Class Notes, Math 462, Winter 2019

Lecture 1: Intro and Cayley’s Theorem

Intro & Discussion of the Outline and class policies.

Recall definition of tree from last quarter.

How many trees can we have on $n$ vertices? Surprisingly, we can answer this question exactly. For example, there is 1 on 2 vertices, three on 3 vertices, and 16 on 4 (draw some); this generalizes as seen below.

**Theorem 1.** (Cayley’s formula) There are $n^{n-2}$ distinct trees on vertex set $[n], n \geq 2$.

Before we prove this theorem, we must understand what functions $f : [n] \rightarrow [n]$ look like.

Let’s do the following thought experiment, with the understanding that it will not impact the proof for Cayley’s theorem: can we use graphs to understand functions?...

A function $f$ from $[n]$ to $[n]$ can be seen as a directed graph, with arrows pointing from $i$ to $f(i)$, for all $i$. This can include loops, but no multi-edges. What does this graph look like? As before, when we talked about permutations, consider the “chain” $i \rightarrow f(i) \rightarrow f^2(i) \rightarrow \ldots$. This chain can end only when we go back to a previously visited vertex. Before, when we talked about permutations, that vertex had to be $i$ because a permutation is a bijection, but this is no longer applicable here.

So the directed graph of a function looks like a directed graph in which each edge has a single “out”, although it may have multiple “in”s; and it will have to be a union of cycles with dangling trees attached to them. Note that any number $i$ which is not in a cycle will be on a path dangling from a cycle.

Let us now step back and construct the proof for Cayley’s formula.

**Proof.** We will show that the number of doubly rooted trees on $n$ vertices (that is, the number of trees on $n$ vertices for which we pick, with replacement, two vertices and call the first Start and the other End) is the same as the number of functions from $[n]$ to $[n]$. Since the number of doubly rooted trees is $n^2$ times the number of trees on $n$ vertices (due to the replacement), the conclusion will follow.

We construct a bijection from the set of functions from $[n]$ to $[n]$ to the doubly rooted trees on $n$ vertices, as follows.

To transform functions into doubly rooted trees, pick $f$ an function from $[n]$ to $[n]$. Let $C = \{x \in [n] : x \text{ is in a cycle in the description above}\}$. Note that if we were to restrict $f$ to $C$, $f$ would be a permutation, and the cycles would be the cycles of the permutation.

Assume $|C| = k$. Let $c_1 \leq c_2 \leq \ldots \leq c_k$. Draw a $k$-path, and write the numbers $f(c_1), f(c_2), \ldots, f(c_k)$ at the nodes. Mark $f(c_1)$ as “Start” and $f(c_k)$ as “End”.

*Note that $f(c_1)f(c_2)\ldots f(c_k)$ is the line-writing of the permutation $f$ on $C$. That is, knowing $f(c_1)f(c_2)\ldots f(c_k)$ we can recover the restriction of $f$ on $C.*

Draw now an additional $n - k$ nodes corresponding to the values in $[n] \setminus C$, and draw an edge between $i$ and $f(i)$. We claim that the resulting graph is a tree.
Note that this must be true, since it must be acyclic (as no cycle can contain the numbers $i \in [n] \setminus C$, and the numbers in cycles have been put in a path), and the graph must necessarily be connected, as all paths formed as $i \to f(i) \to f^2(i) \to \ldots$ must end at a vertex $j \in C$, and so it will be connected to the “central path”.

So this process produces an $n$-vertex doubly rooted tree; it should be clear that different functions will produce different trees. Note that the process produces a different doubly rooted tree for each function, so the function is an injection.

To see that this is a bijection, pick a doubly rooted tree. For vertices $i$ not on the Start-End path, define $f(i)$ to be the neighbor of $i$ in the direction of the path. For the path, see Remark above, and reconstruct $f$ on $C$ from it. Thus we have identified $f$, and that makes the function a surjection.

As explained in the beginning of the proof, this means that $n^2$ times the number of trees is $n^n$, and the conclusion follows. □