K_{3,3} and K_5. We will show now that these graphs are not planar. Note that what this will immediately imply is that any graph containing one of them as a subgraph will also not be planar (in particular, K_{m,n} for all m, n ≥ 3 and K_n for n ≥ 5.)

To see that K_{3,3} is not planar, note that the faces in a bipartite biregular graph have to be even cycles, that is, of length at least 4. K_{3,3} has 9 edges and 6 vertices, so if it were planar, it would have 9 + 2 − 6 = 5 faces. But each face would need at least 4 edges, and each edge could belong to only 2 faces, so at least 4 · 5/2 = 10 < 9 edges would be needed; contradiction.

Similarly, for K_5, since it’s a complete graph, the faces would have to be triangles. If it were planar, Euler’s formula says that with 5 vertices and 10 edges it should have 7 faces. 7 triangles have 21 edges, each edge must belong to 2 faces, contradiction as we have only 10 edges.

The “big picture” implications. In fact, the non-planarity of K_5 and K_{3,3} is at the root of all possible non-planarity. This is a theorem we will not prove, but we state it here (it is due to Kuratowski).

**Theorem 1.** Any non-planar graph “contains” a K_5 or a K_{3,3}.

Here “contains” has a wider meaning than just that of a subgraph; see textbook for a litttle bit more on this (end of chapter 12.1).

Next, we will examine a very close relationship between planar graphs in 2D and convex polyhedra in 3D.

**Convex Polyhedra.** These are 3D bodies which are convex (if two points are in the interior, the entire segment with them as endpoints is in the interior) and whose boundary is a union of convex polygons. E.g, prism, parallelepiped, icosahedron.

Any convex polyhedron can be projected down on a plane as a planar graph. One may obtain this projection by shining a light outside (but close to) a face, and looking at the “shadow” that the polyhedron is making on a plane on the other side of it as the light. By rotating the plane slightly, we can make sure that every vertex has a corresponding projection vertex, and every edge can be seen with no intersections. The face close to the light is the outer face, if the light is close enough.

The projection of the polyhedron onto a plane, as a planar graph, is called the 1-skeleton of the polyhedron.

This means that the Euler Theorem also works for polyhedra!

**Remark 1.** The main difference between planar graphs and convex polyhedra is that the former can have “dangling” trees whereas the latter do not. So: results for planar graphs apply to convex polyhedra, but not necessarily the converse.

Below we have a few theorems that can be deduced with the help of the Euler Formula. They turn out to be very useful in constructing a proof that the number of regular polyhedra is exactly 5 (tetrahedron, cube, octahedron, dodecahedron, icosahedron). By regular polyhedra we mean with all faces regular polygons, and the same number of faces/edges meeting at each vertex. We’ll give more details as we go along.
Lemma 1. In a simple, connected planar graph (hence in a convex polyhedron) with \( V \geq 3 \), \( 3F \leq 2E \).

Proof. Summing all edges over all the faces, we should get exactly \( 2E \), since edges that belong to cycle-faces belong to exactly two faces, and edges that do not (the “dangling tree” edges) get summed twice around their one face (once in each direction). One the other hand, each face has at least 3 edges, so the total number of edges in all the faces is at least \( 3F \). That gives the lemma.

Lemma 2. In a convex polyhedron, \( 3V \leq 2E \). THIS IS NOT TRUE in general planar graphs; e.g., it’s not true for trees.

Proof. It is a similar proof to the previous one. We count the number of edges over all vertices. Each vertex has at least 3 edges (because the polyhedron is 3-dimensional), and if we count all of them, we get \( 2E \) since each edge has 2 endpoints. Thus \( 3V \leq 2E \).

Lemma 3. In a simple, connected planar graph with \( V \geq 3 \), \( E \leq 3V - 6 \). In a convex polyhedron, also \( E \leq 3F - 6 \).

Proof. Start from \( F + V = E + 2 \), and since

\[
2E \geq 3F , \quad F \leq \frac{2}{3}E .
\]

Substituting this in the Euler Formula we get

\[
E + 2 = V + F \leq V + \frac{2}{3}E ,
\]

hence

\[
\frac{1}{3}E + 2 \leq V , \quad V - 2 \geq \frac{1}{3}E ,
\]

hence \( E \leq 3V - 6 \).

For convex polyhedra, we can use the \( 2E \leq 3V \), equivalently

\[
V \leq \frac{2}{3}E ,
\]

and substituting this in the Euler Formula, we get

\[
E + 2 = V + F \leq F + \frac{2}{3}E ,
\]

and the rest follows similarly as above.