Last time, we saw the 6-color Theorem, which shows that $\chi(G) \leq 6$ for any planar graph $G$.

With little effort, one may actually prove that $\chi(G) \leq 5$ (Theorem 12.15 in the book). How low can you go? No lower than 4, for sure, since $K_4$ is planar and $\chi(K_4) = 4$. Turns out 4 is the answer, as has been shown in a series of increasingly simpler proofs (yet proofs that cannot are all computer-assisted, i.e., need a huge number of case-check via software (1936 in the original version from 1976, 633 in 1996). Initial proof required 1200 hours of computer runtime. It is the first example of computer-assisted proof that has been accepted, and it has marked a new era of mathematics.

**Ramsey Theory.** During the last chapter, we have seen what it means to properly color the vertices of a graph, and what potential applications this may have. Starting today, we will completely switch mindsets, and color edges instead. But unlike before, a “proper” coloring will no longer be defined by adjacency; we will instead forbid the presence of certain cliques.

It’s likely most of you have seen this problem before: given 6 people at a party, show that either there is a group of 3 who all know each other, or a group of 3 none of whom know the other two. We will cast this problem today in the form of a graph coloring problem.

$K_6$ **must have monochromatic triangles.** Examine the complete graph $K_6$, and color each one of its edges either Red ($R$) or Blue ($B$). No matter how you choose the colors, there will always exist a monochromatic triangle (a triangle all of whose edges are either all $R$ or all $B$). You may start to suspect this once you try a few colorings and it doesn’t work, but why is it true?

**Proof.** Pick any one of the vertices of $K_6$ and consider the 5 edges coming out of it. By Pigeohole Principle, three will have to have the same color, without loss of generality let’s say $R$. Now consider the three vertices at the end of those edges, and the three edges between each pair of those three vertices. If any of the latter edges is $R$, we have a $R$ triangle. Else they must all be $B$, and now we have a $B$ triangle. Done.

As it happens, $K_6$ is the smallest complete graph which, when its edges are colored with 2 colors, **must** have a monochromatic triangle. Indeed, one may color each of the two cycles of $K_5$ differently, and so obtain no monochromatic triangles; and also, andy $K_n$ with $n \geq 6$ includes $K_6$ and thus inherits $K_6$’s property. So this is kind of a remarkable feat—no $K_n$ with $n < 6$ has the property, and all $K_n$ with $n \geq 6$ do. This means that the property of having a monochromatic triangle in every 2-coloring of the edges is monotonous.

What if instead of monochromatic triangles we want some other kind of monochromatic $K_k$ (aka a “clique”)? Will there be a number $n$ such that this phenomenon repeats?

**Theorem 1.** For any $k, l \geq 2$ positive integers, there exists a minimal number $R(k, l)$ such that if we color the edges of a complete graph with at least $R(k, l)$ vertices Red or Blue, we always get either a completely Red $K_k$ or a completely Blue $K_l$. 
Proof. Note two things. One, it suffices to show that there exists some number $N$ for which $K_N$ has the desired property; this means that the number of $N$’s that do is non-empty, and hence it has a minimal element, which we can call $R(k,l)$. Second, just like before, if $R(k,l)$ is the minimal number with the property, then trivially no $n < R(k,l)$ has it, and ALL $n \geq R(k,l)$ have it (monotonicity).

We will prove the property by induction over the sum $S = k + l$. As we will need a “border” for the region of the $(k,l)$s that work, note that $R(2,l)$ is always $l$, since for every $K_n$ with $n \geq l$, either the coloring has a Red edge, or a complete $K_l$. Similarly and symmetrically $R(k,2) = k$.

Assume now that we know the statement of the theorem for the borders of the triangular region (that is, $(k,2)$ and $(2,l)$) and also for all $(k,l)$ with $k + l \leq S - 1$ within that region.

We will show now that $N = R(k-1,l) + R(k,l-1)$ has the desired coloring property; this will imply that $R(k,l)$ exists and that $R(k,l) \leq R(k-1,l) + R(k,l-1)$.

Let $N = R(k-1,l) + R(k,l-1)$. Just as before, pick a vertex $v$ in $K_N$ and examine the edges coming out of it. There are $R(k-1,l) + R(k,l-1) - 1$ of them.

By a sort of Pigeonhole, either at least $R(k-1,l)$ of them are Red, or at least $R(k,l-1)$ of them are Blue. Indeed, if both of these were to fail, we would have fewer than $R(k-1,l) + R(k,l-1) - 1$ edges.

Assume wlog that $R(k-1,l)$ of these edges are Red. Let us look at the $K_{R(k-1,l)}$ at the end of these edges. By induction, it has to either have a $K_{k-1}$ completely Red, or a $K_l$ completely Blue. If it has the latter, there is a completely Blue $K_l$ in our $K_N$, so we are done. Elsewise, consider the completely Red $K_{k-1}$. As all edges connecting $v$ with the vertices of this $K_{k-1}$ are also Red, adding $v$ to this $K_{k-1}$ builds a completely Red $K_k$, and once again we are done.

The other case is analogous, and this finishes the induction proof. □

Remark 1. Inequalities (13.1) and (13.2) in the book are missing a comma (it should be $R(k,l-1)$, not $R( kl - 1)$).