SINGULAR MONGE-AMPÈRE FOLIATIONS

TOM DUCHAMP AND MORRIS KALKA

ABSTRACT. This paper generalizes results of Lempert and Szöke on the structure of the singular set of a solution of the homogeneous Monge-Ampère equation on a Stein manifold. Their *a priori* assumption that the singular set has maximum dimension is shown to be a consequence of regularity of the solution. In addition, their requirement that the square of the solution be C^3 everywhere is replaced by a smoothness condition on the blowup of the singular set. Under these conditions, the singular set is shown to inherit a Finsler metric, which in the real analytic case uniquely determines the solution of the Monge-Ampère equation. These results are proved using techniques from contact geometry.

1. INTRODUCTION

The *homogeneous Monge-Ampère equation* on the complex n-dimensional complex manifold $\mathcal{M}$ is the equation

$$(1.1) \qquad (dd^{\mathbb{C}}u)^n = 0 \,,$$

where $u : \mathcal{M} \to \mathbb{R}$ and $d^{\mathbb{C}} := i(\overline{\partial} - \partial)$. In the special case where u is at least C^3 and the form $dd^{\mathbb{C}}u$ has constant rank the integral curves of $dd^{\mathbb{C}}u$ foliate $\mathcal{M}$ by complex submanifolds. This foliation is called the *Monge-Ampère foliation* of u and was first studied in [2].

An important class of solutions of (1.1) is the class of plurisubharmonic exhaustion functions for which the sets

$$\{p \in \mathcal{M} : u(p) \le c\}$$

are compact for all $c < \sup u$ and $dd^{\mathbb{C}}u$ is a positive semidefinite form of constant rank $n - 1$. It is known (see for instance [7, Theorem 1.1]) that every such function must fail to be smooth on a non-empty *singular set* $M \subset \mathcal{M}$. In this paper, we study the extent to which the geometry of the singular set determines u.

Previous work. Our work builds on previous results of a number of authors, particularly those of Stoll[10], Burns[3], Wong[14], Patrizio-Wong[9], Lempert-Szöke [7], Szöke [11], and Guillemin and Stenzel [4]. The main result of this paper, Theorem 1.4, was inspired by a question posed in [7].

In the case where u has a logarithmic singularity, and $\tau = \exp(u)$ is a smooth Kähler potential, Stoll [10] showed that M is a point and that $\mathcal{M}$ is biholomorphic to either the unit ball $B^n \subset \mathbb{C}^n$ or to $\mathbb{C}^n$. Burns [3] gave a more geometric proof, exploiting the fact that the leaves of the Monge-Ampère foliation are totally geodesic with respect to the Kähler metric. Wong [14] further explored the geometry of the Monge-Ampère foliation, proving the following theorem.

Theorem 1.2 (Wong)**.** *Let u be a solution of the Monge-Ampère equation and let $\mathcal{F}$ denote the Monge-Ampère foliation. Let $\tau = f \circ u$ be the potential function f for a Kähler metric on $\mathcal{M} \setminus M$, where f is a smooth function satisfying the conditions $f' \circ u > 0$ and $f'' \circ u > 0$. Then the leaves of*

$\mathcal{F}$ *are totally geodesic with respect to the Kähler metric. Moreover if* Z *denotes the complex vector field on* $\mathcal{M} \setminus M$ *defined by*

$$Z \lrcorner \, dd^{\mathbb{C}}\tau = d\tau + id^{\mathbb{C}}\tau \,,$$

then the leaves of $\mathcal{F}$ *coincide with the orbits of the complex flow of* Z*. Finally, the integral curves of the real vector field* $Z_I = \frac{1}{2i}(Z - \overline{Z})$ *are (after reparametrizing) geodesics that intersect the level sets of* u *at right angles.*

The proof is essentially contained in [14] (see also [9] and [7]).

The case where u has a logarithmic singularity and M is a point is an extreme case. At the other extreme is the case where M is assumed to be a compact smooth, real n-dimensional submanifold of $\mathcal{M}$. Assume that u is continuous on all of $\mathcal{M}$ and that the singular set coincides with the zero set of u. Compactness of the set $u^{-1}([c, \infty))$ then implies that u is bounded below, we may therefore assume without loss of generality that u is non-negative and that M is the zero set of u. Assume, in addition, that the function $\tau = u^2$ is C^3 and strictly plurisubharmonic on all of $\mathcal{M}$. Then $dd^{\mathbb{C}}\tau$ defines a Kähler metric on all of $\mathcal{M}$. The singular set M, then inherits a Riemannian metric g. The triple $(\mathcal{M}, M, u)$ is called a *(Riemannian) Monge-Ampère model*. Patrizio and Wong [9] studied the special case where M is a compact symmetric space. Their results were later generalized by Lempert and Szöke [7], and independently (when M is real analytic) by Guillemin and Stenzel [4], to the case where M is an arbitrary compact Riemannian manifold. The results in [7] were extended by Szöke in [11] and [12]. The main results of [7] and [11] are summarized in the following theorem:

Theorem 1.3 (Lempert-Szöke [7] and Szöke [11])**.** *(a) Let* $(\mathcal{M}, M, u)$ *be a Monge-Ampère model. Then the set of curves given by the intersection of* M *with the leaves of the Monge-Ampère foliation is precisely the set of geodesics of* M *with respect to the induced metric* g*.*

(b) Every compact real analytic Riemannian manifold arises from a Monge-Ampère model.

(c) Let $(\mathcal{M}, M, u)$ *and* $(\mathcal{M}', M', u')$ *be two Monge-Ampère models. Suppose that* (M, g) *and* (M', g') *are isometric and* $\sup u = \sup u'$ *then there is a biholomorphic map* $\Phi : \mathcal{M} \to \mathcal{M}'$ *such that* $u = u' \circ \Phi$*.*

In a related paper [6], Lempert showed that the Riemannian manifold M, metric g, and exhaustion function u^2, associated to a Monge-Ampère model are all real analytic. And in [12] Szöke proved a further generalization of part (c) of Theorem 1.3.

Results. Our goal is to understand the structure of the singular set of a solution of the Monge-Ampère equation under weakened smoothness assumptions on u as well as weakened assumptions on the topology of M. Throughout this paper $\mathcal{M}$ denotes a complex n-dimensional Stein manifold. We remark that the assumption that $\mathcal{M}$ is Stein is made to avoid cases such as $\mathcal{M} = X \times Y$ with X Stein and Y compact. When we say that u *is a solution of the homogeneous Monge-Ampère equation* we always mean that u is an everywhere continuous, non-negative, plurisubharmonic exhaustion of $\mathcal{M}$ that is a solution of the equation

$$(dd^{\mathbb{C}}u)^n = 0 \,, \quad (dd^{\mathbb{C}}u)^{n-1} \neq 0$$

on the set $\mathcal{M} \setminus M$, where M, the zero set of u, is assumed to be a smooth compact submanifold. We also assume the u is smooth on the complement of M[1]

Additional smoothness assumptions on u and M are made in both [7] and [4], where u^2 is assumed to be a smooth Kähler potential on all of $\mathcal{M}$ and M is assumed to have real dimension

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[1]Most of our results apply to the case where u is only of class C^3, the minimum smoothness assumption needed to make our geometrical constructions.

n. In Section 4, we show that both of these assumptions can be weakened, Theorem 4.11 shows that the assumption that u^2 is a smooth Kähler potential implies a regularity condition for u on the normal-blowup of M (see below). And Theorem 4.3 shows that the regularity condition implies that the singular set is an n-dimensional, totally real submanifold.

Our regularity condition is expressed in term of the *normal blowup of M in $\mathcal{M}$*, which is a smooth manifold with boundary $\widetilde{\mathcal{M}}$, together with a smooth map

$$\widetilde{\pi} : \widetilde{\mathcal{M}} \to \mathcal{M}$$

such that (i) the space $SM = \widetilde{\pi}^{-1}(M)$ is diffeomorphic to the normal sphere bundle of M in $\mathcal{M}$, and (ii) $\widetilde{\pi} : \widetilde{\mathcal{M}} \setminus SM \to \mathcal{M} \setminus M$ is a diffeomorphism. Section 3 contains a more detailed description. Let $\tilde{u} = u \circ \widetilde{\pi}$ and $\widetilde{\theta} = \widetilde{\pi}^* d^{\mathbb{C}} u$. We assume that $\tilde{u}$ is a smooth function.[2] And we replace the assumption that u^2 is strictly plurisubharmonic by the assumption that the 1-form θ extends smoothly to all of $\widetilde{\mathcal{M}}$ and satisfies the non-degeneracy condition

$$d\tilde{u} \wedge \widetilde{\theta} \wedge (d\widetilde{\theta})^{n-1} \neq 0 \,.$$

When u satisfies these conditions we say that u is *regular on the normal blowup of M*.

Theorem 1.3 generalizes to the case where u is regular on the normal blowup. We prove that regularity implies that the pull-back to SM of the form $\widetilde{\theta}$ is a contact form on SM. Let Q denote the normal bundle of M. Then, by Theorem 4.3, the bundle map

$$TM \xrightarrow{J} T\mathcal{M}|_M \xrightarrow{\pi_Q} Q \,,$$

where J is the complex structure tensor of $\mathcal{M}$ and π_Q is projection onto the normal bundle, is an isomorphism. Therefore, SM can be identified with the projective tangent bundle of M. We show in Section 5 that this form defines a Finsler metric F on M. Following the terminology of [9], we say that the triple $(\mathcal{M}, M, u)$ a *regular Monge-Ampère model* for the Finsler metric F. Our main result is the following theorem.

Theorem 1.4. *Let $\mathcal{M}$ be a Stein manifold and let $u \geq 0$ be a solution of the Monge-Ampère equation $(dd^{\mathbb{C}}u)^n = 0$, $(dd^{\mathbb{C}}u)^{n-1} \neq 0$ on $\mathcal{M} \setminus M$, where $M = \{u = 0\}$. Finally assume that M is a compact, smooth submanifold.*

(a) If u is regular on the normal blowup of M, then $(\mathcal{M}, M, u)$ is a regular Monge-Ampère model for a Finsler metric (M, F). The leaves of the Monge-Ampère foliation intersect M along geodesics.

(b) Every real analytic Finsler metric on M arises from a regular Monge-Ampère model.

(c) Let $(\mathcal{M}, M, u)$ and $(\mathcal{M}', M', u')$ be two real analytic Monge-Ampère models for the real analytic Finsler metrics (M, F) and (M', F'), respectively. Then there is a biholomorphic map $\Phi : \mathcal{M} \to \mathcal{M}'$, defined in a neighborhood of M such that $u = u' \circ \Phi$ if and only if (M, F) and (M, F') are isometric.

The paper is organized as follows. In Section 2 we show that the space $\mathcal{M} \setminus M$ is diffeomorphic to the product of a contact manifold and an open interval. In Section 3 we define the normal blowup. In Section 4, we give a precise definition of regularity on the normal blowup, extend the contact structure in Section 2 to the the blowup, and prove that regularity implies total reality of M. In Section 5, we review Finsler geometry and give the proof that M inherits a Finsler metric. In Section 6, we complete the proof of Theorem 1.4.

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[2]Most of our computations require that $\tilde{u}$ only be C^3.

Remark 1.5. In [7], Lempert and Szőke conjectured that the case when u^2 is not smooth could be studied by replacing the Riemannian metric g by a Finsler metric. Theorem 1.4 confirms this conjecture.

The authors would like to thank the referee for pointing out an error in the original proof of part (b) of Theorem 1.4

2. Contact geometry away from the singular set

Our goal is to understand the relation between solutions of the Monge-Ampère equation on a Stein manifold and the geometry of its singular set.

We assume that $\mathcal{M}$ is a complex n-dimensional Stein manifold and that u is an everywhere continuous, non-negative, solution of the homogeneous Monge-Ampère equation

$$(dd^{\mathbb{C}}u)^n = 0$$

whose zero set $M = \{u = 0\}$ is a smooth, compact submanifold. By restricting to a neighborhood of M if necessary, we assume that u is bounded above by $R > 0$ and that $u^{-1}(r)$ is compact for all $0 \le r < R$. Finally, we assume that $\tau = u^2/2$ is strictly plurisubharmonic on $\mathcal{M} \setminus M$.[3]

Because τ is strictly plurisubharmonic, the two form $dd^{\mathbb{C}}\tau$ has rank n away from M, and the computation

$$(dd^{\mathbb{C}}\tau)^n = \left(du \wedge d^{\mathbb{C}}u + u\, dd^{\mathbb{C}}u\right)^n = (du \wedge \theta + u\, d\theta)^n = n\, u^{n-1}\, du \wedge \theta \wedge (d\theta)^{n-1}\,.$$

shows that u satisfies the non-degeneracy condition

$$(2.1) \qquad\qquad du \wedge d^{\mathbb{C}}u \wedge (dd^{\mathbb{C}}u)^{n-1} \neq 0\,.$$

It follows that the level set $M_\epsilon = \{u = \epsilon\}$ is a smooth, contact manifold for all ϵ between 0 and R. The contact form on M_ϵ is the pull-back of the one-form

$$\theta = d^{\mathbb{C}}u\,.$$

Because u has no critical points, the level sets M_ϵ are all diffeomorphic. Indeed, they are isomorphic as calibrated, contact manifolds. (A *calibrated contact manifold* is a contact manifold with a distinguished contact form.) To see this, let Y be the vector field characterized by the conditions

$$(2.2) \qquad\qquad Y \lrcorner\ \theta = 0\,, Y \lrcorner\ du = 1\,, \text{ and } Y \lrcorner\ d\theta = 0\,.$$

Let μ_t denote the flow of Y. Because $du(Y) = 1$, μ_t maps level sets of u to level sets, and, therefore, defines a diffeomorphism

$$(2.3) \qquad\qquad \mu : M_{R/2} \times (0, R) \to \mathcal{M} \setminus M : (p, t) \mapsto \mu_{t-R/2}(p)$$

satisfying the identity

$$(2.4) \qquad\qquad u \circ (\mu(p, t)) = t\,.$$

The computation of the Lie derivative

$$\mathcal{L}_Y \theta = d(Y \lrcorner\ \theta) + Y \lrcorner\ d\theta = 0$$

then shows that μ_t restricts to a contact diffeomorphism

$$\mu_{\epsilon_2 - \epsilon_1} : M_{\epsilon_1} \to M_{\epsilon_2}\,.$$

between each pair of level sets.

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[3]We gain no further generality by replacing $\tau = u^2/2$, by a more general function $\tau = f(u)$, with $f'(u), f''(u) > 0$ for $u > 0$: For $(dd^{\mathbb{C}}\tau)^n = n(f')^{n-1}f'')du \wedge d^{\mathbb{C}}u \wedge (dd^{\mathbb{C}}u)^{n-1}$, which is a positive multiple of $(dd^{\mathbb{C}}(u^2/2))^n$.

Remark 2.5. The form θ satisfies an even stronger condition: The identities $Y \lrcorner \theta = 0$ and $\mathcal{L}_Y \theta = 0$ together imply that θ descends to a contact form on the orbit space $(\mathcal{M} \setminus M)/Y \simeq M_{R/2}$. This implies that $\mathcal{M} \setminus M$ has the structure of the product of a contact manifold with the interval $(0, R)$ and that the pull-back $\mu^* \theta$ extends smoothly to $M_{R/2} \times [0, R)$ with $\mu^*(\theta \wedge (d\theta)^{n-1}) \neq 0$ everywhere.

3. The normal blowup of M

Because u is continuous, the level sets M_ϵ approach M as ϵ approaches 0. When u is sufficiently well-behaved, the contact structures on the level sets converge to a limiting contact structure on the projective normal bundle of M. The normal blowup of M, defined below, is our main tool for formalizing this behavior.

We first discuss the simpler case of the blowup of the origin in $\mathbb{R}^m$. In this context, blowing up is just the transformation to spherical coordinates. Spherical coordinates, which we formalize by the *blowdown map*

$$\widetilde{\pi} : \widetilde{\mathbb{R}}^q := S^{q-1} \times [0, \infty) \to \mathbb{R}^m : (v, r) \mapsto r \cdot v \,,$$

where S^{q-1} is the unit sphere in $\mathbb{R}^q$. The preimage $\widetilde{\pi}^{-1}(0)$ is called the *blowup of the origin*. Notice that any smooth curve satisfying the conditions $\gamma(t)$, $t \geq 0$ with $\gamma(0) = 0$, $\gamma'(0) \neq 0$, and $\gamma(t) \neq 0$ for $t > 0$, has a unique lift to a smooth curve on the blowup defined by

$$\tilde{\gamma}(t) = \begin{cases} \left(\frac{\gamma(t)}{\|\gamma(t)\|}, \|\gamma(t)\| \right) & \text{for } t > 0, \\ \left(\frac{\gamma'(0)}{\|\gamma'(0)\|}, 0 \right) & \text{for } t = 0. \end{cases}$$

Remark 3.1. We want to emphasize the following three obvious properties of the lift:

  (i) $\tilde{\gamma}(t)$ intersects the boundary of $\widetilde{\mathbb{R}}^m$ transversely;
  (ii) $\tilde{\gamma}(0)$ depends only on the oriented ray generated by $\gamma'(0)$;
  (iii) $\tilde{\gamma}(0) = \gamma'(0)/\|\gamma'(0)\|$.

Roughly speaking, the normal blowup of a submanifold M is obtained by replacing each point of M by the blowup of the origin of the vector space of normal vectors to M in $\mathcal{M}$. We now present a more formal description.

Consider first the case where V is a q-dimensional vector space and M is the origin. Let V_0 be the set of non-zero vectors, and let SV denote the space of oriented rays through the origin. We call SV the *(oriented) projectivization* of V. The *blowup* of V at the origin is the subspace

$$\widetilde{V} = \{([v], r \cdot v) \in SV \times V \ : \ v \in V, v \neq 0, \ r \in [0, \infty)\} \,,$$

where $[v]$ denotes the oriented ray defined by the non-zero vector $v \in V$. This definition generalizes fiber wise to a vector bundle E in the standard way. In this case SE denotes the oriented projective bundle of E and $\widetilde{E}$ denotes the blowup of the set of zero vectors of E. There is a natural *blowdown* map $p : \widetilde{E} \to E$. It is easy to check that if E is equipped with a norm, we can identify SE with the set of unit length vectors, and the map

$$SE \times [0, \infty) \to \widetilde{E} \ : \ (v, r) \mapsto ([v], r \cdot v)$$

is a diffeomorphism. In particular, SE is a sphere bundle over M. The canonical map

$$\widetilde{\pi} : \widetilde{E} \to E$$

sending $([v], v)$ to v and $([v], 0)$ to the zero vector is called the *blowdown* map.

The normal blowup of a submanifold is the non-linear version of the blowup of the zero section of a vector bundle. We give two equivalent constructions here. The first highlights the role of the normal bundle and uses the exponential map of an auxiliary metric, the second is based on

local coordinate charts and does not rely on an explicit choice of metric. The proof that these constructions are equivalent is an exercise in differential geometry, which we leave to the reader.

Let Q denote the normal bundle of M. Then there is a short exact sequence of vector bundles

$$0 \to TM \longrightarrow T\mathcal{M}|_M \xrightarrow{\pi_Q} Q \to 0\,.$$

A choice of a Riemannian metric on $\mathcal{M}$ gives a splitting, under which Q can be identified with the orthogonal complement of TM in $T\mathcal{M}$. The exponential map defines a diffeomorphism between an ϵ-neighborhood of the zero-section of Q and a neighborhood of M in $\mathcal{M}$. Let $\widetilde{B}_\epsilon \subset \widetilde{Q}$ be a neighborhood of the blowup of the zero-section of Q. The *normal blowup* of $\mathcal{M}$ along M is the manifold $\widetilde{\mathcal{M}}$ obtained by identifying points in the manifold $\mathcal{M} \setminus M$ with points in $\widetilde{B}_\epsilon$ by the exponential map. Let

$$\widetilde{\pi} : \widetilde{\mathcal{M}} \to \mathcal{M}$$

be the blowdown map, defined in the obvious way. Notice that $\widetilde{\mathcal{M}}$ is a smooth manifold whose boundary is the subspace $SM = \widetilde{\pi}^{-1}(M)$. By definition, $SM = SQ$. Observe also that the distance to SM is comparable to the distance to M with respect to the Riemannian metric on $\mathcal{M}$. We call the submanifold SM the *normal blowup of M* (or less formally, the *blowup of M*).

Our second construction of the blowup begins with a collection U_α of open subsets of $\mathcal{M}$ whose union contains M, together with a collection of coordinates charts

$$\phi_\alpha : U_\alpha \to V_\alpha \times B_\epsilon^q \ : \ p \mapsto (x, y),$$

which satisfy the compatibility condition $M \cap U_\alpha = V_\alpha \times \{0\}$, where V_α is an open subset of $\mathbb{R}^n$ and B_ϵ^q denotes the ball of radius ϵ in $\mathbb{R}^q$ centered at the origin. The transition functions are maps of the form

$$\phi_{\alpha,\beta} = \phi_\beta \circ \phi_\alpha^{-1} \ : \ V_{\alpha,\beta} \times B_\epsilon^q \to: \ V_{\beta,\alpha} \times B_\epsilon^q \ : \ (x,y) \mapsto (X(x,y), Y(x,y))$$

where $V_{\alpha,\beta} = \phi_\alpha\left(M \cap U_\alpha \cap U_\beta\right)$. By virtue of the compatibility condition, the y-component of the transition functions can be written in the form

$$(3.2) \qquad\qquad Y^k(x,y) = a_i^k(x)y^i + R_{i,j}^k(x,y)y^i y^j$$

where $A = (a_j^i(x))$ is a smooth family of invertible $q \times q$ matrices and $R_{i,j}^k(x,y)$ are smooth functions, the indices i, j, k ranging between 1 and q with the summation convention in force. Thus, for $t \geq 0$ sufficiently small, the transition functions induce maps

$$(3.3) \qquad\qquad \widetilde{\phi}_{\alpha,\beta} : V_{\alpha,\beta} \times S^{q-1} \times [0,\epsilon) \to V_{\beta,\alpha} \times S^{q-1} \times [0,\epsilon)\,,$$

defined by the formula

$$(3.4) \qquad \widetilde{\phi}_{\alpha,\beta}(x,v,r) = \begin{cases} \left(X(x,rv), \frac{Y(x,rv)}{\|Y(x,rv)\|}, \|Y(x,rv)\|\right) & r > 0\,, \\ \left(X(x,0), \frac{A \cdot v}{\|A \cdot v\|}, 0\right) & r = 0\,. \end{cases}$$

A straightforward computation shows that these functions satisfy the cocycle condition

$$\widetilde{\phi}_{\beta,\gamma} \circ \widetilde{\phi}_{\alpha,\beta} = \widetilde{\phi}_{\alpha,\gamma}\,.$$

Let $\sim$ denote the equivalence relation on the disjoint union $(\mathcal{M} \setminus M) \ \dot\cup \ \bigcup_\alpha \left(V_\alpha \times S^{q-1} \times [0,\epsilon)\right)$ generated by the relations $(x,v,r) \sim \widetilde{\phi}_{\alpha,\beta}(x,v,r)$ and $p \sim \phi_\alpha \circ \widetilde{\pi}(p)$. The *normal blowup of M in $\mathcal{M}$* is defined to be the quotient space

$$(\mathcal{M} \setminus M) \ \dot\cup \ \bigcup_\alpha \left(V_\alpha \times S^{q-1} \times [0,\epsilon)\right) \ / \sim \ .$$

The cocycle condition guarantees that $\widetilde{\mathcal{M}}$ is a smooth $(n+q)$-dimensional manifold with boundary diffeomorphic to SQ; and the blowdown map $\widetilde{\pi}$ is smooth by construction.

The verification that two definitions of normal blowup are equivalent is an elementary exercise in differential geometry, which we leave to the reader.

3.1. **Blowup coordinates.** We will often have to work in local homogeneous coordinates centered at an oriented normal ray in SM. More specifically, we shall choose a local coordinate chart $\phi : U \to V \times B_\epsilon^q$ with local coordinate functions

$$(x, y) = \left(x^1, \ldots, x^n, y^1, \ldots, y^q \right)$$

on $\mathcal{M}$ such that M intersects U in the set $\{y = 0\}$. The collection of points of $\widetilde{\mathcal{M}}$ over U is then a set of the form

$$\left(x, \frac{y}{\|y\|}, \|y\| \right) \in V \times S^{q-1} \times [0, \epsilon) \, .$$

We shall choose ϕ so that the ray of interest is defined by $y = (0, \ldots, 0, 1)$. The map

$$\left(x, \frac{y}{\|y\|}, \|y\| \right) \mapsto (x, p, r) = (x, \frac{y^1}{y^q}, \ldots, \frac{y^{q-1}}{y^q}, y^q) \in V \times R^{q-1} \times [0, \infty)$$

is clearly a coordinate chart for $\widetilde{\mathcal{M}}$ centered at the ray. We shall refer to such coordinates as *blowup coordinates.* In blowup coordinates, the blowdown map assumes the form

$$(3.5) \qquad \widetilde{\pi} : (x, p, r) = (x, p^1, \ldots, p^{q-1}, r) \mapsto (x, y) = (x, (r\, p^1, \ldots, r\, p^{q-1}, r)) \, .$$

The following lemma summarizes some of the elementary properties of the blowup that we need. It is an obvious extension of Remark 3.1. The proof is an elementary exercise, which we leave to the reader.

Lemma 3.6. *Let $\gamma(t)$, $t \geq 0$, be a smooth curve in $\mathcal{M}$ intersecting M transversely at $t = 0$, with $\gamma(t) \notin M$ for $t > 0$.*

   (i) *Then $\gamma(t)$ has a unique lift to a smooth curve $\widetilde{\gamma}(t)$ in $\widetilde{\mathcal{M}}$ defined by letting $\tilde{\gamma}(0) \in SQ$ be the oriented ray generated by $\pi_Q(\gamma'(0))$.*

   (ii) *Let f be a smooth function on $\widetilde{\mathcal{M}}$ that vanishes on SM, then the quantity $df(\widetilde{\gamma}'(0))$ depends only on $\pi_Q \gamma'(0)$.*

   (iii) *Let Y be a vector based at a point $p \in SM = SQ$ such that $\pi_Q(\widetilde{\pi}_* Y) \neq 0$. Then the ray defined by the normal vector $\pi_Q(\widetilde{\pi}_* Y)$ is p, itself.*

4. The structure of the singular set

In this section, we give a regularity condition on u that generalizes the one given in [7] and explore some of its implications. Set $\tilde{u} = \widetilde{\pi}^* u$ and $\widetilde{\theta} = \widetilde{\pi}^* \theta$. We say that u is *regular on the normal blowup of M* (or more simply *regular on the blowup*) if and only if is satisfies the following two conditions:

   (i) $\tilde{u}$ and $\widetilde{\theta}$ extend smoothly to all of $\widetilde{\mathcal{M}}$,

   (ii) the form $d\tilde{u} \wedge \widetilde{\theta} \wedge (d\widetilde{\theta})^n$ is non-vanishing on all of $\widetilde{\mathcal{M}}$.

The next proposition roughly states that regularity on the blowup is equivalent to the condition that $\widetilde{\mathcal{M}}$ be the product of a contact manifold with an interval. This is the main geometric fact underlying all of our results.

Proposition 4.1. *The diffeomorphism of Equation* (2.3) *extends to a diffeomorphism*

$$\widetilde{\mu} : M_\epsilon \times [0, R) \to \widetilde{\mathcal{M}}$$

if and only if u is regular on the blowup.

Proof. Assume that μ extends to a diffeomorphism $\widetilde{\mu}$ as above; then, by virtue of Equation (2.4), $\tilde{u} = \pi_2 \circ \widetilde{\mu}$, where $\pi_2(p, t) = t$. Because π_2 is smooth and $d\pi_2 = dt$, the function $\tilde{u}$ is smooth and $d\tilde{u}$ never vanishes. Recall from Remark 2.5, that the form $\mu^*(\theta)$ extends smoothly to all of $M_\epsilon \times [0, R)$ and restricts to a contact form on $M_\epsilon \times \{0\}$. This implies that $\widetilde{\theta}$ is smooth on all of $\widetilde{\mathcal{M}}$ and restricts to a contact form on SM.

Conversely, suppose that $\tilde{u}$ is regular on the blowup and that the form $\widetilde{\theta}$ is extends smoothly to all of $\widetilde{\mathcal{M}}$ and restricts to a contact form on SM. Then because $d\tilde{u}$ is non-vanishing, $M_{R/2}$ is diffeomorphic to SM. It also follows that the construction of the vector field Y given in Section 2 extends to define a vector field $\widetilde{Y}_u$ on all of $\widetilde{\mathcal{M}}$. Since $\widetilde{Y}_u$ is transverse to SM, the map

$$\widetilde{\mu} : SM \times [0, R) \to \widetilde{\mathcal{M}} \ : \ (p, t) \mapsto \widetilde{\mu}_t(p)$$

where $\widetilde{\mu}_t$ is the flow of $\widetilde{Y}_u$ is a diffeomorphism. By uniqueness of integral curves, $\widetilde{\mu}$ agrees with μ on the interior of $\widetilde{\mathcal{M}}$. $\square$

Remark 4.2. A result very much like this appears in the paper of Burns [3].

Recall that the Theorems of Stoll [10] and Lempert-Szöke [7] concern the structure of the singular set of u in the extreme cases where its dimension is either 0 or n. Our next result shows that under mild regularity conditions on u, no other dimensions are possible.

Theorem 4.3. *Suppose that u is a solution of the Monge-Ampère equation that is regular on the normal blowup of M. Then M is an n-dimensional, totally real submanifold of $\mathcal{M}$.*

Our proof proceeds by studying the lift of the Monge-Ampère foliation $\mathcal{F}$ to $\widetilde{\mathcal{M}}$. Assume that u is regular on the blowup. Then by Proposition 4.1, the closed form $d\widetilde{\theta}$ has rank $n - 1$ everywhere on $\widetilde{\mathcal{M}}$, as does its restriction to SM, the boundary of $\widetilde{\mathcal{M}}$. Consequently, $\mathcal{F}$ lifts to a non-singular foliation $\widetilde{\mathcal{F}}$ of $\widetilde{\mathcal{M}}$ by (real) surfaces, and the leaves of $\widetilde{\mathcal{F}}$ intersect SM transversely in curves.

Each leaf of $\mathcal{F}$ has a holomorphic parameterization expressed in terms of the complex flow of the complex vector field

$$(4.4) \hspace{4cm} Z = X + iY,$$

where X and Y are real vector fields on $\widetilde{\mathcal{M}}$ characterized by the conditions

$$(4.5) \hspace{1cm} X \lrcorner \, \widetilde{\theta} = -1, X \lrcorner \, d\tilde{u} = 0, X \lrcorner \, d\widetilde{\theta} = 0 \ \text{ and } \ Y \lrcorner \, \widetilde{\theta} = 0, Y \lrcorner \, d\tilde{u} = 1, Y \lrcorner \, d\widetilde{\theta} = 0.$$

Notice that Y is the extension to all of $\widetilde{\mathcal{M}}$ of the vector field defined in Equation (2.2). Let ν_t and μ_t be the flows of X and Y, respectively. For each point $\tilde{p} \in SM$, consider the map

$$(4.6) \hspace{2.5cm} \tilde{\phi}_{\tilde{p}} : H \to \widetilde{\mathcal{M}} \ : \ \zeta = s + ir \mapsto \nu_s \circ \mu_r(\tilde{p}),$$

where $H = \{s + ir \in \mathbb{C} \ : \ 0 \le r < R\}$, and set

$$(4.7) \hspace{3cm} \phi_{\tilde{p}} = \widetilde{\pi} \circ \tilde{\phi}_{\tilde{p}} : H \to \mathcal{M}.$$

Lemma 4.8. *For each $\tilde{p} \in SM$, the map $\tilde{\phi}_{\tilde{p}}$ is well-defined and the map $\phi_{\tilde{p}}$ is holomorphic and non-singular at all points of H. The collection of images of $\phi_{\tilde{p}}$ as $\tilde{p}$ ranges over all of SM spans the*

Monge-Ampère foliation. Finally, the leaf of $\mathcal{F}$ defined by $\phi_{\tilde{p}}$ intersects M along the non-singular curve

$$s \mapsto \phi_{\tilde{p}}(s)\,, s \in \mathbb{R}\,.$$

Proof. We claim that $X = JY$ away from SM and that X and Y commute everywhere. To verify the first condition, recall that $\widetilde{\theta}$ is the extension of $\theta = d^{\mathbb{C}}u$ to $\widetilde{\mathcal{M}}$, and that $d^{\mathbb{C}} = d \circ J$, where J is the complex structure tensor; the identity $Y = JX$ follows from the definition (4.5). To see that the vector fields X and Y commute, recall that because $d\tilde{u} \wedge \widetilde{\theta} \wedge (d\widetilde{\theta})^{n-1}$ is a volume form on $\widetilde{\mathcal{M}}$, we need only prove that the Lie bracket $[X, Y]$ is in the kernel of each of the forms $d\tilde{u}$, $\widetilde{\theta}$, and $d\widetilde{\theta}$. But

$$0 = d^2\tilde{u}(X, Y) = Xd\tilde{u}(Y) - Yd\tilde{u}(X) - d\tilde{u}([X, Y]) = -d\tilde{u}([X, Y])$$

and

$$0 = d\widetilde{\theta}(X, Y) = X\widetilde{\theta}(Y) - Y\widetilde{\theta}(X) - \widetilde{\theta}([X, Y]) = -\widetilde{\theta}([X, Y])\,,$$

and, because the kernel of $d\widetilde{\theta}$ is an involutive distribution and X and Y are both in the kernel, so is $[X, Y]$. Because $\mathcal{M} = u^{-1}([0, R))$ and the level sets of u are all compact, $\tilde{\phi}$ is well defined on all of H. Because $Y = JX$ on $\mathcal{M} \setminus M$, ϕ is a holomorphic curve in $\mathcal{M}$.

By construction, the vectors $\widetilde{\pi}_*(X)$ and $\widetilde{\pi}_*Y(p)$ are non vanishing for all $p \in \widetilde{\mathcal{M}}$. Consequently, ϕ is a non-singular parameterization of a leaf of the Monge-Ampère foliation.

To see that every leaf of $\mathcal{F}$ is contained in the image of $\phi_{\tilde{p}}$ for some $\tilde{p} \in SM$, choose a point in $p \in \mathcal{M} \setminus M = \widetilde{\mathcal{M}} \setminus SM$. Then $p = \mu_r(\tilde{p})$ for a unique point $\tilde{p} \in SM$. Hence, the leaf of $\mathcal{F}$ through p is contained in the image of $\phi_{\tilde{p}}$.

Finally, to verify that the leaves of $\mathcal{F}$ intersect M along non-singular curves of the form $s \mapsto \phi_{\tilde{p}}(s)$, recall that $X(\tilde{u}) = 0$. This shows that the curve is contained in M. Moreover, by construction,

$$\phi'_{\tilde{p}}(s) = J\left(\widetilde{\pi}_*Y_{\tilde{\phi}(s)}\right) \neq 0\,,$$

showing that the curve is non-singular. $\qquad\square$

Proof of Theorem 4.3. Assume that u is regular on the normal blowup. First observe that the flow of Y induces a continuous deformation retract $\rho : \mathcal{M} \to M$ defined as follows

$$\rho(p) = \begin{cases} p & \text{for } p \in M \\ \widetilde{\pi} \circ \mu_{-u(p)}\left(\widetilde{\pi}^{-1}(p)\right) & \text{for } p \in \mathcal{M} \setminus M. \end{cases}$$

Hence, M and $\mathcal{M}$ have the same homotopy type. By the theorem of Andreotti-Frankel [1], the Stein manifold $\mathcal{M}$ has the homotopy type of an n-dimensional cell complex. Consequently, M can have dimension at most n.

Let $T\mathcal{M}$ denote the tangent bundle of M, and let $J : T\mathcal{M} \to T\mathcal{M}$ denote the complex structure tensor of $\mathcal{M}$. We claim that the composition

$$(4.9) \qquad\qquad TM \overset{J}{\longrightarrow} T\mathcal{M}|_M \overset{\pi_Q}{\longrightarrow} Q$$

is a surjective map onto the normal bundle of M in $\mathcal{M}$. Because the dimension of M is at most n, this claim implies, that the map (4.9) is an isomorphism of vector spaces, hence, that M is totally real.

To prove that the map (4.9) is surjective, first choose a point $p \in M$ and a non-zero vector $v \in Q_p$. We need only show that a multiple of v is in the image of this map. But the vector v defines an

oriented ray, which by definition of SM is a point $\tilde{p} \in SM$ with $\widetilde{\pi}(\tilde{p}) = p$. By Lemma 3.6(iii), the oriented rays defined by $\pi_Q\left(J\widetilde{\pi}_* X_{\tilde{p}}\right)$ and v coincide. $\qquad\square$

Remark 4.10. A theorem of Harvey and Wells [5] states that the zero set of a non-negative, strictly plurisubharmonic function is locally contained in a totally real submanifold. Because u^2 may not be smooth on $\mathcal{M}$, the theorem does not apply.

Our next theorem shows that Proposition 4.1 is, indeed, a generalization of the requirement in [7] that u^2 be a smooth Kähler potential. More generally, one could assume that the function $\tau = f(u)$ is a smooth potential function for a Kähler metric. The next theorem shows that all such conditions imply that u is regular on the blowup.

Theorem 4.11. *Let $u \geq 0$ be a solution of the Monge-Ampère equation on $\mathcal{M}$. Assume that the singular set $M = \{u = 0\}$ a smooth submanifold. Suppose that $\tau = f \circ u \geq 0$ is a smooth, strictly plurisubharmonic exhaustion function for $\mathcal{M}$, where f is a real analytic function with $f(0) = 0$ and with $f'(u)$ and $f''(u)$ both positive for $u > 0$. Then u is regular on the blowup and M is a totally real submanifold of maximum dimension.*

Proof. We first claim that the form $\widetilde{\theta}$ extends smoothly to all of $\widetilde{\mathcal{M}}$. To see this, give $\mathcal{M}$ the Kähler metric defined by the Kähler potential τ. One easily verifies that the vector field $Z = X + iY$ defined in (4.4) satisfies the identity

$$Z \lrcorner\, dd^{\mathbb{C}}\tau = \frac{f''(u)}{f'(u)}\left(d\tau + id^{\mathbb{C}}\tau\right)$$

on $\mathcal{M} \setminus M$. Therefore by Theorem 1.2, the gradient vector field $\nabla\tau$ is a scalar multiple of Y, and each integral curve of Y is contained in a geodesic of $\mathcal{M}$ that intersects the level sets of τ orthogonally.

Consequently, these geodesics lift to the blowup and intersect the boundary of $\widetilde{\mathcal{M}}$ transversely. The union of all of these curves forms a one dimensional foliation of $\widetilde{\mathcal{M}}$ with tranversal intersection with the boundary of $\widetilde{\mathcal{M}}$. Moreover, the leaves of this foliation are (by construction) the closures of the integral curves of Y. The identities $Y \lrcorner\, \widetilde{\theta} = \mathcal{L}_Y \widetilde{\theta} = 0$ then show that the form $\widetilde{\theta} = \widetilde{\pi}^* d^{\mathbb{C}} u$ extends smoothly to all of $\widetilde{\mathcal{M}}$ and is non-vanishing at all points of SM.

Let θ_S denote the pullback of $\widetilde{\theta}$ to the boundary $SM \subset \widetilde{\mathcal{M}}$. The non-degeneracy condition $\widetilde{\theta} \wedge (d\widetilde{\theta})^{n-1} \neq 0$, implies that θ_S is a contact form on SM. Therefore, to conclude the proof of regularity, we need only show that $\tilde{u}$ is smooth on all of $\widetilde{\mathcal{M}}$ and that $d\tilde{u}$ is non-vanishing near SM. We do this obtaining explicit formulas for $d^{\mathbb{C}} u$ and $\tilde{u}$ in blowup coordinates adapted to the complex structure on $\mathcal{M}$.

By a theorem of Harvey and Wells [5] (see also [7]), M, the zero set of a smooth strictly plurisubharmonic function, is totally real. Let $m \leq n$ be the dimension of M, and let $q = n - m$, and let the indices j and a range between 1 and m and 1 and q, respectively.

We choose holomorphic coordinates

$$\mathcal{M} \supset U \to \mathbb{C}^{m+q} \,:\, p \mapsto (z^1, \ldots, z^n)$$

with $z = x + iy$ and a smooth function $H : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{C}^q$ such that

$$M \cap U = \{z \in \mathbb{C}^{m+q} \,:\, (y^1, \ldots, y^m, z^{m+1}, \ldots, z^{m+q}) = H(x^1, \ldots, x^m)\}.$$

Because M is totally real, we may choose coordinates so that H vanishes to arbitrarily high order at $x = 0$. These coordinates are not adapted to M, so they must be replaced by the adapted

coordinates $(x^1, \ldots, x^m, v^1, \ldots, v^m, w^1, \ldots, w^q)$ defined by

$$v^j = y^j - H^j(x^1, \ldots, x^m) \quad w^a = z^{m+a} - H^{m+a}(x^1, \ldots, x^m).$$

Blowup coordinates are then given by the formulas

$$v^\alpha = rp^\alpha, \quad v^m = r, \quad w^a = r\zeta = r(\xi^a + i\eta^a),$$

where Greek indices range between 1 and $m - 1$. Since we only have to compute $d^{\mathbb{C}}u$ on the set $x^j = 0$, and since H vanishes to high order, we may assume that $H(x)$ is identically zero in any finite order computation along $x^j = 0$. In particular, up to first order along the set $x^j = 0$, $j = 1, \ldots, n$, we have

$$z^\alpha = x^\alpha + irp^\alpha, \quad z^m = s + ir, \quad z^{m+a} = r\zeta = r(\xi^a + i\eta^a).$$

(To highlight the special role played by the radial parameter r, we have written $z^m = s + ir$.) A straightforward computation using the chain rule, shows that

$$\widetilde{\pi}^*(d^{\mathbb{C}}u) = -\left(\frac{\partial u}{\partial r} - \frac{p^\alpha}{r}\frac{\partial u}{\partial p^\alpha} - \frac{\xi^a}{r}\frac{\partial u}{\partial \xi^a} - \frac{\eta^a}{r}\frac{\partial u}{\partial \eta^a}\right) ds + \left(\frac{\partial u}{\partial s} + rp^\alpha\frac{\partial u}{\partial p^\alpha} - \frac{\xi^a}{r}\frac{\partial u}{\partial \eta^a} + \frac{\eta^a}{r}\frac{\partial u}{\partial \xi^a}\right) dr$$

$$- \frac{1}{r}\frac{\partial u}{\partial p^\alpha}\, dx^\alpha + r\frac{\partial u}{\partial x^\alpha}\, dp^\alpha - \frac{\partial u}{\partial \eta^a}\, d\xi^a + \frac{\partial u}{\partial \xi^a}\, d\eta^a.$$

We claim that $\tilde{u}$ can we written in the form

(4.12) $$\tilde{u} = r^{2/k}\, U(x', s, p, \xi, \eta, r^{2/k}),$$

where $k > 0$ is an integer, $x' = (x^1, \ldots, x^{m-1})$, and $U(x', s, p, \xi, \eta, t)$ is a differentiable function of t such that

$$U(x', s, p, \xi, \eta, 0) > 0.$$

Assume this claim for the moment. Then, substituting (4.12) into the formula for $\widetilde{\theta} = \widetilde{\pi}^* d^{\mathbb{C}}u$ and simplifying gives

$$\widetilde{\theta} = -r^{\frac{2}{k}-1}\left(\frac{2}{k}\, U + r\frac{\partial U}{\partial r} - p^\alpha\frac{\partial U}{\partial p^\alpha} - \xi^a\frac{\partial U}{\partial \xi^a} - \eta^a\frac{\partial U}{\partial \eta^a}\right) ds$$

$$+ r^{\frac{2}{k}-1}\left(r\frac{\partial U}{\partial s} + r^2 p^\alpha\frac{\partial U}{\partial x^\alpha} + \eta^a\frac{\partial U}{\partial \xi^a} - \xi^a\frac{\partial U}{\partial \eta^a}\right) dr$$

$$- r^{\frac{2}{k}-1}\left(\frac{\partial U}{\partial p^\alpha}\, dx^\alpha\right) + r^{\frac{2}{k}+1}\left(\frac{\partial U}{\partial x^\alpha}\, dp^\alpha\right) - r^{\frac{2}{k}}\left(\frac{\partial U}{\partial \eta^a}\, d\xi^a - \frac{\partial U}{\partial \xi^a}\, d\eta^a\right)$$

But we have already proved that $\widetilde{\theta}$ extends smoothly to the set $r = 0$ and is nowhere-vanishing. Inspection of the above formula for $d^{\mathbb{C}}u$ shows that this implies that $k = 2$. Thus, $\tilde{u} = rU(x^\alpha, p^\alpha, s, \xi, \eta, r)$, which is smooth on all of $\widetilde{\mathcal{M}}$. The formula $d\tilde{u} = U\, dr$ for $r = 0$ shows that u is regular. That that M is totally real and has dimension n follows from Theorem 4.3.

It remains only to prove that $\tilde{u}$ is of the form (4.12). Because f is real analytic, τ has a series expansion of the form

$$\tau = f(u) = au^k\, (1 + g(u))$$

where $a > 0$ and $g(u)$ is a smooth function such that $g(0) = 0$. Therefore, the equation

$$\tau^{1/k} = a^{1/k}\, u\, (1 + g(u))^{1/k}$$

can be inverted to show that u is of the form

(4.13) $$u = \tau^{1/k}\, G(\tau^{1/k})$$

for $G(t)$ a smooth (in fact, analytic) function satisfying the condition $G(0) > 0$.

On the other hand, $\tau \geq 0$ is smooth and vanishes precisely on M. This, together with the positivity condition $dd^{\mathbb{C}}\tau > 0$, implies that τ vanishes precisely to order 2 on M. Therefore, τ can be expressed in the form

$$(4.14) \qquad\qquad \tau = r^2\, T(x', s, p, \xi, \eta, r)\,,$$

where $T(x', p, \xi, \eta, r)$ is a smooth function and $T(x', p, \xi, \eta, 0) > 0$. Combining (4.13) and (4.14), and setting $U = \sqrt{T}$ results in the expression (4.12).

Setting $k = 2$ in the above formula for $\widetilde{\theta}$ and simplifying yields the identity

$$(4.15) \qquad \theta_S = -\left(U - p^\alpha \frac{\partial U}{\partial p^\alpha} - \xi^a \frac{\partial U}{\partial \xi^a} - \eta^a \frac{\partial U}{\partial \eta^a} \right) ds - \frac{\partial U}{\partial p^\alpha}\, dx^\alpha\,.$$

At this point, we invoke Theorem 4.11 to conclude that $m = n$ and $q = 0$, and that M is totally real. $\qquad\qquad\square$

Remark 4.16. For later reference, we note that because $q = 0$, Formula 4.15 reduces to the identity

$$\theta_S = -\left(U - p^\alpha \frac{\partial U}{\partial p^\alpha} \right) ds - \frac{\partial U}{\partial p^\alpha}\, dx^\alpha\,.$$

5. The Finsler metric on M

When u is regular on the blowup, the restriction of $\widetilde{\theta}$ to SM is a contact form. We now show that that where u is regular on the blowup, it induces a *Finsler metric* on M and that the leaves of the Monge-Ampère foliation intersect M along geodesics.

5.1. Review of Finsler geometry.
We begin with a quick review of Finsler geometry from the perspective of contact geometry. For a more complete and more general, exposition of these ideas the reader should consult the paper of Pang [8]. Let $\pi : TM \to M$ denote the tangent space of M and let $T_0 M \subset TM$ denote the set of non-zero tangent vectors. A *Finsler metric* on M is a smooth, positive function $F : T_0 M \to R$ that satisfies the following two conditions:

(i) For all $X \in T_0 M$ and all $t > 0$, $F(tX) = tF(X)$.

(ii) The set $S_p = \{X : F(X) = 1\}$ is strongly convex and diffeomorphic to a sphere.

Let x^j, $j = 1, \ldots, n$ be local coordinates on M and let $(x^j, \dot{x}^j)$ be the induced coordinates on the tangent bundle. The *Hilbert form* θ_F on $T_0 M$ is the 1-form defined by the local formula

$$\theta_F = \frac{\partial F(x, \dot{x})}{\partial \dot{x}^j}\, dx^j\,,$$

where the summation conventions are in force. It is not difficult to show that the convexity condition (ii) is equivalent to the condition that

$$\theta_F \wedge (d\theta_F)^{n-1}$$

be non-vanishing.

The homogeneity of F implies that θ_F is the pullback of a 1-form on the projective tangent bundle SM, which by abuse of notation we also denote by θ_F. To see this, let

$$X_R = \dot{x}^j \frac{\partial}{\partial \dot{x}^j}$$

denote the radial vector field. We must only show that

$$X_R \lrcorner\, \theta_F = 0 \text{ and } \mathcal{L}_{X_R} \theta_F = 0\,.$$

The first identity is obvious. To prove the second, compute as follows:

$$\mathcal{L}_{X_R}\theta_F = X_R \lrcorner \, d\theta_F + d(X_R \lrcorner \, \theta_F) = X_R \lrcorner \, d\theta_F = \dot{x}^j \frac{\partial^2 F}{\partial \dot{x}^j \partial \dot{x}^k} \, dx^k = 0 \,,$$

where the last equality on the right follows by differentiating Euler's identity, $\dot{x}^j \dfrac{\partial F}{\partial \dot{x}^j} = F$, with respect to $\dot{x}^k$. We have proved the following lemma:

Lemma 5.1. *The function F is a Finsler metric if and only if the form θ_F on SM is a contact form, i.e.*

$$\theta_F \wedge d\theta_F^{n-1} \neq 0 \,.$$

Remark 5.2. The geodesics of a Finsler manifold have an elegant formulation in terms of the Hilbert form. The *Reeb vector field* of the contact contact manifold (SM, θ_F) is the vector field X_F characterized by the conditions:

$$\theta_F(X_F) = 1 \text{ and } X_F \lrcorner \, d\theta_F = 0 \,.$$

The geodesics of (M, F) are the images under the projection map $\pi : SM \to M$ of the integral curves of X_F. In fact, if $t \mapsto \nu_t(\tilde{p})$ is the integral curve of X_F starting at $\tilde{p} \in SM$, then $\gamma : t \mapsto \pi \circ \nu_t(\tilde{p})$ is the unit speed geodesic with $[\gamma'(0)] = \tilde{p}$.

5.2. Construction of the metric.
Let u be a solution of the Monge-Ampère equation and assume that u is regular on the blowup of M. Recall that this implies that M is a maximal, totally real submanifold of $\mathcal{M}$. Thus the composition of the maps in (4.9) is an isomorphism of vector bundles.

Define $F : TM_0 \to \mathbb{R}$ as follows. Let X be a non-zero tangent vector based at a point $p \in M$. Let $\gamma(t)$ be curve such that $\gamma(0) = p$ and $\gamma'(0) = JX$. Then we set

$$(5.3) \qquad F(X) = \lim_{t \to 0^+} \frac{u \circ \gamma(t)}{t} \,.$$

The next proposition shows that F is a Finsler metric on M.

Proposition 5.4. *Suppose that u is regular on the normal blowup of M and let θ_S denote contact form on SM obtained by pulling-back the form $\widetilde{\theta}$ to SM. Then F is a Finsler metric and its Hilbert form θ_F coincides with $-\theta_S$.*

Proof. Let $\widetilde{\gamma}(t)$ be the lift of $\gamma(t)$ to $\widetilde{\mathcal{M}}$ defined in Lemma 3.6(i). Then

$$F(X) = d\tilde{u}(\widetilde{\gamma}'(0)) \,.$$

By (3.6(i)), $F(X)$ depends only on X; thus, F is well defined. Homogeneity of F follows from the definition of F. To see that $F(X)$ is positive, write $\tilde{u}$ in the form

$$\tilde{u}(x, p, r) = r \, U(x, p, r) \,,$$

where (x, p, r) are blowup coordinates as in 3.6. Because u is regular on the blowup, $U(x, p, 0)$ is strictly positive. Consequently,

$$F(X) = r'(0) \, U\left(x(0), p(0), 0\right)$$

where $\widetilde{\gamma}(t) = (x(t), p(t), r(t))$. Finally, observe that $r'(0)$ is positive because $JX = \gamma'(0)$ is transverse to M.

By Lemma 5.1, to conclude the proof we need only show that $-\theta_S$ coincides with the Hilbert form of F. We prove equality via explicit formulas for both forms using blowup coordinates centered at an arbitrary point $p_0 \in M$. Because p_0 is arbitrary, we need only verify equality on the fiber $\widetilde{\pi}^{-1}(p_0)$.

Choose holomorphic coordinates $z^j = x^j + i\, y^j$, $j = 1, \ldots, n$, centered at p_0 as in the proof of Theorem 4.11. In these coordinates, u assumes the form

$$(5.5) \qquad\qquad u(x, p, r) = rU(x, p, r)$$

and, by Remark 4.16, the form θ_S assumes the form

$$(5.6) \qquad\qquad \theta_S = -\left(U - p^\alpha \frac{\partial U}{\partial p^\alpha}\right) ds - \frac{\partial U}{\partial p^\alpha} dx^\alpha \,.$$

We next focus on the computation of F and θ_F. Let

$$X = \dot{x}^1 \frac{\partial}{\partial x^1} + \cdots + \dot{x}^{n-1} \frac{\partial}{\partial x^{n-1}} + \dot{s} \frac{\partial}{\partial s}$$

denote a tangent vector to M at p_0. If the ray generated by JX is in the coordinate patch of $\widetilde{\mathcal{M}}$, then $\dot{s} > 0$. Then by (5.5) and Lemma 3.6,

$$F(X) = \dot{s}\, U\left(x, p, 0\right) \,,$$

where $p^\alpha = \dot{x}^\alpha/\dot{s}$. The Hilbert form of F is therefore given by

$$\theta_F = \frac{\partial F}{\partial \dot{x}^j}\, dx^j = \left(U - p^\alpha \frac{\partial U}{\partial p^\alpha}\right) ds + \frac{\partial U}{\partial p^\alpha} dx^\alpha \,.$$

Comparing this formula with (5.6) yields the equality $\theta_F = -\theta_S$ and concludes the proof of the proposition. $\qquad\qquad\square$

Proof of Theorem 1.4(a). The proof is a corollary to Proposition 5.4. Because the forms $-\theta_S$ and θ_F coincide, the Reeb vector field of θ_F coincides with the restriction to SM of the vector field X defined by Equation (4.5). But Lemma 4.8 shows that the projection onto M of the integral curves of X are the intersections of leaves of the Monge-Ampère foliation with M. $\qquad\square$

6. Construction of regular Monge-Ampère models

In this section, we prove parts (b) and (c) of Theorem 1.4. Our proof is a generalization of a construction of Lempert-Szöke [7].

Before beginning the proof, we make a few preliminary observations. Recall that, because M is totally real, the complex structure tensor J induces an analytic isomorphism between the projective tangent bundle of M and the projective normal bundle SQ, which is, by construction, the boundary of $\widetilde{\mathcal{M}}$. We may, therefore, identify the boundary SM of $\widetilde{\mathcal{M}}$ with the projective tangent bundle of M.

Thus far, we have worked in the smooth category; we now introduce the further assumption that all data are real analytic. Specifically, let (M, F) denote a compact, real analytic manifold with a real analytic Finsler metric. Then the oriented projective tangent bundle SM is also real analytic, as are the Hilbert form θ_F and the Reeb vector field X_F. If follows that the flow of X_F,

$$\nu : SM \times \mathbb{R} \to SM \,:\, (p, t) \mapsto \nu_t(p)$$

defines a real analytic family of diffeomorphisms of SM.

Next let $\mathcal{M}$ denote the complexification of M. By construction, M is an analytic, n-dimensional, totally real submanifold of its complexification $\mathcal{M}$, and any real analytic atlas for M extends to define a holomorphic atlas for $\mathcal{M}$. Using this atlas to define the normal blowup as in Section 3 immediately shows that $\widetilde{\mathcal{M}}$ has real analytic boundary and that the blowdown map

$$\widetilde{\pi} : \widetilde{\mathcal{M}} \to \mathcal{M}$$

is real analytic. With this identification, we have the following diagram of real analytic maps:

$$(6.1) \qquad SM \times \mathbb{R} \xrightarrow{\nu} SM \hookrightarrow \widetilde{\mathcal{M}} \xrightarrow{\widetilde{\pi}} \mathcal{M} \,.$$

We are now going to extend this map to the domain $SM \times \mathbb{C}$ by analytic continuation and use the extension to define a solution u of the Monge-Ampère equation. The map (6.1) gives a real-analytic family of curves, $\gamma_{\tilde{p}}$, $\tilde{p} \in SM$, defined by

$$(6.2) \qquad \gamma_{\tilde{p}} : \mathbb{R} \to M \subset \mathcal{M} \,:\, t \mapsto \widetilde{\pi} \left(\nu_t(\tilde{p}) \right) \,,$$

and each curve is both a geodesic in the Finsler manifold (M, F) and a real analytic curve in $\mathcal{M}$. By virtue of the second property, each of these curves can be holomorphically extended to a holomorphic curve defined on a neighborhood of of $\mathbb{R}$ in $\mathbb{C}$. The next lemma shows that the extension is uniform over all of SM.

Lemma 6.3. *There exists a real number $R > 0$ and a real analytic extension*

$$\nu^{\mathbb{C}} : SM \times H_R \to \widetilde{\mathcal{M}}$$

of ν, where $H_R = \{s + ir \,:\, 0 \le y < R\}$. The map $\nu^{\mathbb{C}}$ has the following properties:

(i) *For each $p \in SM$, the map $z \mapsto \widetilde{\pi} \circ \nu_C(p, z)$ is a holomorphic immersion.*

(ii) *The map $\mu : SM \times [0, R) \to \widetilde{\mathcal{M}}$ defined by the formula*

$$\mu(p, r) = \nu^C(p, 0 + ri)$$

is a real analytic diffeomorphism onto its image.

Proof. Choose a point $p \in SM$. Because γ_p is real analytic, for sufficiently small $\epsilon > 0$, it has a holomorphic extension $\gamma_p^{\mathbb{C}} : V \to \mathcal{M}$, where $V_{\epsilon} = \{z = s + ir \,:\, |s| < \epsilon, \, 0 \le r < \epsilon$. It is easy to check that $\gamma_p^{\mathbb{C}}$ lifts to a real analytic map

$$\tilde{\gamma}_p^{\mathbb{C}} : V \to \widetilde{\mathcal{M}}$$

which is an extension ν. By analytic dependence of γ_p on p and compactness of SM, there exists a real number $R > 0$ such that $\gamma_p^{\mathbb{C}}$ is defined on V_R for all $p \in SM$. We now have a real analytic map

$$\nu^{\mathbb{C}} \,:\, SM \times V_R \to \widetilde{\mathcal{M}} \,,$$

which is holomorphic in the second factor. The one-parameter identity $\nu_{t+s} = \nu_t \circ \nu_s$ then allows us to extend the map to all of $SM \times H_R$ as the composition

$$\nu_{s+ir}^{\mathbb{C}}(p) = \nu_{s/k+ir}^{\mathbb{C}} \circ \cdots \circ \nu_{s/k+ir}^{\mathbb{C}}(p) \,,$$

where the integer k is chosen so that $|s/k| < R$.

Property (i) of ν follows by construction. To prove property (ii), first observe that μ is the identity map on $SM \times \{0\}$. We, therefore, need only show that the derivative of μ is injective on all of SM. It then follows (after shrinking R if necessary) that μ is a diffeomorphism, as claimed. But because μ is the identity on SM, it follows that μ_* is injective if and only if the vector field $\mu_*(\partial/\partial r)$ is transverse to SM. It suffices to show that the projection $\widetilde{\pi}_* \mu_*(\partial/\partial r)$ is transverse to M. But this is clear, for by construction

$$(\widetilde{\pi} \circ \mu)_*(\partial/\partial r) = \frac{d}{dr} \gamma_p^{\mathbb{C}}(ir) = J\nu_t'(p) \,.$$

This completes the proof of the lemma. $\qquad \qquad \square$

Replace $\mathcal{M}$ by the image of μ, and let $\tilde{u} : \widetilde{\mathcal{M}} \to \mathbb{R}$ be the smooth function defined by the formula

$$(6.4) \qquad\qquad \tilde{u} : \widetilde{\mathcal{M}} \overset{\mu^{-1}}{\longrightarrow} SM \times [0, R) \overset{\pi_2}{\longrightarrow} \mathbb{R},$$

where π_2 is projection onto the second factor. Because $\tilde{u}$ vanishes on SM, it descends to a continuous function u on $\mathcal{M}$. To complete the proof of Theorem 1.4(b), we need only show that $(\mathcal{M}, M, u)$ is a real analytic Monge-Ampère model for (M, F). We need only check that the following conditions are satisfied (after possibly further shrinking R) :

  (i) $\tilde{u}$ is smooth on all of $\widetilde{\mathcal{M}}$;
  (ii) u induces the Finsler metric F. Specifically, choose a tangent vector X and let $\tilde{\gamma}(t)$ be the lift to $\widetilde{\mathcal{M}}$ of a smooth curve $\gamma(t)$ with $\gamma'(0) = JX$, then $F(X) = d\tilde{u}(\tilde{\gamma}'(0))$;
  (iii) $dd^{\mathbb{C}} u^2 > 0$ on $\mathcal{M} \setminus M$;
  (iv) $(dd^{\mathbb{C}} u)^n = 0$ on $\mathcal{M} \setminus M$;
  (v) $\theta = d^{\mathbb{C}} u$ lifts to a smooth form $\widetilde{\theta}$ which extends smoothly to all of $\widetilde{\mathcal{M}}$ and which satisfies the inequality $d\tilde{u} \wedge \widetilde{\theta} \wedge (d\widetilde{\theta})^{n-1} \neq 0$.

Properties (i) and (ii) follow immediately from the constructions above.

To verify condition (iii), choose an arbitrary point $p \in SM$ and choose blowup coordinates $(x^\alpha, s, p^\alpha, r)$ centered at p with $z^\alpha = x^\alpha + irp^\alpha$, $z^n = s + ir$ holomorphic coordinates on $\mathcal{M}$. We claim that complex Hessian of

$$H_{\mathbb{C}}(u^2) = \begin{pmatrix} \frac{\partial^2 u^2}{\partial z^\alpha \partial \bar{z}^\beta} & \frac{\partial^2 u^2}{\partial z^\alpha \partial \bar{z}^n} \\ \frac{\partial^2 u^2}{\partial z^n \partial \bar{z}^\beta} & \frac{\partial^2 u^2}{\partial z^n \partial \bar{z}^n} \end{pmatrix}$$

extends continuously to a positive definite matrix on a neighborhood of p. Assume the claim for the moment. By compactness of SM, there is an open neighborhood $U \subset \mathcal{M}$ of M such that $dd^{\mathbb{C}} u^2 > 0$ on $U \setminus M$. By shrinking R if necessary, we may assume that $U = \mathcal{M}$.

To prove the claim, observe that by construction $u = r U(s, r, x^\alpha, p^\alpha)$, where U is smooth and $U(s, 0, x^\alpha, p^\alpha) > 0$. Noting that

$$x^\alpha = \frac{1}{2}(z^\alpha + z^{\bar{\alpha}}), \quad p^\alpha = \frac{z^\alpha - z^{\bar{\alpha}}}{z^n - z^{\bar{n}}}, \quad s = \frac{1}{2}(z^n + z^{\bar{n}}), \quad r = \frac{1}{2i}(z^n - z^{\bar{n}})$$

and applying the chain rule to u^2 yields the formulae

$$(6.5) \qquad \begin{cases} \dfrac{\partial^2 u^2}{\partial z^\alpha \partial \bar{z}^\beta} &= \dfrac{1}{4}\dfrac{\partial^2 (U^2)}{\partial p^\alpha \partial p^\beta} \\ \dfrac{\partial^2 u^2}{\partial z^\alpha \partial z^{\bar{n}}} &= \dfrac{1}{4}\dfrac{\partial}{\partial p^\alpha}\left(2U^2 - p^\beta \dfrac{\partial (U^2)}{\partial p^\beta}\right) \\ \dfrac{\partial^2 u^2}{\partial z^n \partial z^{\bar{n}}} &= \dfrac{1}{4}\left(2U^2 - 3p^\alpha \dfrac{\partial (U^2)}{\partial p^\alpha} + p^\alpha p^\beta \dfrac{\partial^2 (U^2)}{\partial p^\alpha \partial p^\beta}\right) \end{cases}$$

for $r = 0$.

On the other hand, (ii) implies that the Finsler metric F has the form

$$F(x, \dot{x}) = \dot{x}^n U(s, 0, x^\alpha, \dot{x}^\alpha / \dot{x}^n).$$

The convexity for F implies that the real Hessian

$$H_{\mathbb{R}}(F^2) = \begin{pmatrix} \frac{\partial^2 F}{\partial \dot{x}^\alpha \partial \dot{x}^\beta} & \frac{\partial^2 F}{\partial \dot{x}^\alpha \partial \dot{x}^n} \\ \frac{\partial^2 F}{\partial \dot{x}^n \partial \dot{x}^\alpha} & \frac{\partial^2 F}{\partial \dot{x}^n \partial \dot{x}^n} \end{pmatrix}$$

is positive definite for all $\dot{x} \neq 0$. A straightforward computation shows that

$$
(6.6) \quad
\begin{cases}
\dfrac{\partial^2(F^2)}{\partial \dot{x}^\alpha \partial \dot{x}^\beta} &= \dfrac{\partial^2(U^2)}{\partial p^\alpha \partial p^\beta} \\[2mm]
\dfrac{\partial^2(F^2)}{\partial \dot{x}^\alpha \partial \dot{x}^n} &= \dfrac{\partial}{\partial p^\alpha}\left(2U^2 - p^\beta \dfrac{\partial(U^2)}{\partial p^\beta} \right) \\[2mm]
\dfrac{\partial^2(F^2)}{\partial \dot{x}^n \partial \dot{x}^n} &= 2U^2 - 3p^\alpha \dfrac{\partial(U^2)}{\partial p^\alpha} + p^\alpha p^\beta \dfrac{\partial^2(U^2)}{\partial p^\alpha \partial p^\beta}
\end{cases}
$$

Comparison of (6.5) and (6.6) shows that $H_\mathbb{C}(u^2) = \frac{1}{4} H_\mathbb{R}(F^2)$ at $r = 0$. Consequently, the complex Hessian of u^2 is positive definite in a neighborhood of $p \in SM$.

To verify condition (iv), recall that by Lemma 6.3 every point of $\mathcal{M} \setminus M$, is contained in the image of a holomorphic curve of the form $z \mapsto \nu^\mathbb{C}(p, z)$. By definition, $u \circ \nu^\mathbb{C}(p, z) = \Im(z)$, showing that the pull-back of $dd^\mathbb{C} u$ to the curve vanishes. Together with (iii), this shows that $dd^\mathbb{C} u$ has rank strictly less than n.

To verify condition (v), we first show that the form $\widetilde{\theta} = \widetilde{\pi}^* d^\mathbb{C} u$ extends to all of $\widetilde{\mathcal{M}}$. To see this note that by construction, the vector field on $\widetilde{\mathcal{M}}$ $Y = \mu_* \frac{\partial}{\partial r}$ satisfies the identity $Y \lrcorner \, \widetilde{\theta} = 0$ on the set $\widetilde{\mathcal{M}} \setminus SM$, and the computation

$$
\mathcal{L}_Y \widetilde{\theta} = Y \lrcorner \, d\widetilde{\theta} = 0
$$

shows that the Lie derivative vanishes. Consequently, we need only verify (iv) on SM. But since $Y(\tilde{u}) = 1$, we need only show that θ_S, the pull back of $\widetilde{\theta}$ to SM, is a contact form. But the computations leading to the formula (5.6) all apply here, showing that $\theta_F = -\theta_S$. Non-degeneracy follows, concluding the proof of part (b) of Theorem 1.4.

To prove part (c) of Theorem 1.4, first suppose that $\Phi : \mathcal{M} \to \mathcal{M}$ is a biholomorphism between two Monge-Ampère models $(\mathcal{M}, M, u)$ and $(\mathcal{M}', M', u')$ such that $u = u' \circ \Phi$. Equation (5.3) shows the Φ restricts to an isometry between (M, F) and (M', F'). Conversely, any analytic isometry between real analytic Finsler manifolds (M, F) and (M', F') extends uniquely to a biholomorphism $\Phi : \mathcal{M} \to \mathcal{M}'$ between their complexifications.

Therefore, we need only show that $u = u' \circ \Phi$, which we can do by proving equality on each leaf of $\mathcal{F}$. To this end, let $\gamma^\mathbb{C} : H_R \to \mathcal{M}$ be the holomorphic parameterization of a leaf given above. Then $U(z) = u \circ \gamma^\mathbb{C}(z)$ and $U'(z) = u' \circ \Phi \circ \gamma^\mathbb{C}(z)$ are both real analytic solutions of the initial value problem

$$
\frac{\partial^2 U}{\partial r^2} = -\frac{\partial^2 U}{\partial s^2}\,, \quad U(s,0) = 0, \quad \frac{\partial U(s,0)}{\partial r} = 1\,.
$$

By the Cauchy-Kovaleskaya Theorem, it follows that $U(s,r) = r$. Uniqueness follows, completing the proof of Theorem 1.4.

References

[1] A. Andreotti and T. Frankel, *The Lefschetz theorem on hyperplane sections*, Annals of Math., **69** (1959), 713–717.

[2] E. Bedford and M. Kalka, *Foliations and the complex Monge-Ampère equation*, Commun. Pure Appl. Math. **30** (1977), 543–571.

[3] D. Burns, *Curvatures of Monge-Ampère foliations and parabolic manifolds*, Annals of Mathematics **115** (1982), 348–373.

[4] V. Guillemin and M. Stenzel, *Grauert tubes and the homogeneous Monge-Ampère equation*, JDG **34** (1991), 561–570.

[5] F. Reese Harvey and R. O. Wells, Jr., *Zero sets of non-negative strictly plurisubharmonic functions*, Math. Ann. **201** (1973), 165–170.

[6] L Lempert, *Complex structures on the tangent bundle of Riemannian manifolds*, in "Complex analysis and geometry", Vincenzo Ancona and Alessandro Silva, eds., Plenum, New York, 1993, 235–251.

[7] L. Lempert and R. Szöke, *Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundle of riemannian manifolds*, Math. Ann. **290** (1991), 689–712.

[8] M.-Y. Pang, *The structure of Legendre foliations*, Trans. Amer. Math. Soc. **320** (1990), 417–455.

[9] G. Patrizio and P. M. Wong, *Stein manifolds with compact symmetric center.*, Math. Ann. **289** (1991), 355–382.

[10] W. Stoll, *The characterization of strictly parabolic manifolds*, Ann Scuola. Norm. Sup. Piza, **7** (1980), 87–154.

[11] R. Szöke, *Complex structure on tangent bundles of Riemannian manifolds*, Math. Ann. **291** (1991), 409–428.

[12] R. Szöke, *Automorphisms of certain Stein manifolds*, Math. Z. **219** (1995), 357–385.

[13] J.H.C. Whitehead, *Convex regions in the geometry of paths*, Quart. J. Math. Oxford Ser. **3** (1932),33–42.

[14] P.-M. Wong, *Geometry of the homogeneous Monge-Ampère equation*, Invent. Math. **67** (1982), 261–274.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, BOX 354350, SEATTLE, WA 9819-43505
E-mail address: duchamp@math.washington.edu

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
E-mail address: kalka@math.tulane.edu