proofs for both values.

What may be very surprising is that these are the only other values (other than $R(3, 3)$) which are known exactly. In fact, Paul Erdős used to have a great illustrative joke; if some super-race of aliens came to Earth and requested (on point of annihilation) that within the year we tell them precisely what $R(5, 5)$ is, all of the world’s computers and all of the world’s mathematicians and computer scientists could come together and work on this problem to get the answer. But if instead, same aliens would require us to find $R(6, 6)$, then the whole world should come together and use the year to figure out how to get rid of the aliens.

While this is a fanciful story, it underlines a very deep idea. Why is it that $R(k, k)$ is hard to find? After all, can’t we just test all of the colorings, for $n = 1, 2, \ldots$ and stop when we find $R(k, k)$?

Well, to answer that, let’s begin by giving $R(k, k)$ a simple bound.

Theorem 1. $R(k, l) \leq \binom{k+l-2}{l-1}$.

Proof. We construct the proof inductively, the same way we did with the existence proof. Note that if $k = 2$ or $l = 2$, the bound becomes $R(k, 2) = k \leq \binom{k+2-2}{2-1} = \binom{k}{1} = k$, and that is right (for $R(2, l)$ it works in the same way).

Assume now that we have shown that the bound works up to $k + l \leq S - 1$, and we will show it for $k + l = S$.

We have already shown that $R(k, l) \leq R(k-1, l) + R(k, l-1)$. By induction, this implies that

$$R(k, l) \leq \binom{k-1+l-2}{l-1} + \binom{k+l-1+2}{l-2} = \binom{k+l-3}{l-1} + \binom{k+l-3}{l-2} = \binom{k+l-2}{l-1},$$

where the last equality is true courtesy of Pascal’s triangle. Thus, induction is complete and the proof is finished.

Letting now $k = l$ we can obtain the following bound.

Corollary 2. $R(k, k) \leq 4^{k-1}$. 

The reason is simple, \( \binom{2k - 2}{k - 1} \leq 2^{2k - 2} = 4^{k - 1} \). So \( R(k, k) \) may grow exponentially. Even so, why would a simple algorithm for checking not work? The reason is that we color edges. There are \( \binom{n}{2} \) edges in a complete graph, and that means that there are \( 2^{\binom{n}{2}} \) possible colorings. Add in now the fact that \( n \) is exponential, and we have a double exponential function to contend with. To give you an example of how this grows, for \( k = 5 \), \( 4^4 = 256 \), and \( 2^{256} \) is, shall we say, very large...

But this exponential upper bound need not be a tight one. Perhaps \( R(k, k) \) grows slower, polynomially, maybe even linearly? The answer, sadly, is going to be no, as we will see when we talk about the Probabilistic Method.

**Other generalizations.** Why stop at 2 colors? We could examine the same kind of problem, allow for some number \( k \) of colors, and define the generalized Ramsey number \( R(n_1, n_2, \ldots, n_k) \). The proof for its existence follows the same pattern as before, and it relies on an upper bound of the form

\[
R(n_1, n_2, \ldots, n_k) \leq \sum_{i=1}^{k} R(n_1, \ldots, n_i - 1, \ldots, n_k) - k + 2.
\]

There will be a problem on the homework related to forbidding monochromatic triangles in colorings of \( K_n \). There are further generalizations relating to hypergraphs. See the textbook for more details.