THE CLASSIFICATION OF LEGENDRE IMMERSIONS

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ABSTRACT. The main result of this paper is a homotopy theoretic classification of Legendre immersions from a compact manifold into a contact manifold. The paper also includes normal form theorems for Legendre submanifolds as well as a multi-jet transversality theorem for Legendre immersions.

CONTENTS

1. INTRODUCTION

A smooth manifold M^{2n+1} equipped with a one form of satisfying the non-degeneracy condition that $\eta \wedge d\eta^n$ never vanishes is called a *contact manifold* and an immersion $\varphi : \Sigma^n \to M^{2n+1}$ satisfying the condition $\varphi^* \eta = 0$ is called a *Legendre immersion*.

Such immersions arise in both classical and quantum mechanics, [AM], [Ar1] and [E], in the study of partial differential equations, [Ly] and in complex analysis as peak-interpolation sets for the algebra $A(D)$ of functions holomorphic on the strictly pseudoconvex domain D, continuous on $\overline{D}$ [BS], [CC1], [CC2], [HS], [R1]. A classification of Legendre immersions is, therefore, of some interest and is the subject of the present paper. The related problem of classifying Lagrangian immersions into symplectic manifolds was studied by Lees [L] using ideas of M. Gromov[Gr].

In [L], Lees classified homotopy classes of Lagrangian immersions into a symplectic manifold by using the work of Weinstein [W1] to adapt the proof of Haefliger and Poenaru [HP] of the classification theorem for combinatorial immersions to the symplectic setting. The strategy here is the same. The main technical result needed

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to use the Haefliger-Poenaru machine is a certain homotopy extension lemma (Section 5 of this paper). Lees' proof does not readily adapt to our case because it is based on the construction of symplectic isotopies of a symplectic manifold which are localized on a small open set and the corresponding construction does not work on contact manifolds. This necessitates a different proof of the homotopy extension theorem. Apart from this, we proceed as Lees does; but because we do not have to consider the cohomological condition of Lees our proof is more straightforward. Because we wish to prove certain extension theorems with no analogue in [L] and because the results here are of interest to complex analysts, we present here a relatively detailed exposition.

The outline of the paper is as follows. Section 2 contains a statement of the main results. In Section 3 several applications of the results stated in Section 2 are presented. Section 4 contains various theorems on normal forms for Legendre immersions in the spirit of Weinstein [W1]. In Section 5 a homotopy extension theorem is proved. Section 6 contains the statement and proof of the classification theorem for Legendre immersions. In Section 7 a version of the Thom transversality theorem for Legendre immersions is proved and used to show that in the compact case every Legendre immersion is homotopic to a Legendre embedding.

I wish to thank E. L. Stout for many helpful discussions. This paper arose out of an effort to answer several questions raised by him in [St]. Discussions with M. Bendersky, A. M. Chollet, R. Hain, and D. Ravenel were quite useful. D. Bennequin carefully read an early draft of the paper, who found an error in my original proof of the Homotopy Extension Lemma in Section 5. Claude Viterbo fixed the error, and the proof given in Section 5 is essentially his.

Remark [added June, 1996] The original version of the paper was distributed in preprint form in 1982. In view of the imminent publication of Gromov's book [Gr] in which he presents a general theory of which the main result of the present paper is a corollary, I chose not to publish. Gromov's book, however, refers to the preprint, and the repeated requests for copies of the preprint indicate that it should be made more widely available. Apart from correcting the error in the my original proof of the Homotopy Extension Lemma and updating the bibliography, I have kept the paper pretty much in its original form. The reader should consult Gromov's book [Gr] and the paper of Audin [Au] for more recent bibliography.

2. Definitions and Statement of Results

In this section we review the definition of Legendre immersion, define the various types of homotopies of such immersions and state our main results concerning the homotopy classification of Legendre immersions.

A **contact manifold** is a pair (M, η) consisting of a manifold M of dimension $2n+1$ and a one form η with the property that the $(2n+1)$-form $\eta \wedge (d\eta)^n$ is a volume form. The **Reeb vector field** of (M, η) is a vector field X_η determined by the conditions $\eta(X_\eta) = 1$ and $X_\eta \rfloor d\eta = 0$. A **horizontal vector** is a vector annihilated by η and the rank $2n$ subbundle of $T(M)$ consisting of all such vectors will be denoted by $H(M)$.

Example 2.1 (Circle Bundles)**.** A good example to keep in mind it the case where M is the total space of the canonical circle bound of a Kähler manifold and η its holomorphic connection one form. In this case X_η is the vertical vector field defined by the circle action on M and $H(M)$ is the bundle of horizontal vectors of the connection.

It is not hard to see that the form $d\eta$ defines a symplectic structure on the bundle $H(M)$ and it can be shown [W1, p. 8] that one can always find a complex structure $J : H(M) \to H(M)$ on $H(M)$ which is compatible with the symplectic structure, i.e., if X and Y are vectors based at $p \in M$ then $d\eta(JX, JY) = d\eta(X, Y)$. *We will assume that such a complex structure has been chosen; and in the case where M is the boundary of a strictly pseudoconvex domain, we assume that J is the restriction of the complex structure map of the ambient complex manifold. We let g denote the unique Riemannian metric on M in which X_η is a unit vector orthogonal to $H(M)$ and such that $g(X, Y) = d\eta(JX, Y)$ for all $p \in M$ and $X, Y \in T(M)_p$.* Note that the form $h = g + id\eta$ defines a Hermitian structure on $H(M)$.

Let Σ be an n-dimensional manifold. An immersion $\varphi : \Sigma \to M$ is called a **Legendre immersion** if $\varphi^*\eta = 0$. If $\Sigma_0 \subseteq \Sigma$ is a closed subset of Σ a **germ at Σ_0 of a Legendre immersion** $\varphi : V \to M$, where V is a neighborhood of Σ_0, is an equivalence class of Legendre immersions of neighborhoods of Σ_0 (two immersions are said to be equivalent if they agree on a neighborhood of Σ_0). The germ of φ at Σ_0 is denoted by $[\varphi]$. An **ℓ-regular homotopy relative** to $[\varphi]$ is a smooth family $\varphi_t : \Sigma \to M$ of Legendre immersions, $t \in [0, 1]$, with $[\varphi_t] = [\varphi]$ for all t.

Notice that if $\varphi : \Sigma \to M$ is a Legendre immersion then its derivative $\varphi : T(\Sigma) \to T(M)$ in injective, it takes values in $H(M)$, and it satisfies the condition $\varphi^* d\eta = 0$ so that $\varphi_* T(\Sigma)_p$ is a Lagrangian subspace of the symplectic vector space $H(M)_{\varphi,p)}$ for all $p \in \Sigma$. We view the derivative $\varphi_* : T(\Sigma) \to H(M)$ as an infinitesimal version of φ. Notice that homotopies of immersions induce homotopies of their derivatives. We formalize this observation in the following definitions which are the infinitesimal analogues of Legendre immersion and homotopy:

For $\Sigma_0 \subseteq \Sigma$ a subset of Σ an **ℓ-bundle injection of $T(\Sigma)_{|\Sigma_0}$ into $H(M)$** is a vector bundle injection $\Phi : T(\Sigma)_{|\Sigma_0} \to H(M)$ such that $\Phi^* d\eta = 0$. The ℓ-bundle injection Φ is called an **integrable ℓ-bundle injection** if there is an immersion $\varphi : V \to M$ defined on a neighborhood of Σ_0 with $\Phi = \varphi_{*|\Sigma_0}$.

Remark 2.2. If $\Sigma_0 \subseteq \Sigma$ is an embedded submanifold and $\Phi : T(\Sigma)_{|\Sigma_0} \to H(M)$ is an ℓ-bundle injection over an immersion $\varphi : \Sigma_0 \to M$ such that $\Phi_{|T(\Sigma_0)} = \varphi$ then Φ is integrable. (To prove this, use the exponential map to find a suitable immersion of a neighborhood of Σ_0 into M.)

Two ℓ-bundle injections $\Phi_j : T(\Sigma)_{|V_j} \to H(M)$, $j = 1, 2$ defined on neighborhoods V_j, $j = 1, 2$ of a closed set Σ_0 are said to be **equivalent** if they agree over some neighborhood of Σ_0. The equivalence class of Φ_1 is called a **germ at Σ_0 of an ℓ-bundle injection** and is denoted by $[\Phi_1]$. Let $[\Phi]$ be the germ at Σ_0 of an ℓ-bundle injection, then an **ℓ-homotopy of ℓ-bundle injections relative to $[\Phi]$** is a smooth family $\Phi_t : T(\Sigma) \to H(M)$, $t \in [0, 1]$, of ℓ-bundle injections with $[\Phi_t] = [\Phi]$.

Our first result is a semi-local classification theorem for Legendre immersions:

Theorem 2.3 (Semi-local Classification)**.**

1. *Let $\Sigma_0 \subseteq \Sigma$ be a closed subset of Σ and let $\Phi_0 : (T\Sigma)_{|\Sigma_0} \to H(M)$ be an integrable ℓ-bundle inclusion over a map $\varphi_0 : \Sigma_0 \to M$. The map φ_0 extends to a Legendre immersion $\varphi : V \to M$ defined on a neighborhood of $\Sigma_0(M)$.*

2. *Moreover, there is a regular homotopy $\psi_t : V \to M$ of immersions extending φ with the following properties:*
   (a) $\psi_{0*}|\Sigma_0 = \Phi_0,\ \psi_1 = \varphi$; *and*
   (b) $\psi_{t*} : T(\Sigma)_{|\Sigma_0} \to H(M)$ *an ℓ-bundle injection for all $t \in [0,1]$.*

Our main result is Theorem 2.4. It is a global immersion theorem which classifies relative isotopy classes of Legendre immersions. Recall that a subset $\Sigma_0 \subset \Sigma$ is called a **smooth neighborhood retract** if there is a smooth nonnegative function $f : \Sigma \to \mathbf{R}$ with $\Sigma_0 = f^{-1}(0)$ and with df nowhere zero on the open set $f^{-1}([0,1]) - \Sigma_0$. The sets $V_\varepsilon = f^{-1}([0,\varepsilon]),\ 0 < \varepsilon \leq 1$ are called **tubular neighborhoods** of Σ_0.

Theorem 2.4 (Homotopy Classification)**.** *Let $\Sigma_0 \subseteq \Sigma^n$ be a compact, smooth neighborhood retract and let $[\varphi_0]$ be the germ at Σ_0 of a Legendre immersion into the contact manifold (M^{2n+1}, η).*

1. *If $\Phi : T(\Sigma) \to H(M)$ is an ℓ-bundle injection with $[\Phi] = [\varphi_{0*}]$ then there is a Legendre immersion $\varphi : \Sigma \to M$ with $[\varphi] = [\varphi_0]$ and φ and Φ are ℓ-homotopic relative to $[\Phi]$.*
2. *Moreover, the mapping $d : \varphi \mapsto \varphi$ induces a bijection between ℓ-regular homotopy classes of Legendre immersions relative to $[\varphi_0]$ and ℓ-regular homotopy classes of ℓ-bundle injections relative to $[\varphi_{0*}]$.*
3. *The subspace of injective, Legendre immersions is dense in the space of all Legendre immersions with the Whitney C^∞-topology.*
4. *If $\psi : \Sigma \to M$ to homotopic to a Legendre immersion φ then there is another Legendre immersion φ' which is C^0-close to ψ and ℓ-homotopic to φ.*

We now wish to relate the notion of Legendre immersion to the complex structure defined by J. Notice that if $\varphi : \Sigma \to M$ is a Legendre immersion then the derivative $\varphi_* : T(\Sigma) \to H(M)$ extends as follows

$$\begin{aligned} \varphi_*^{\mathbf{C}} : T(\Sigma)^{\mathbf{C}} &=: T(\Sigma) \otimes \mathbf{C} \to H(M) \\ X + iY &\mapsto \varphi_x(\chi) + J\varphi_*(Y) \end{aligned}$$

to an injection of complex vector bundles—in fact $\varphi^{\mathbf{C}*} : T(\Sigma)^{\mathbf{C}} \to \varphi_* H(M)$ is a complex bundle isomorphism. We may thus give complex analogues of ℓ-bundle injections, ℓ-regular homotopies, etc. A vector bundle map $\Phi : T(\Sigma)_{|\Sigma_0} \to H(M)$ over a subset $\Sigma_0 \subseteq \Sigma$ is called a **C-bundle injection** if the map $\Phi^{\mathbf{C}} : T(\Sigma)_{|\Sigma_0} \to H(M)$ is an injection (and hence an isomorphism) of complex vector bundles. A **C**-bundle injection $\Phi : T(\Sigma) \to H(M)$ is called **integrable** over $\Sigma_0 \subseteq \Sigma$ if there is an immersion φ of a neighborhood of Σ_0 into M such that $\varphi_{*|\Sigma_0}$ is an ℓ-bundle injection with $\Phi_{|\Sigma_0} = \varphi_{*|\Sigma_0}$. The definitions of **C-bundle homotopy** and **germs of C-bundle maps** should be clear.

Note that a **C**-bundle injection is not necessarily an ℓ-bundle injection. However, the next lemma shows that in the above theorems (and elsewhere in this paper) all ℓ-bundle injections and homotopies of ℓ-bundle injections can be replaced by **C**-bundle injections and homotopies of **C**-bundle injections.

Lemma 2.5. *Let Σ be an n-dimensional manifold and (M^{2n+1}, η) a contact manifold with an almost complex structure map $J : H(M) \to H(M)$. Then there is a map $\Phi \mapsto \Phi^\ell$ which assigns to each* **C***-bundle injection $\Phi : T(\Sigma) \to H(M)$ an*

ℓ-bundle injection $\Phi^\ell : T(\Sigma) \to H(M)$ which is $\mathbf{C}$-homotopic to Φ. Moreover, if $\Phi^* d\eta = 0$ at $x \in \Sigma$ then $\Phi_x = \Phi^\ell_x$. The assignment $\Phi \mapsto \Phi^\ell$ depends smoothly on Φ.

Proof. Let $\{V_\alpha\}_{\alpha=1}^N$ be a finite cover of Σ with $\overline{V}_\alpha \subseteq U_\alpha$ for $U_\alpha \subset \Sigma$ open, let $\rho_\alpha : \Sigma \to [0,1]$ be a smooth function with support contained in U_α and with $\rho_{\alpha|V_\alpha} = 1$ and let $\{b_{\alpha 1}, \ldots b_{\alpha n}\}$ be a framing of $T(\Sigma)$ over U_α for $1 \leq \alpha \leq N$. Denote the Hermitian inner product of $H(M)$ by h.

Now suppose inductively that $\Phi_{k-1} : T(\Sigma) \to H(M)$ is a $\mathbf{C}$-bundle injection, $\mathbf{C}$-homotopic to $\Phi = \Phi_0$ and that Φ_{k-1} is an ℓ-bundle injection on $\bigcup_{j=1}^{k-1} \begin{matrix} V \\ 0 \end{matrix}$. We will define a $\mathbf{C}$-bundle map $\Phi_k : T(\Sigma) \to H(M)$ which is $\mathbf{C}$-homotopic to Φ_{k-1} with Φ_k an ℓ-bundle injection on $\bigcup_{j=1}^{k} V - j$ and with $\Phi_{k,x} = \Phi_{k-1,x}$ at all $x \in \Sigma$ with $\Phi^*_{k-1x}(d\eta) = 0$. First set $\widetilde{b}_{kj} = \Phi_{k-1}(b_{kj}),\ j = 1, 2, \ldots, n$ and define $\{\widetilde{b}^\ell_{k1}, \ldots, \widetilde{b}^\ell_{kn}\}$ inductively by: $\widetilde{b}^\ell_{k1} = \widetilde{b}_{k1}$ and

$$\widetilde{b}_{kj} = \widetilde{b}_{kj} - \sum_{i=1}^{j-1} \frac{h(\widetilde{b}^\ell_{ki}, \widetilde{b}_{kj})}{h(\widetilde{b}^\ell_{ki}, \widetilde{b}^\ell_{ki})} \widetilde{b}^\ell_{kj}$$

for $j = 2, 3, \ldots, n$. Finally define Φ_k on U_k by

$$\Phi_k(b_{kj}(x)) = (1 - \rho_k(x))\widetilde{b}_{kj} + \rho_k(x)\widetilde{b}^\ell_{kj}$$

for $x \in U_k$ and extend to $T(\Sigma)_{|U_k}$ by linearity, and extend Φ_k to all of Σ by defining $\Phi_k = \Phi_{k-1}$ on $\Sigma - U_k$. □

We close this section with two corollaries.

Corollary 2.6. *Let Σ^n be a compact manifold whose complexified tangent bundle $T(\Sigma)^{\mathbf{C}}$ is trivial. Then Σ embeds as a Legendre submanifold of every contact manifold M of dimension $2n+1$.*

Proof. Take $\varphi : \Sigma \to M$ to be the constant map. Then the pull-back $\varphi^* H(M) \to \Sigma$ is trivial. Hence, there is a $\mathbf{C}$-bundle injection $\Phi : T(\Sigma) \to H(M)$ over φ. Now apply part (1) of Theorem 2.4. □

Corollary 2.7. *Let $\varphi_0 : \Sigma_0 \to M^{2n+1}$ be an immersion of the smooth p-dimensional manifold Σ_0 into the contact manifold (M, η), such that $p < n$ and $\varphi_0^* \eta = 0$. Then there is an n-dimensional manifold Σ containing Σ_0 and a Legendre immersion $\varphi : \Sigma \to M$ extending φ_0 if and only if the quotient bundle $\varphi_0^* H(M)/T(\Sigma_0)^{\mathbf{C}} \to \Sigma_0$ is the complexification of a real vector bundle over Σ_0.*

Proof. Suppose an immersion as in the corollary exists. Then since $\varphi_*^{\mathbf{C}} : T(\Sigma)^{\mathbf{C}} \to \varphi^* H(\Sigma)$ is an isomorphism extending the complex vector bundle inclusion $\varphi_0^{\mathbf{C}} : T(\Sigma_0)^{\mathbf{C}} \to \varphi^* H(M)$ it follows that $\varphi^* H(M)$ and $T(\Sigma)^{\mathbf{C}}_{|\Sigma_0}/T(\Sigma_0)^{\mathbf{C}} = (T(\Sigma)_{|\Sigma_0}/T(\Sigma_0))^{\mathbf{C}}$ are isomorphic.

Conversely, let $\varphi_0^* H(M)/T(\Sigma_0)^{\mathbf{C}}$ be isomorphic to $E \otimes \mathbf{C} \to \Sigma_0$ for $E \to \Sigma_0$ a real vector bundle. The Hermitian inner product on $H(M)$ yields an ℓ-bundle injection $\Phi : E \oplus T(\Sigma_0) \to H(M)$ extending φ_{0*}. Now identify Σ_0 with the image of the zero section of $E \to \Sigma_0$.

By Remark 2.2 the map $\Phi : T(E)_{|\Sigma_0} \to H(K)$ is integrable and Theorem 2.4 (2) applies. Let $\Sigma \subseteq E$ be the neighborhood of Σ_0 of Theorem 2.3 (1). □

Remark 2.8. Let $\varphi_0 : \Sigma_0 \to M$ be as in Section 2 and suppose that Σ_0 is an embedded submanifold of a fixed manifold Σ^n. Let $N \to \Sigma_0$ be the normal bundle of Σ_0 in Σ. If the map φ_0 extends to a Legendre embedding of a neighborhood of Σ_0, $\varphi : V \to M$, it follows that the derivative map $\varphi_* : T(V)_{|\Sigma_0} \to H(M)$ induces an isomorphism $\Phi L N^{\mathbf{C}} \to \varphi^* H(M)/T(\Sigma_0)^{\mathbf{C}}$. The above corollary shows that the existences of such an isomorphism is sufficient for φ_0 to so extend.

3. Examples and Applications

In this section we will consider several examples of contact manifolds and apply the results stated in Section 2 to characterize their Legendre submanifolds.

The universal local model of a contact manifold is the space $\mathbf{R}^{2n+1}$ with contact form.

$$\eta_0 = du - \sum_{i=1}^{n} y_i dx^i \tag{3.1}$$

where $(x, y, u) = (x^1, \dots, x^n, y_1, \dots, y_n, u)$ are coordinates on $\mathbf{R}^{2n+1}$. Darboux's Theorem states that every contact manifold is locally equivalent to an open submanifold of $(\mathbf{R}^{2n+1}, \eta_0)$. Suppose now that $f : \mathbf{R}^n \to \mathbf{R}$ is a smooth function. Here, and elsewhere, we will use the notation $df(x) \equiv \frac{\partial f(x)}{\partial x^1}, \dots \frac{\partial f(x)}{\partial x^n})$ when no confusion is likely to arise. The map

$$j(f) : \begin{cases} \mathbf{R}^n \to \mathbf{R}^{2n+1} \\ x \mapsto (x, df(x), f(x)) \end{cases} \tag{3.2}$$

is easily seen to be a Legendre embedding.

A construction due to Weinstein [W1, p. 25] extends this example as follows. Let Σ be the hypersurface in $\mathbf{R}^{n+1}$ given as the zero set of the function $g : \mathbf{R}^{n+1} \to \mathbf{R}$ with dg never zero on Σ. Let $(x, v) \in \mathbf{R}^n \times \mathbf{R} = \mathbf{R}^{n+1}$ be coordinates and let $F : \mathbf{R}^{n+1} \to \mathbf{R}$ be a function with $g = \frac{\partial F}{\partial v}$. Then the following map is a Legendre immersion

$$\begin{cases} \Sigma \hookrightarrow \mathbf{R}^{n+1} \to \mathbf{R}^{2n+1} \\ (x, v) \mapsto (x, dF_v(x), F(x, v)) \end{cases}$$

where $F_v = F(-, v) : \mathbf{R}^n \to \mathbf{R}$ is smooth. This construction can be used to give an explicit Legendre embedding of the unit sphere $S^n \subset \mathbf{R}^{n+1}$ into $\mathbf{R}^{2n+1}$: choose $(F(x, v) = (||x||^2 - 1)v + v^3/3$.

Example 3.3 (Legendre Submanifolds of $\mathbf{R}^5$)**.** It is possible to determine precisely which surfaces admit injective immersions into $\mathbf{R}^5$. (Arnold [Ar2] gives an independent and different proof of an equivalent result.)

First observe that oriented surfaces all embed in $\mathbf{R}^5$. To see this represent an oriented surface Σ as the zero set of a function g which is negative on the bounded component of $\mathbf{R}^3/\Sigma$ and such that the lines x^i = constant, $v \in \mathbf{R}$ intersect Σ at most twice (and at those points once with $v > 0$ and once with $v < 0$). Setting $F(x, v) = \int_0^v g(x, t) dt$ one obtains a Legendre embedding of Σ into $(\mathbf{R}^5, \eta_0)$.

To determine which unoriented surfaces Σ admit injective, Legendre immersions into $\mathbf{R}^5$ observe that by virtue of Corollary 2.6 we need only determine which surfaces have trivial complexified tangent bundles. This problem is solved by a routine application of obstruction theory. The only obstruction to triviality of $T(\Sigma)^{\mathbf{C}}$ is its first Chern class. However, if Σ is not compact or has non-empty boundary

$H^2(\Sigma; \mathbf{Z})$ vanishes. Therefore, we may assume that Σ is a closed unorientable manifold.

Since the bundles $T(\Sigma)^{\mathbf{C}}$ and $\overline{T(\Sigma)}^{\mathbf{C}}$ are isomorphic and $c_1(\overline{T(\Sigma)}^{\mathbf{C}}) = -c_1(T(\Sigma)^{\mathbf{C}})$ we can identify $C_1(T(\Sigma)^{\mathbf{C}})$ with its mod 2 reduction $w_2(T(\Sigma)^{\mathbf{C}}) = w_2(T(\Sigma) \oplus T(\Sigma)) = w_1^2(\Sigma)$ where w_i denotes the i^{th} Stiefel-Whitney class. However, every closed unorientable surface can be expressed as a direct sum $N \# T_g$ where T_g is an oriented surface of genus g and N is either K, the Klein bottle, or $\mathbf{RP}^2$, real projective space. It is now an easy exercise in algebraic topology to show that $w_1(\Sigma) = \pi^* w_1(N)$ where $\pi^* : N \# T_g \to N \cong N \# T_g / N$ is the quotient map. Again it is not difficult to show that $\pi^* : H^2(N; \mathbf{Z}_2) \to H^2(\Sigma; \mathbf{Z}_2)$ is injective and that $w_1(K)^2 = 0$ but $w_1(\mathbf{RP}^2)^2 \neq 0$. **Therefore, the only surfaces which do not immerse into $\mathbf{R}^5$ as Legendre submanifolds are the direct sums of $\mathbf{RP}^2$ and closed orientable surfaces**.

Example 3.4 (Exact Lagrangian Submanifolds)**.** A Legendre immersion $\widetilde{\varphi} : \Sigma^n \to \mathbf{R}^{2n+1}$ gives rise to a Lagrangian immersion $\varphi : \Sigma^n \to \mathbf{R}^{2n}$ where $\mathbf{R}^{2n}$ is equipped with the standard symplectic form $\omega = \sum_{j=1}^{n} dx^i \wedge dy_i$. For if $\pi : \mathbf{R}^{2n+1} \to \mathbf{R}^{2n}$ is the projection $\pi(x, y, u) = (x, y)$ then setting $\varphi = \pi \circ \widetilde{\varphi}$ gives the desired immersion. However, not every Lagrangian immersion is of the form $\pi \circ \widetilde{\varphi}$. To see this set $\theta = \sum_{i=1}^{n} y_i dx^i$ and note that $\eta = du - \pi^* \theta$. Then observe that the condition $\widetilde{\varphi}^* \eta = 0$ implies that for $\varphi = \pi \circ \widetilde{\varphi}$, $d(\widetilde{\varphi}^* u) = \varphi^* \theta$ and hence the pull-back $\varphi^* \theta$ must be exact for φ to be of the form $\pi \circ \widetilde{\varphi}$. If $\varphi^* \theta = df$ for some function f on Σ then the map $\widetilde{\varphi}(p) = (p, f(p))$ defines a Legendre immersion of the required form.

More generally, let $(N^{2n}, d\theta)$ be a symplectic manifold, set $M = N \times \mathbf{R} = \{(p, u)\}$ and set $\eta = du - \pi^* \theta$ with $\pi : M \to N$ the project map. An **exact Lagrangian immersion** is an immersion $\varphi : \Sigma^n \to N$ for which $\varphi^* \theta$ is exact. It is easy to see from the argument above that **a necessary and sufficient condition for a Lagrangian immersion φ to lift to a Legendre immersion $\varphi : \Sigma \to M$ is that $\varphi^* \theta$ be exact**. If Σ is connected then $\widetilde{\varphi}$ is uniquely defined up to addition of a constant.

An important special case of the above construction is the following. Let Σ be an n-dimensional manifold without boundary. Let $J(\Sigma)$ be the space of one-jets of real-valued functions on Σ and denote the one jet of the function f at the point $p \in \Sigma$ by $j(f)_p$. Since $j(f)_p$ is completely characterized by the pair $(df_p, f(p))$, there is a diffeomorphism

$$\begin{cases} J(\Sigma) \to T^*(\Sigma) \times \mathbf{R} \\ j(f)_p \mapsto (df_p, f(p)) \end{cases}$$

Therefore, the vector bundle $J(\Sigma)$ will be identified with the bundle $T^*(\Sigma) \times \mathbf{R} \to \Sigma$ and Σ will be identified with the image of the zero section $i : \Sigma \hookrightarrow T^*(\Sigma) \times 0 \hookrightarrow J(\Sigma)$. There are projection maps $\pi_T : J(\Sigma) \to T^*(\Sigma)$ and $u : J(\Sigma) \to \mathbf{R}$. The form η_Σ on $J(\Sigma)$ defined by

$$\eta_\Sigma(X) = du(X) - \alpha(\pi_* X)$$

for $X \in T(J(\Sigma))_{(\alpha, u_0)}$ makes $J(\Sigma)$ into a contact manifold. If $x^1, \ldots, x^n)$ are coordinates on $J(\Sigma)$ then $\eta_\Sigma = du - \sum_{i=1}^{n} y_i dx^i$. In particular $(J(\mathbf{R}^n), \eta_{\mathbf{R}} n)$ coincides with $(\mathbf{R}^{2n+1}, \eta_0)$. Note also that if θ_Σ is the fundamental 1-form on $T^*(\Sigma)$ then

$$\eta_\Sigma = du - \pi_T^* \theta_\Sigma$$

It is easy to see that given a smooth function $f : \Sigma \to \mathbf{R}$ and a diffeomorphism $\tau : \Sigma \to \Sigma$ the mapping

$$\varphi,(f,\tau) = j(f) \circ \tau : \Sigma \to J(\Sigma)$$

is a Legendre embedding since $j(f)^* du = df$ and $j(f)^* \pi_T^* \theta_\Sigma = df^* \theta_\Sigma = df$.

The next theorem was stated and proved in [W2] with the condition $n \neq 3$. We give a proof valid for all dimensions.

Theorem 3.5 (Weinstein [W2]). *Let Σ^n be a smooth manifold with Euler characteristic zero and let $i_T : \Sigma \to T^*\Sigma$ denote the zero section. Then there is a smooth family of exact Lagrangian immersions $\varphi_t : \Sigma \to T^*\Sigma$, $t \in [0,1]$, with $\varphi_1 = i_T$ and $\varphi_1(\Sigma) \cap i_T(\Sigma) = \psi$. Moreover, φ_t can be made arbitrarily C^0-close to i_T for all t. Finally, if φ_t' is another such family then φ_1 and φ_1' can be connected by a smooth family $\varphi_t : \Sigma \to T^*(\Sigma)/\tau(\Sigma)$, $t \in [0,1]$ of exact Lagrangian immersions such that φ_t is C^0-close to i_T for all t.*

Proof. Set $M = J(\Sigma)/\Sigma$, and homotope the ℓ-bundle injection $i_* : T(\Sigma) \to H(J(\Sigma))$ to an ℓ-homotopic ℓ-bundle injection $\Psi : T(\Sigma) \to H(M)$ that is C^0-close to i_*. (This can be done because the Euler characteristic of Σ is zero.) Let $\Psi : \Sigma \to M$ denote the map of base spaces associated to Ψ. Next apply Theorem 6.14 to obtain a Legendre immersion $\tilde{\varphi} : \Sigma \to\subseteq J(\Sigma)$ which is C^0-close to $i : \Sigma \to J(\Sigma)$ and a smooth family, $\tilde{\varphi}_t : \Sigma \to J(\Sigma), t \in [0,1]$, of Legendre immersions, C^0-close to i. Given two such families $\tilde{\varphi}$ and $\tilde{\varphi}_t'$ apply Theorem 6.14 to obtain a third family $\tilde{\Psi}_t : \Sigma \to M$ of Legendre immersions connecting $\tilde{\varphi}_1$ and $\tilde{\varphi}_1'$ such that $\tilde{\Psi}_t$ is C^0-close to i. □

Example 3.6 (Periodic Lagrangian Submanifolds). Yet another class of Legendre submanifolds arose in Einstein's treatment [E] of the Bohr-Sommerfeld quantization conditions. Let $(N^{2n}, d\theta)$ be an exact symplectic manifold and let $S^1 = \mathbf{R}/2\pi$ with coordinate function $u(\operatorname{mod} 2\pi)$. Set $M = N \times S^1$ and $\eta = du - \pi^*\theta$ with $\pi : M \to N$ the projection map. Then a necessary and sufficient condition for a Lagrangian immersion $\varphi : \Sigma \to N$ to lift to a Legendre immersion $\tilde{\varphi} : \Sigma \to M$ is that the cohomology class $\frac{1}{2\pi}[v^*\theta] \in H^1(\Sigma; \mathbf{R})$ be integral (i.e. for all closed loops γ in Σ the integral $\frac{1}{2\pi}\int_\gamma \varphi^*$ in an integer). The lift $\tilde{\phi} : \Sigma \to M$ determines a global phase function $e^{i\varphi}$ on Σ and that such a function exist is Einstein's quantization condition in [E].

This example can be generalized as follows. Let (N, ω) be a symplectic manifold with $[\omega/2\pi] \in H^2(N; \mathbf{R})$ an integral cohomology class. Then there is a circle bundle $\pi : M \to N$ with connection form η such that $d\eta = \pi^*\omega$. An immersion $\varphi : \Sigma \to N$ is Lagrangian precisely when $\varphi^*(M) \to \Sigma$ with the induced connection form η_φ is flat. In that case, the holonomy representation $\pi_1(M) \to U(1)$ can be interpreted as a cohomology class $\frac{1}{2\pi}[\eta_\varphi] \in H^1(M; \mathbf{R}/\mathbf{Z})$ which vanishes if and only if the connection η_φ has no holonomy. *The vanishing of $\frac{1}{2\pi}[\eta_\varphi]$ is a necessary and sufficient condition for the map φ to lift to a Legendre immersion $\tilde{\varphi} : \Sigma \to M$.* Note that the image of $\frac{1}{2\pi}[\eta_\varphi]$ in $H^2(\Sigma, \mathbf{Z})$ under the Bockstein homomorphism of the exact sequence $0 \to \mathbf{Z} \to \mathbf{R} \to \mathbf{R}/\mathbf{Z} \to 0$ is the Chern class of $\varphi^*(M) \to \Sigma$ and $\frac{1}{2\pi}[\eta_\varphi]$ is an example of a Chern-Simons invariant.

Stout considered the example of the Hopf bundle $S^2n + 1 \to \mathbf{CP}^n$ in [St]. The natural inclusion $\varphi : \mathbf{RP}^n \to \mathbf{CP}^n$ is a Lagrangian immersion with respect to the standard Kähler form $\mathbf{CP}^n$. One can show that $\frac{1}{2\pi}[n_\varphi]$, reduced $\pmod 2$, is the standard generator of $H^*(\mathbf{RP}^n; \mathbf{Z}/2)$ and, therefore, φ does not lift to a Legendre immersion into S^{2n+1}. In fact, $\mathbf{RP}^n$ seldom embeds as a Legendre submanifold of S^{2n+1}.

Proposition 3.7. *The only integers for which $\mathbf{RP}^n$ embeds as a Legendre submanifold of S^{2n+1} are $n = 1, 3$ and 7.*

Proof. (This argument is due to D. Ravenel). First note that on any map φ from $\mathbf{RP}^n$ into S^{2n+1} is homotopic to a constant map and, therefore, the pull-back bundle $\varphi^* H(S^{2n+1}) \to \mathbf{RP}^n$ is trivial. Hence, by Corollary 2.6 we need only determine the integers for which $T(\mathbf{RP}^n)^C$ is trivial.

It is well-known that $T(\mathbf{RP}^n)$ is trivial for $n = 1, 3, 7$. (For $n = 7$ note that the framing of the tangent bundle of the unit Cayley numbers is invariant under derivative of the antipodal map and for $n = 3$ recall that all orientable three-folds have trivial tangent bundles.)

Next observe from [MS, pp.45–47] that the total Stiefel-Whitney class of $T(\mathbf{RP}^n)^{\mathbf{C}}$ is $(l + w)^{2n+2} = 1$ from which the $\pmod 2$ binomial theorem yields $n = 2^m - 1$, m an integer.

We now show that if n is larger that 7 then $T(\mathbf{RP}^n)^{\mathbf{C}} \oplus \mathbf{1}^{\mathbf{C}}$ (and, hence, $T(\mathbf{RP}^n)^{\mathbf{C}}$) is not trivial. ($\mathbf{1}^C$ denotes the trivial complex line bundle.) Adams [Ad] showed that the reduced complex K-group $\tilde{K}^0(\mathbf{RP}^n)$ is an additive cyclic group of order $2^{[n/2]}$ generated by $\nu^{\mathbf{C}}$, the complexification of the canonical line bundle on $\mathbf{RP}^n$. However, by [MS, p. 45], $T(\mathbf{RP}^n)^{\mathbf{C}} \oplus \mathbf{1}^{\mathbf{C}} = (n+1)\nu^{\mathbf{C}}$. Since $(n + 1) < 2^{[n/2]}$ for $n > 7$ it follows that $T(\mathbf{RP}^n)^{\mathbf{C}}$ is not trivial (or even stably trivial) for $n > 7$. □

4. Normal Forms for Legendre Immersions

In [W1] Weinstein obtained various generalizations of the classical Darboux lemma and used them to give normal forms for Lagrangian immersions. In this section the methods of [W1] are used to obtain analogous results for Legendre immersions.

The following version of the Darboux lemma is fundamental to our subsequent development.

Lemma 4.1. *Let η_0 and η_1 be two contact forms on a manifold M which agree on the closed set $C \subseteq M$. On a sufficiently small neighborhood N of C the forms $\eta_t = \eta_0 + t(\eta_1 - \eta_0)$, for $t \in [0, 1]$ are all contact form. Let X_{η_t} denote the Reeb vector field on N associated to the contact form Suppose that C is contained in an open $2n$-dimensional submanifold $U \subseteq N$ and that X_{η_t} are transverse to U for all t. Then there is an open neighborhood $N' \subset N$ of C and a vector field Z on N' whose unit time flow μ satisfies the equation $\mu^*\eta_1 = \eta_0$ in a neighborhood of C. The vector field Z—and hence its flow—depend smoothly on η_1 and on the submanifold U.*

Proof. First let η be any contact form on N with X_η transversal to U and let α be a one form vanishing on C. There is a unique decomposition $\alpha = g \cdot \eta + \beta$ where $\beta(X_\eta) = 0$ (see Section 2). Because X_η is transversal to U the equation $X_\eta(f) = g$ can be integrated to yield a smooth function f defined on a neighborhood V of C

in N with $f = 0$ on $U \cap V$. Since $X_\eta \rfloor (df - g\eta) = 0$ and $d\eta_{|H(M)}$ is nondegenerate, there is a unique vector field Z_α characterized by the conditions $Z_\alpha \rfloor \eta = f$ and $Z_\alpha \rfloor d\eta = \alpha - df = \beta + (g\eta - df)$. The Lie derivative $\mathcal{L}_{Z_\alpha}$ can be computed as follows

$$\mathcal{L}_{Z_\alpha}\eta = d(Z_\alpha \rfloor \eta) + Z_\alpha \rfloor d\eta = df + \alpha - df = \alpha\,.$$

Note that Z_α vanishes on C.

Apply the considerations of the above paragraph to the forms $\eta = \eta_t$ and $\alpha = \eta_1 - \eta_0$ to obtain vector fields $Z_z, 0 \leq t \leq 1$, vanishing on C and defined on a neighborhood $V \subseteq M$ of C. Let $\nu_t(p,s)$ denote the flow of Z_τ and set $\mu_t(p) = \nu_t(p,t)$. (Since Z_t is zero on C the functions μ_t are defined on a neighborhood N' of C.) Following the proof given in [W1], one shows that $\mu_t^*\eta_t = \eta_0$. Now set $\mu = \mu_1$. By construction μ_t depends smoothly on η_1 and U and μ_t is the identity on C. □

Remark 4.2. The fact that the condition $L_{Z_1}(\eta_1) = 0$ holds on C implies that the flow of Z_t leaves η_t fixed on C. But in general the flow of Z_1 will not fix $d\eta_1$ even at points of C. The flow of Z_1 will only keep $d\eta_1$ fixed at points for which $L_{Z_1}d\eta_1 = d\eta_1 - d\eta_0 = 0$.

Theorem 4.3 (Normal Form for Legendre Immersions). *Let $\varphi : \Sigma^n \to M^{2n+1}$ be an immersion into the contact manifold (M, η) with $\varphi^*\eta = 0$ and $\varphi^*d\eta = 0$ on a closed set $\Sigma_0 \subseteq \Sigma$. Then there is a neighborhood $N \subseteq J(\Sigma)$ of Σ_0 and a local diffeomorphism $\tilde{\varphi} : N \to M$ extending $\varphi_{|\Sigma_0}$ with $\tilde{\varphi}^*\eta = \eta_\Sigma$. The assignment $\varphi \mapsto \tilde{\varphi}$ depends smoothly on φ.*

Proof. We first define a local diffeomorphism $\psi : V \to M$ from a neighborhood of Σ_0 in $J(\Sigma)$ into M with $\psi_{|V\cap\Sigma} = \varphi_{|V\cap\Sigma}$. Use a fixed Riemannian metric on Σ to identify $T^*(\Sigma)$ with $T(\Sigma)$ and let $\psi : J(\Sigma) \cong T(\Sigma) \times \mathbf{R} \to M$ be the map defined by

$$\psi(y, s) = \exp(J(\varphi_* Y) + sX_{\eta,\varphi(p)}, 1) \tag{4.4}$$

for $Y \in T(\Sigma)$, $p \in \Sigma$ where $\tilde{J} : T(M) \to H(M) \subseteq T(M)$ is the composition of the projection $T(\Sigma) = \mathbf{R} \cdot X \oplus H(M) \to H(M)$ and the complex structure map $J : H(M) \to H(M)$ and $\exp : T(M) \times \mathbf{R} \to M$ is the exponential map of the Riemannian manifold M—of course ψ is only defined on a neighborhood V of Σ_0 in $J(\Sigma)$. A standard computation with the exponential map which utilizes the facts that $\varphi^*\eta_{|\Sigma_0} = 0$, $\varphi^*d\eta_{|\Sigma_0} = 0$ and that $g(Y, JZ) = d\eta(Y, Z)$ for $Y, Z \in H(M)$ (see Section 2) shows that $\psi^*\eta = \eta_\Sigma$ and $\psi^*d\eta = d\eta_\Sigma$ on Σ_0. Clearly ψ depends smoothly on φ.

Now set $\eta_0 = \eta_\Sigma$ and $\eta_1 = \psi^*\eta$ and let $U = V \cap (T^*\Sigma \times (0)) \subseteq J(\Sigma)$. Note that, after shrinking V if necessary, it follows from the equations $\eta_0 = \eta_1$ and $d\eta_0 = d\eta_1$ on Σ_0 that the vector fields X_{η_0} and X_{η_1} agree on Σ_0 and are transverse to U. Lemma 4.1 now applies to give a diffeomorphism μ of a neighborhood N of Σ_0 into V with $\mu^*\eta_1 = \eta_0$. Finally set $\tilde{\varphi} = \psi \circ \mu : N \to M$. Since ψ and μ depend smoothly on φ so does $\tilde{\varphi}$. □

Corollary 4.5. *Let $\varphi_0 : \Sigma^n \to M^{2n+1}$ be an immersion into the contact manifold (M, η) with $\varphi^*\eta = 0$ and $\varphi^*d\eta = 0$ on a closed set Σ_0. Then there is a smooth family of immersions $\varphi_t : V \to M, t \in [0, 1]$, defined on a neighborhood of Σ_0 with*

$\varphi_t^*\eta = 0$ and $\varphi_t^* d\eta = 0$ on Σ_0 and φ_1 a Legendre immersion.[1] *The homotopy of φ_t depends smoothly on φ_0 and $\varphi_t(p) = \varphi_0(p)$ at all points $p \in V$ for which $\varphi_{0p}^*(\eta) = 0$ and $\varphi_{0p}^*(d\eta) = 0$.*

Proof. Since $\psi_{|\Sigma} = \varphi$ and $\tilde{\varphi}$ is regularly homotopic to ψ, it follows that the immersions $\varphi_0 = \varphi_{|N\cap\Sigma} : N \cap \Sigma \to M$ and $\varphi_1 : N \cap \Sigma \hookrightarrow N \xrightarrow{\tilde{\varphi}} M$ are regularly homotopic. In fact, in the notation of the proof of Lemma 4.1, the family of maps $\varphi_t : N \cap \Sigma \to M$ defined by $\varphi_t(p) = \psi \circ \nu_1(p,t), 0 \le t \le 1$ is a regular homotopy between φ_0 and φ_1 with $\varphi_{t|\Sigma_0} = \varphi_{0|\Sigma_0}$ and $\varphi_t^*\eta$ and $\varphi_t^* d\eta$ vanishing on Σ_0 for all $t \in [0,1]$. Note that, given φ, a Riemannian metric on Σ and the Hermitian metric on $H(M)$, the above arguments give a *constructive* method of obtaining the maps $\tilde{\varphi}$ and φ_t which depends smoothly on φ. □

We close this section with two technical results which we need in Section 5 and Section 6.

Proposition 4.6.

(1) Let Σ and M be as above with $\partial\Sigma = \emptyset$ and let K be a finite cell complex. Let $\psi : \Sigma \times K \to M$ be a continuous map, smooth on each cell of K, with $\psi_s = \psi(-,s) : \Sigma \to M$ a Legendre immersion for all $s \in K$. Then there is a neighborhood $N \subseteq J(\Sigma)$ of Σ and a map $\Psi : N \times K \to M$, extending ψ, smooth on each cell of K and such that the local diffeomorphism $\Psi_s = \Psi(\ ,s) : N \to M$ satisfies $\Psi_s^*\eta = \eta_S$ for all $s \in K$.

(2) If $\partial\Sigma \ne \emptyset$ and $\psi : \Sigma \times K \to M$ is as before then there is a collaring Σ' of Σ and an extension $\psi' : \Sigma' \times K \to M$ with ψ'_s a Legendre immersion for $s \in K$ and φ' smooth on cells of K and ψ' extends as in the previous paragraph.

(3) Now choose a point $s_0 \in K$ and set $\Psi_0 = \Psi_{s_0}$ (by (2) we may assume that $\partial\Sigma = \emptyset$) and let $\Sigma_0 \subseteq \Sigma$ be compact. Then there are neighborhoods $V \subseteq \Sigma$ of Σ_0 and U of s_0 with the property that there are (cell-wise) smooth families of functions $f_s : \Sigma \to \mathbf{R}$ and diffeomorphisms $\tau_s : V \to \Sigma$ in Σ for $s \in U$ with f_s near zero and τ_s near the identity in the C^∞-topology such that

$$\psi_s = \psi_0 \circ j(f_s) \circ \tau_s, s \in U \tag{4.7}$$

on V.

Proof. (1) This is an immediate consequence of Theorem 4.3 with $C = \Sigma$.

(2) First extend ψ to a cell-wise smooth map $\varphi : \Sigma' \times K \to M$ where Σ' is a smooth collaring of Σ. Now apply Corollary 4.5 to the family $\varphi_s, s \in K$ with $\Sigma_0 = \Sigma$.

(3) By compactness of Σ_0 and continuity of ψ there are neighborhoods V' of Σ_0 and U' of s_0 such that there is a smooth family $\varphi_s : V' \to N, s \in U'$ of Legendre immersions such that $\varphi_{s_0} = i : V' \to N$, the zero section and $\psi_s = \Phi_{s_0} \circ \varphi_s$. After shrinking V' and U' we may assume (again by compactness of Σ_0) that $\varphi_s = \alpha_s \circ \tau_s$ where $\tau_s : V \to \Sigma$ is a diffeomorphism into Σ and $\alpha_s : V \to J(\Sigma)$ is a section. Because φ_s is a Legendre immersion it follows that $\alpha_s = j(f_s)$ for a unique function $f_s : \tau_s(V) \to \mathbf{R}$. By shrinking V and U if necessary we can extend f_s to Σ. Clearly, the families τ_s and f_s are cell-wise smooth. □

[1] In [CC2] Chaumat and Chollet give an example of a subspace of M which is locally contained in Legendre submanifolds of M but not globally contained in any Legendre submanifold.

Theorem 4.8. *Let K be a finite cell complex, $L \subset K$ a closed subcomplex, Σ_0 an embedded submanifold of a manifold $\Sigma^n, \partial\Sigma = \emptyset$, and (M^{2n+1}, η) a contact manifold. Let $\varphi : \Sigma_0 \times K \cup \Sigma \times L \to M$ be a simplex-wise smooth map with $\varphi(-, s) = \varphi_s$ satisfying $\varphi_s^* \eta = 0$ and let $\tilde{\psi} : T(\Sigma)_{|\Sigma_0} \times K \to H(M)$ be piece-wise smooth family of ℓ-bundle injections extending $\varphi_* : T(\Sigma_0) \times K \cup T(\Sigma)_{|\Sigma_0} \times L \to H(M)$. Then there is a neighborhood V of Σ_0 and a family of Legendre immersions $\psi : V \times K \to M$, agreeing with φ on the intersection of their domains and with* *$\psi : V \times K \to M$, agreeing with φ on the intersection of their domains and with* *$\psi_* : T(\Sigma)_{|\Sigma_0} \times K \to H(M)$ ℓ-homotopic to $\tilde{\psi}$.*

Proof. Because Σ_0 is an embedded submanifold we can extend $\varphi : \Sigma_0 \times K \to M$ to a simplex-wise smooth family of immersions $\psi' : V' \times K \to M$ with $\psi' = \varphi$ on $V' \times L$ and $\psi'_* = \tilde{\psi}$ on $T(\Sigma)_{|\Sigma_0} \times K, V'$ a neighborhood of Σ_0. Now apply Corollary 4.5. □

Normal form for transverse curves. Although we do not need it later, we include here a normal form theorem for curves which are transverse to the contact distribution. This result was used by Globevnic and Stout [GS].

Identify $\mathbf{R}$ with the u-axis $\{(0, 0, u)\} \subset \mathbf{R}^{2n+1}$, and let $\eta_0 = du - \sum y_i dx^i$ be as in equation (3.1).

Theorem 4.9. *Let (M, η) be a contact manifold of dimension $2n + 1$ and let $\gamma : \mathbf{R} \to M$ be a curve such that the pullback $\gamma^*\eta$ never vanishes. Then there is an extension of γ*

$$\Phi : U \to M$$

to an open neighborhood $U \subset \mathbf{R}^{2n+1}$ such that

$$\Phi^*\eta = f\eta_0$$

where f is a smooth positive function on U.

Proof. We will express Φ as a composition of diffeomorphisms $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1 \circ \Phi_0$, where $\Phi_0 : U_0 \to M$ is a diffeomorphism extending γ and $\Phi_j : U_j \to U_{j-1}$, $j = 1, 2, 3$ are diffeomorphisms between open neighborhoods U_j of $\mathbf{R} \subset \mathbf{R}^{2n+1}$.

Step 0: Choose a framing for the contact distribution along γ, say $Z_k : \mathbf{R} \to H \subset T(M)$, $k = 1, 2, \ldots, 2n$. This gives an identification of $\mathbf{R}^{2n+1}$ with the pullback bundle $\gamma^* H \to \mathbf{R}$, which is isomorphic to the normal bundle of the curve. Use the exponential map to construct a local diffeomorphism $\Phi_0 : U_0 \to M$ with the property that the vectors $\Phi_{0*}\partial/\partial x^i$ and $\Phi_{0*}\partial/\partial y_i$, $i = 1, \ldots, n$, span the contact distribution at all points in the image of γ. It follows that

$$\Phi_0^*\eta = g\eta_0 \text{ at all points of } \{0\} \times \{0\} \times \mathbf{R}.$$

where g is a positive function defined on U_0.

Step 1: By virtue of Step 0, we may replace M by U_0, and assume that η is a contact form on U_0, which agrees with $f\eta_0$ along the u-axis. Moreover, by restricting U_0 if necessary, we may assume that η is of the form

$$\eta = g\left\{du - \sum_{i=1}^{n} \left(a_i\, dx^i + b^i\, dy_i\right)\right\}$$

where a_i and b^i are smooth functions which vanish along the u-axis, and g is a positive smooth function. Set

$$\theta = du - \sum_{i=1}^{n} \left(a_i\, dx^i + b^i\, dy_i\right) .$$

The functions a_i and b^i are uniquely determined. Because we are only interested in $\eta = g\theta$ up to a non-zero multiple, we are free to set

$$g(x,y,u) = 1 - \sum_{i=1}^{n} \left(\frac{\partial a_i(0,0,u)}{\partial u}\, x^i - \frac{\partial b^i(0,0,u)}{\partial u}\, y_i \right) .$$

With this choice of g, the form $\eta = g\theta$ satisfies the identities

$$\frac{\partial}{\partial u} \lrcorner\, d\,\eta = 0 \text{ and } \eta = \eta_0$$

at all points of the u-axis.

Step 2: Let X_η be the Reeb field of η. Recall that this means that X_η satisfied the identities

$$X_\eta \lrcorner\, d\eta = 0 \text{ and } X_\eta \lrcorner\, \eta = 1 .$$

Observe that by Step 1, $X_\eta = \frac{\partial}{\partial u}$ along the u-axis.

Let $\nu_t : U_1 \to \mathbf{R}^{2n+1}$ be the restriction of the flow of X_η to an open subset of U_0. Now define a map

$$\Phi_1 : U_1 \to \mathbf{R}^{2n+1}$$

by the formula $\phi_1(x,y,u) = \nu_u(x,y,0)$. (That a suitable neighborhood U_1 exists follows from the fact the $X_\eta(0,0,u) = \frac{\partial}{\partial u}$, and hence $\nu_t(0,0,u)$ is defined for all $t, u \in \mathbf{R}$.)

By construction,

$$\Phi_1^* du = du \text{ and } \Phi_{1,*}\left(\frac{\partial}{\partial u}\right) = X_\eta . \tag{4.10}$$

Set $\eta_1 = \Phi_1^*\eta$. Then $\eta_1 = du$ along the u-axis. Moreover, equations (4.10) imply that η_1 satisfies the identities

$$\frac{\partial}{\partial u} \lrcorner\, d\eta_1 = 0 \text{ and } \frac{\partial}{\partial u} \lrcorner\, \eta_1 = 1 \tag{4.11}$$

on U_1.

Step 3: Equations (4.11) and the standard identity

$$\mathcal{L}_{\partial/\partial u}\eta_1 = d\left(\frac{\partial}{\partial u} \lrcorner\, \eta_1\right) + \frac{\partial}{\partial u} \lrcorner\, d\eta_1$$

imply that the Lie derivative $\mathcal{L}_{\partial/\partial u}\eta_1$ vanishes. Thus, η_1 is independent of u and its exterior derivative defines a symplectic form on a neighborhood of the origin of $\mathbf{R}^{2n}$. By Darboux's theorem there is a diffeomorphism and after shrinking U_1 if necessary, there is therefore a diffeomorphism of the form

$$\Phi_2 : U_2 \to U_1 \; : \; (x,y,u) \mapsto (\phi(x,y), \psi(x,y), u)$$

such that the 1-form $\eta_2 = \Phi_2^*\eta_1$ satisfies the identity $d\eta_2 = \sum_i dx^i \wedge dy_i$. By shrinking again if necessary, we may assume that U_2 is contractible.

Step 4: The form η_2 satisfies the conditions

$$\eta_2 = \eta_0 \text{ on the } u\text{-axis}$$

and

$$d\eta_2 = \sum_i dx^i \wedge dy_i = d\eta_0 \text{ on all of } U_2.$$

Hence, there is a function h such that

$$dh = \eta_2 - \eta_0$$

and $dh_{(0,0,u)} = 0$. We may therefore choose h so that it vanishes along the u-axis.

Next let $\Phi_3 : U_3 \to U_2$ be the diffeomorphism whose inverse is defined by the formula

$$\Phi_3^{-1} : (x, y, u) \mapsto (x, y, u + h(x, y, u)),$$

where $U_3 = \Phi_3^{-1}(U_2)$. Notice that $\left(\Phi_3^{-1}\right)^* \eta_0 = \eta_2$. By construction, the map Φ_3 satisfies the two conditions

$$\Phi_3^* \eta_2 = \eta_0 \text{ and } \Phi_3(0, 0, u) = (0, 0, u)$$

Step 5: Set $U = U_3$ and $\Phi = \Phi_0 \circ \Phi_1 \circ \Phi_2 \circ \Phi_3 : U \to \mathbf{R}^{2n+1}$. □

Remark 4.12. Suppose that γ and (M, η) are real-analytic. Then the metric on M can be chosen to be real-analytic and all constructions in the proof yield real-analytic objects. Consequently Φ is real-analytic as well.

5. The Homotopy Extension Theorem

In this section we prove a homotopy extension theorem for families of Legendre immersions which will be needed in Section 6 to prove the classification theorem for Legendre immersions. As mentioned in the introduction, my original proof was incorrect; the proof given here is essentially due to Claude Viterbo.

Theorem 5.1 (Homotopy Extension Theorem)**.** *Let (M, η) be a $(2n+1)$-dimensional contact manifold. Let $\Sigma_0 \subseteq \Sigma$ be a compact subset of an n-dimensional manifold Σ^n with open neighborhood $V \subseteq \Sigma$. Let*

$$\psi : V \times K \times I \to M$$

be a cell-wise smooth family[2] of Legendre immersions, where K is a finite complex and I is the unit interval $[0, 1]$, and let $\tilde{\psi}_0 : \Sigma \times K \to M$ be a cell-wise smooth family of Legendre immersions extending the the restriction of ψ to $V \times K \times \{0\}$. Then there is a cell-wise smooth family of Legendre immersions

$$\tilde{\psi} : \Sigma \times K \times I \to M$$

such that

$$\tilde{\psi}(x, s, t) = \begin{cases} \psi(x, s, t) & \text{for } (x, s, t) \in V_1 \times K \times I \\ \tilde{\psi}_0(x, s) & \text{for } (x, s, t) \in (\Sigma - \overline{V_2}) \times K \times I, \end{cases}$$

where V_1 and V_2 are open sets with $\Sigma_0 \subset V_1 \subset \overline{V_1} \subset V_2 \subset \overline{V_2} \subset V$ and $\overline{V_2}$ compact.

Remark 5.2. There is no loss of generality in assuming that Σ_0 is a compact, n-dimensional submanifold with smooth boundary N. Next observe that we can choose a collaring of N and identify V with the manifold $[-2, 2) \times N$. Moreover, without loss of generality we replace Σ_0 by $[-2, 0] \times N$ and Σ by $[-2, 2) \times N$.

[2]By *cell-wise smooth* we mean that ψ is continuous and that there is a partition of I into closed subintervals, $I_1, \ldots, I_N$, so that ψ is smooth on $V \times \Delta \times I_j$ for each cell Δ of K and all j.

Notation. The following notation will be in force throughout this section. Points on Σ will be written in the form $x = (x_0, x') \in [-2,2) \times N$ when necessary, and points on $J(\Sigma) \cong T^*([-2,2)) \times \mathbf{R} \times T^*(N)$ will be written $p = (x_0, y_0, u, p'), x_0 \in [-2,2), (y_0, u) \in \mathbf{R}^2$, $p' \in T^*(N)$. In particular, the contact form η_Σ on $J(\Sigma)$ assumes the form

$$\eta_\Sigma = du - y_0 dx_0 - \theta_N$$

where θ_N is the canonical 1-form on $T^*(N)$. We will make use of the following natural inclusions and projections (see Example 3.4):

$$\Sigma \subseteq T^*(\Sigma) \approx T^*(\Sigma) \times (0) \subseteq J(\Sigma),\ j_0 : \Sigma \hookrightarrow J(\Sigma),$$
$$\pi : J(\Sigma) \to \Sigma,\ \pi_T : J(\Sigma) \to T^*(\Sigma),\ \pi_\Sigma : T^*(\Sigma) \to \Sigma\,.$$

The symbol $V_\delta, \delta \in (0,2)$ denotes the open neighborhood of Σ_0, $N| \times [-2,\delta)$. The symbol s denotes an arbitrary element of the finite cell complex K. Families of maps $\psi : X \times K \times I \to Y$, X and Y smooth manifolds will *always* be assumed to be cell-wise smooth and we will use the notations $\psi_{s,t}(p), \psi(p,s,t)$ and $\psi_s(p,t)$ interchangeably where (p,s,t) is an element of the space $X \times K \times I$.

The proof of Theorem 5.1 requires the following lemma.

Lemma 5.3 (Micro-compressibility). *Let $\psi_{s,t} : V \to M, (s,t) \in K \times [a, a+b]$ be a family of Legendre immersions. Then there is a number $\varepsilon_a > 0$ such that for each $\delta \in (0, 1/4)$ there is another family of Legendre immersions $\tilde{\psi}_{s,t} : V \to M$, $(s,t) \in K \times [a, a+\varepsilon_a]$ with*

1. *$\tilde{\psi}_{s,t}(x) = \psi_{s,t}(x)$ for $x \in V_\delta$,*
2. *$\tilde{\psi}_{s,t}(x) = \psi_{s,a}(x)$ for $x \in V - V_{3\delta}$, and*
3. *$\tilde{\psi}_{s,a}(x) = \psi_{s,a}(x)$ for $x \in V$.*

Given the lemma the proof of Theorem 5.1 is as follows:

Proof. First observe that, after shrinking V if necessary, we may assume that ψ is defined on the set $V \times K \times [0, 1+b)$ for some $b > 0$. To see this apply Theorem 4.3 to obtain a family of contact diffeomorphisms $\Psi : U \times K \to M$, $U \subseteq J^1(V)$ a neighborhood of $\Sigma_0 \subseteq J^1(\Sigma)$ with $\psi(x,s,1) = \Psi(x,s), x \in V, s \in K$. Then, after a possible shrinking of V, there are a number $c > 0$, a family $f : V \times K \times [1-c, 1] \to \mathbf{R}$ of functions with $f(x,s,1) = 0$ for all (x,s) and a family $\tau : V \times K \times [1-c,1] \to \Sigma$ of diffeomorphisms with $\tau(x,s,1) = x$ for all (x,s) such that the identity

$$\psi(x,s,t) = \Psi(j(f(x,s,t) \circ \tau(x,s,t)) \tag{5.4}$$

holds for $(x,s,t) \in V \times K \times [1-c, 1]$. Now extend f and τ in any way to the interval $[1-c, 1+b]$ for $b > 0$ sufficiently small. Then, by virtue of the compactness of $\Sigma_0 \times K$ and the equations $f(x,s,1) = 0$ and $\tau(x,s,1) = x$, and after another shrinking of V, the required extension is given by the formula (5.4).

Next apply Lemma 5.3 to each number $a \in [0, 1+b)$ to obtain intervals $[a, a+\varepsilon_a]$ on which conditions (1), (2) and (3) of Lemma 5.3 hold. By compactness of $[0,1]$ it is covered by a finite set of such intervals, say

$$I_0 = [0, \varepsilon_0], I_1 = [a_1, a_1 + \varepsilon_1], \ldots, I_m = [a_m, a_m + \varepsilon_m]\,.$$

Start by applying Lemma 5.3 to $\psi : V \times K \times I_0 \to M$ with $\delta = \delta_0 \equiv 1/4$ to obtain a family $\psi_0 : V \times K \times I_0 \to M$ with $\psi_0' = \psi$ on $V_{1/4} \times K \times I_0$ and $\psi_0'(x,s,t) = \psi(x,s,0)$ on $V - V_{1/2}$.

Then apply Lemma 5.3 to $\psi : V_{\delta_0} \times K \times I_1 \to M$ with $\delta = \delta_1 \equiv \delta_0/3$ to obtain a family $\tilde{\psi}_1' : V_{\delta_1} \times K \times I_1 \to M$ with $\tilde{\psi}_1' = \psi$ on $V_{\delta_1} \times K \times I_1$ and $\tilde{\psi}_1'(x,s,t) = \psi(x,s,a_1) = \psi_0'(x,s,a_1)$ for $(x,s) \in (V_{\delta_0} - V_{\delta_1}) \times K, t \in I_1$ and $\psi_1'(x,s,a_1) = \psi_0'(x,s,a_1)$ for $(x,s) \in V_{\delta_0} \times K$. Therefore, we can extend ψ_1' to $V \times I_1$ by the formula $\psi_1'(x,s,t) = \tilde{\psi}_0'(x,s,a_1)$ for $x \in V - V_{\delta_1}$. Define $\tilde{\psi}_1 : V \times K \times [0, a_1 + \varepsilon_1] \to M$ by

$$\tilde{\psi}_1(x,s,t) = \begin{cases} \tilde{\psi}_0(x,s,t) & \text{for } t \in [0,a_1] \\ \psi_1'(x,s,t) & \text{for } t \in I_1. \end{cases}$$

Now inductively assume that we have constructed a family $\tilde{\psi}_j : V \times K \times [0, a_j + \varepsilon_j] \to M$ with

$$\tilde{\psi}_j(x,s,t) = \begin{cases} \psi(x,s,t) & \text{for } (x,s,t) \in V_{\delta_j} \times K \times [0, a_j + \varepsilon_j] \\ \tilde{\psi}_0(x,s) & \text{for } (x,s,t) \in (V - V_{\delta_{j-1}}) \times K \times [0, \varepsilon_j],\ \delta_j = 1/3^{s+2}. \end{cases}$$

Apply Lemma 5.3 to the interval I_{j+1} with $\delta = \delta_{j+1} = \delta_j/3$ to obtain a family $\psi_{j+1}' : V_{\delta_{j-1}} \times K \times I_{j+1} \to M$ with

$$\psi_{j+1}'(x,s,t) = \begin{cases} \psi(x,s,t) & \text{for } (x,s,t) \in V_{\delta_{j+1}} \times K \times I_{j+1} \\ \psi(x,s,a_{j+1}) & \text{for } (x,s,t) \in (V_{\delta_{j-1}} - V_{\delta_j}) \times K \times I_{j+1}. \end{cases}$$

Use this last formula to define ψ_{j+1}' on $V \times K \times I_{j+1}$ and define $\tilde{\psi}_{j+1} : V \times K \times [0, a_{j+1} + \varepsilon_{j+1}] \to M$ by the equation

$$\tilde{\psi}_{j+1}(x,s,t) = \begin{cases} \tilde{\psi}_j(x,s,t) & \text{for } t \in [0, a_{j+1}] \\ \psi_{j+1}'(x,s,t) & \text{for } t \in [a_{j+1}, a_{j+1} + \varepsilon_{j+1}] \end{cases}$$

After m repetitions the required family is obtained. □

We now turn to the proof of Lemma 5.3.

Proof. Without loss of generality assume $a = 0$ and $b = 1$. The proof consists of a series of reductions to increasingly simpler cases.

Reduction 1. It is sufficient to prove the lemma in the case where M is an open neighborhood of $\Sigma \subseteq J(\Sigma)$ and where $\psi_{s,t}$ is of the form

$$\psi_{s,t} = j(f_{s,k}) \circ \tau_{s,t} : V_1 \to M \subseteq J(\Sigma) \tag{5.5}$$

with $\tau_{s,t} : V_1 \to V_{3/2}$ a family of diffeomorphisms into Σ with $\tau_{s,0}(x) = x$ for all $x \in V_1$ and $f_{s,t} : V_{3/2} \to \mathbf{R}$ a family of functions with $f_{s,0}(x) = 0$ for all x.

To see this apply Theorem 4.3 to the family $\psi_{s,0} : \Sigma \to M$ to obtain a family of diffeomorphisms $\Psi_s : M' \to M$ with $\Psi_s^*(\eta) = \eta_\Sigma$ where M' is an open neighborhood of Σ in $J(\Sigma)$ and where the equation $\psi_{s,0} = \Psi_s \circ j_0 : V \to M$ holds. By choosing $\varepsilon > 0$ sufficiently small, it follows from the compactness of the sets $K \times \overline{V}_\delta$, $\delta \in (0,2)$ that $\psi_{s,t}$ can be written in the form

$$\psi_{s,t} = \Psi_s \circ j(f_{s,t}) \circ \tau_{s,t}$$

for $\tau_{s,t}$ and $f_{s,t}$ as above. Hence we may replace M by M' and $\psi_{s,t}$ by the family (5.5).

Reduction 2. In this step we show that, given $\varepsilon > 0$ sufficiently small and $M' \subset M \subset J(\Sigma)$ a sufficiently small neighborhood of $\Sigma \subset J(\Sigma)$, the following condition holds: For all $\delta \in (0, 1/4)$ there is a family of diffeomorphisms $\nu^\delta_{s,t} : M' \to M$, preserving η_Σ such that for $t \in [0, \varepsilon]$ the family $\psi'_{s,t}$ given by the composition

$$\psi'_{s,t} : V_1 \xrightarrow{\tau_{s,t}} \Sigma \overset{j_0}{\hookrightarrow} M' \xrightarrow{\nu^\delta_{s,t}} M$$

satisfies the conditions:

$$\begin{aligned} &\psi'_{s,0} \equiv \psi_{s,0} \\ &\psi'_{s,t}(x) = \psi'_{s,t}(x) \quad \text{for } x \in V_\delta \\ &\psi'_{s,t}(x) = \psi_{s,0}(x) \quad \text{for } x \in V_1 - V_{2\delta}. \end{aligned} \tag{5.6}$$

It follows that we may once more replace M by M' and that we may assume that the family $\psi_{s,t}$ is of the form

$$\psi_{s,t} = j_0 \circ \tau_{s,t} : \Sigma \to M \subseteq J(\Sigma). \tag{5.7}$$

We will construct the maps $\nu^\delta_{s,t}$ as the unit time flows of a family of vector fields $X^\delta_{s,t}$. These vector fields are infinitesimal automorphisms of the pair $(J(\Sigma), \eta_\Sigma)$ and are the analogues of symplectic vector fields.

To construct $X^\delta_{s,t}$, begin by letting g be a real-valued function on the contact manifold (M, η) with $X_\eta(g) = 0$, where X_η is the Reeb vector field of M. Then there is a unique vector field on M, written X_g, characterized by the conditions $i(X_g)\eta = g$ and $i(X_g)d\eta = -dg$. In terms of local coordinates (x, y, u) with $\eta = du - \sum_{i=1}^n y_i dx^i$, $X_\eta = \dfrac{\partial}{\partial u}$ and $g = g(x, y)$ the vector field X_g is given by the formula

$$X_g = \left(g - \sum_i y_i \frac{\partial g}{\partial y_i}\right)\frac{\partial}{\partial u} + \sum_i \left(\frac{\partial g}{\partial x^i}\frac{\partial}{\partial y_i} - \frac{\partial g}{\partial y_i}\frac{\partial}{\partial x^i}\right), \tag{5.8}$$

and an easy computation shows that the Lie derivative $\mathcal{L}_{X_g}\eta$ vanishes. The vector fields $X^\delta_{s,t}$ will be of the form $X_{g^\delta_{s,t}}$ for $g^\delta_{s,t}$ a suitable family of functions, which we now construct.

Let U be a neighborhood of $j_0(\overline{V}_1)$ with $\overline{U} \subset M$ compact and such that $\pi(\overline{U}) \subseteq V_{3/2}$, $U = U_0 \times (-c, c) \subseteq T^*(\Sigma) \times \mathbf{R}$, $c > 0$ and such that for all $x \in \pi(U)$ the sets $\pi^{-1}(x) \cap U$ are convex neighborhoods of $j_0(x)$ in the vector space $\pi^{-1}(x)$. Let $\rho_0 : T^*(M) \to [0, 1]$ be a smooth function with $\rho_0 \equiv 1$ on U_0 and with compact support supp (ρ_0) contained in $\pi_T(M)$ and such that $\pi_\Sigma(\text{supp}(\rho_0))$ is contained in $V_{3/2}$. Set $\rho = \rho_0 \circ \pi_T : M \to [0, 1]$. By compactness of $\overline{V}_1 \times K$ and the condition $f_{s,0} \equiv 0$ it follows that for $\varepsilon > 0$ sufficiently small there are inclusions $j(f_{s,t})(\overline{V}_1) \subset U_0 \times [-c/2, c/2]$ for all $(s, t) \in K \times [0, \varepsilon]$.

Define a family of functions $g_{s,t} : M \to \mathbf{R}$ by the formula

$$g_{s,t}(p) = \rho(p) f_{s,t}(\pi(p)), \quad (s, t) \in K \times [0, \varepsilon]$$

for $p \in \text{supp}(\rho)$ and extend by zero to all of M. Let $X_{s,t}$ be the family of vector fields associated to $g_{s,t}$ and use the product decomposition $J(\Sigma) = T(\Sigma) \times \mathbf{R}$ to write $X_{s,t}$ in the form:

$$X_{s,t} = Y_{s,t} + h_{s,t}\frac{\partial}{\partial u}.$$

The local formula (5.8) shows that $h_{s,t}$ is independent of u, as is $Y_{s,t}$, and that both $h_{s,t}$ and $Y_{s,t}$ vanish outside of the support of ρ_0. Since $X_{s,0}$ is identically zero, the compactness of K insures that by choosing $\varepsilon > 0$ sufficiently small, the inequality

$$\max_{(v,s,t)\in T^*(\Sigma)\times K\times[0,\varepsilon]} |h_{s,t}(v)| \leq c/2$$

holds. It is now clear that the unit time flow of $X_{s,t}$ yields a family of diffeomorphisms

$$\nu_{s,t} : M' \to M, \quad (s,t) \in K \times [0,\varepsilon]$$

where M' is the open neighborhood of $j_0(\overline{V}_1)$ given by the formula

$$M' = \{v \in T^*(\Sigma) | \rho_0(v) > 0\} \times (-c/2, c/2) \cup M - \mathrm{supp}(\rho).$$

Now choose any $\delta \in (0, 1/2)$. Let $k : \Sigma \to [0,1]$ be a function whose support is contained in $V_{2\delta}$ and such that $k \equiv 1$ on $\overline{V}_\delta$. Let $g^\delta_{s,t} = (k \circ \pi) g_{s,t}$, and denote the associated vector fields by

$$X^\delta_{s,t} = Y^\delta_{s,t} + h^\delta_{s,t} \frac{\partial}{\partial u}$$

with notation as before. Because k depends only on x in local coordinates (x,y,u), it follows from formula (5.8) that the equation $h^\delta_{s,t} = (k\circ\pi)h_{s,t}$ holds and, therefore, so does the inequality $|h^\delta_{s,t}| < c/2$. The unit time flow of $X^\delta_{s,t}$ then furnishes a family of diffeomorphisms

$$\nu^\delta_{s,t} : M' \to M.$$

On the set $U \cap \pi^{-1}(V_\delta)$ the function $\rho_0 = 1$ and, consequently, the formula

$$X^\delta_{s,t} = f_{s,t}(x)\frac{\partial}{\partial u} + \sum_i \frac{\partial f_{s,t}}{\partial x^i}\frac{\partial}{\partial y_i}$$

holds. It follows then from the convexity of $\pi^{-1}(x) \cap U$ that the formula

$$\nu^\delta_{s,t}(x,y,u) = (x, y + df_{s,t}, u + f_{s,t}(x))$$

is satisfied for sufficiently small (y,u) and $x \in V_\delta$. Also, since $X^\delta_{s,t}$ has support contained in $\pi^{-1}(V_{2\delta})$ and since $X^\delta_{s,0} = 0$ for all $s \in K$ it is now clear that conditions (5.6) hold.

Reduction 3. Replace M by M' as in step 2 and suitably restrict ε. Then by virtue of the previous step we need only prove the lemma for the family of Legendre immersions

$$\psi_{s,t} : V_1 \xrightarrow{\tau_{s,t}} \Sigma \xhookrightarrow{j_0} M \subseteq J(\Sigma).$$

Since N is compact and $\tau_{s,0}(x) = x$ for $(x,s) \in V_1 \times K$, for $\varepsilon > 0$ small the maps $\tau_{s,t}$ restricted to the set $[-1/2, 1/2] \times N$ assume the form

$$\tau_{s,t} : [-1/2, 1/2] \times N \to (-2,2) \times N$$

$$\tau_{s,t}(x_0, x') = (k_{s,t}(x_0,x'), \varphi_{s,t}(x_0,x'))$$

where

$$\varphi_{s,t,x_0} : N \to N : x' \mapsto \varphi_{s,t}(x_0, x')$$

is a family of diffeomorphisms of N with $\varphi_{s,0,x_0} = id_N$ and

$$k_{s,t,x'} : [-1/2, 1/2] \to (-2,2) : x_0 \mapsto k_{s,t}(x_0, x')$$

is a family of injective immersions with $k_{s,t,x'}(x_0) = x_0$.

Reduction 4. In this step we show that we may further assume that the family

$$\psi_{s,t} : [-1/2, 1/2] \times N \to (-1,1) \times N \overset{j_0}{\hookrightarrow} M$$

is of the form

$$\psi'_{s,t}(x_0, x') = (h_{s,t}(x) + x_0, x') \text{ on } V_{1/2} - V_{3\delta/2}\,.$$

Since the family $\varphi_{s,t}$ is cell-wise smooth, there is a smooth bijection $\gamma : [0,1] \to [0,\varepsilon]$ such that the map

$$\Phi : [-1/2, 1/2] \times N \times K \times [0,1] \to N$$

$$\Phi(s_0, x', s, z) = \varphi_{s,\gamma(z)}(x_0, x')$$

is smooth on each of the sets $[-1/2, 1/2] \times N \times \Delta \times [0,1]$, where Δ is a cell of K. To ensure that Φ is smooth, it is only necessary to choose γ so that all derivatives of γ vanish on the finite set of values at which φ is not smooth in t.)

Pick $\delta \in (0, 1/4)$ and let $\rho : (-1/2, 1/2) \to [0,1]$ be a smooth function with support in $(-1/2, 3\delta/3)$ and with $\rho \equiv 1$ on $(-1/2, \delta)$. Observe that the family

$$\varphi'_{s,t} : [-1/2, 1/2] \times N \to N$$

$$\varphi'_{s,t}(x_0, x') = \varphi_{s,t\cdot\gamma(\rho(x_0))/\varepsilon}(x_0, x')$$

is cell-wise smooth and satisfies the conditions:

$$(5.9)\qquad \begin{aligned} &\varphi'_{s,0} = \varphi_{s,0} \\ &\varphi'_{s,t}(x_0, x) = \varphi_{s,t}(x_0, x) \qquad \text{for } (x_0, x') \in V_\delta \\ &\varphi'_{s,t}(x_0, x') = \varphi_{s,0}(x_0, x') \quad \text{for } (x_0, x') \in V_{1/2} - V_{3\delta/2}. \end{aligned}$$

Next observe that for $\varepsilon > 0$ sufficiently small, the inequality

$$\max_{(x_0,x',s,t)\in[-1/2,1/2]\times N\times K\times[0,\varepsilon]} \left| k_{s,t}(x_0, x') \right| \le 3/4$$

holds. For this value of ε and for any $\delta \in (0, 1/4)$ there is a family

$$k'_{s,t} : [-1/2, 1/2] \times N \to (-2, 2)$$

satisfying the conditions

$$(5.10)\qquad \begin{aligned} &k'_{s,0} = k_{s,0} \\ &k'_{s,t}(x_0, x') = k_{s,t}(x_0, x') \qquad \text{for } (x_0, x') \in V_\delta \\ &k'_{s,t}(x_0, x') = h_{s,t}(x') + x_0 \quad \text{for } x_0 \in (3\delta/2, 1/2), \end{aligned}$$

where the map $h_{s,t} : N \to (-1,1)$ is defined by the equation

$$h_{s,t}(x') = k_{s,t}(x', 3\delta/2) - 3\delta/2.$$

The family of Legendre immersions

$$\psi'_{s,t} : [-1/2, 1/2] \times N \to (-1,1) \times N \overset{j_0}{\hookrightarrow} M$$

$$\psi'_{s,t}(x_0, x') = j_0(k'_{s,t}(x_0, x'), \varphi'_{s,t}(x_0, x'))$$

satisfies the conditions

$$(5.11)\qquad \begin{aligned} &\psi'_{s,0} = \psi_{s,0} \\ &\psi'_{s,t} = \psi_{s,t} \text{ on the set } V_\delta \\ &\psi'_{s,t}(x_0, x') = (h_{s,t}(x) + x_0, x') \text{ on } V_{1/2} - V_{3\delta/2} \end{aligned}$$

(The second condition is obtained by extending $\psi'_{s,t}$ to Σ_0 by the formula $\psi'_{s,t}(x) = \psi_{s,t}(x)$ for $x \in \Sigma_0$.)

At this point we have reduced the problem to the following special case: *The family of Legendre immersions $\psi_{s,t}$ is of the form on the submanifold $N \times (3\delta/2, 1/2)$:*

$$\psi_{s,t}(x_0, x') = j_0(k_{s,t}(x_0, x'), x') \in M \subseteq J(\Sigma)$$

where $k_{s,t}(-, x') : (3\delta/2, 1/2) \to (-2, 2)$ is a family of injective immersions satisfying the conditions:

$$\begin{aligned} &k_{s,0}(x_0, x') = x_0, \ \textit{for}\ (s, x_0, x') \in K \times (3\delta/2, 1/2) \times N \\ &k_{s,t}(x_0, x') = h_{s,t}(x') + x_0 \\ &\max_{(s,t,x') \in K \times [0,\varepsilon] \times N} |h_{s,t}(x')| < 3/4. \end{aligned}$$

We will show how to modify $\psi_{s,t}$ on $N \times (3\delta/2, 1/2)$ to obtain another family of Legendre immersions $\psi'_{s,t}$ such that the following conditions hold

$$\begin{aligned} \psi'_{s,t}(x) &= \psi_{s,t}(x) && \text{for } x \in V_\delta \\ \psi'_{s,t}(x) &= j_0(x) && \text{for } x \in V_{1/2} - V_{2\delta} \\ \psi'_{s,0}(x) &= \psi_{s,0}(x) && \text{for } x \in V_{1/2}. \end{aligned} \tag{5.12}$$

Choose a neighborhood U of $N \times [-1, 1] \subseteq M \subseteq J(\Sigma)$ in M of the form $U = U' \times U''$ where

$$U' = \{(x_0, y_0, u) \in \mathbf{R}^3 \mid |x_0| < a,\ |y_0| < b,\ |u| < b\},$$

where $2 > a > 1$, $b > 0$ and U'' is an open neighborhood of the zero section $N \subseteq T^*N$ and we are using the coordinates of Remark 5.2. We will be done if we construct a family of curves

$$\gamma_{s,t,x} : (3\delta/2, 1/2) \to U' \text{ for } (s, t, x') \in K \times [0, \varepsilon] \times N$$

with $\gamma^*_{s,t,x}(du - y_0 dx_0) = 0$ and satisfying the conditions

$$\begin{aligned} &\gamma_{s,t,x'}(\zeta) = h_{s,t}(x') + \zeta && \text{for } \zeta \in (3\delta/2, 7\delta/4) \\ &\gamma_{s,t,x}(\zeta) = (\zeta, 0, 0) && \text{for } \zeta \in (2\delta, 1/2) \\ &\gamma_{s,0,x}(\zeta) = \zeta && \text{for } -\zeta \in (3\delta/2, 1/2) \end{aligned}$$

For then the family $\psi'_{s,t}$ will be given by the formula

$$\psi'_{s,t}(x_0, x') = (\gamma_{s,t,x}(x_0), x') \in U' \times N \subseteq U' \times U'' \subseteq M.$$

Let $f', g' : \mathbf{R} \to [-1, 1]$ be two even smooth functions (see Figure 1) with support on the interval $[-1, 1]$ and satisfying the conditions:

$$\begin{aligned} &f'(0) = 1, & & g'(\zeta) < 0 && \text{for } |\zeta| < 1 - c, \\ &f'(\zeta) \geq 0 & \text{and} \quad & g'(\zeta) > 0 && \text{for } 1 - c < |\zeta| < 1 \\ &\textstyle\int_{-1}^{1} f'(\zeta) d\zeta = 1 & & \textstyle\int_{-1}^{+1} g'(\zeta) d\zeta = 0 \end{aligned}$$

where $c \in (0, 1/2)$ is be determined shortly.

Next set

$$f(\zeta) = \int_{-1}^{\zeta} f'(z) dz \text{ and } g(\zeta) = \int_{-1}^{\zeta} g'(z) dz.$$

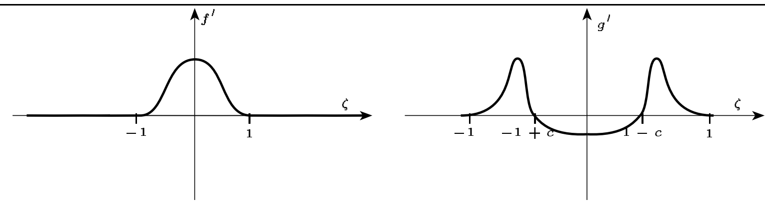

FIGURE 1. The graphs of $f'(\zeta)$ and $g'(\zeta)$.

The x_0, y_0 and u components of the family of curves $\gamma_{s,t,x'}$ are defined by the equations (see Figure 2)

$$\begin{aligned} x_0 =& X_{s,t,x'}(\zeta) = \zeta = h_{s,t}(x')f\Big(-\frac{8}{\delta}(\zeta - 7\delta/4)\Big) \\ y_0 =& Y_{s,t,x'}(\zeta) = \frac{b\delta}{8}h_{s,t}(x')g\Big(\frac{8}{\delta}(\zeta - 7\delta/4)\Big) \\ u =& U_{s,t,x'}(\zeta) = \int_{-1}^{\zeta} Y_{s,t,x'}(z)\frac{d}{d\zeta}X_{s,t,x'}(z)dz. \end{aligned}$$

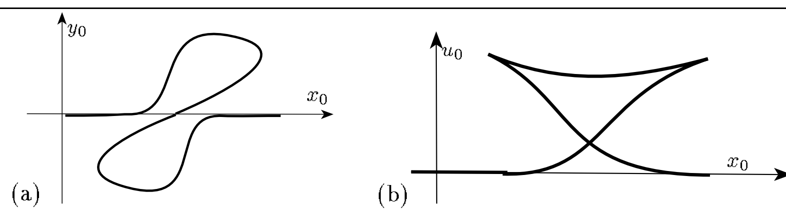

FIGURE 2. The image of the curve $\gamma_{s,t,x'}$

To ensure that the map $\zeta \mapsto \gamma_{s,t,x'}(\zeta)$ defines an immersion we must choose the constant c so that the functions $X_{s,t,x'}(\zeta)$ and $Y_{s,t,x'}(\zeta)$ have no common critical points. It is an easy exercise to see that for each (s,t,x') the derivative $X_{s,t,x'}(\zeta)$ has at most two zeros contained in the interval $(13\delta/8, 15\delta/8)$ and, therefore, by compactness of $K \times [0,\varepsilon] \times N$ there is a number $c \in (0, 1/2)$ such that for all $(s,t,x') \in K \times [0,\varepsilon] \times N$ the zero set of $X'_{s,t,x'}(\zeta)$ is contained in the interval $((13+c)\delta/8, (15-c)\delta/8)$. Because the critical points of $Y_{s,t,x'}(\zeta)$ lie outside of that interval, the map $\zeta \mapsto \gamma_{s,t,x'}(\zeta)$ is an immersion.

It is an easy exercise to see that $\gamma_{s,t,x'}$ has all of the required properties. $\square$

6. THE CLASSIFICATION OF LEGENDRE IMMERSIONS

In this section we prove the classification theorem for Legendre immersions. As mentioned in the introduction, the proof here closely parallels the proof in [HP] of the classification theorem for combinatorial immersions. Lees [L] has used the techniques in [HP] to prove a classification theorem for Lagrange immersions.

To state the classification theorem we must define several semi-simplicial complexes. Semi-simplicial complexes are sometimes called simplicial sets. The geometric realization of the semi-simplicial complex K will be written $|K|$. For basic facts about semi-simplicial complexes see [Z].

Definition 6.1. Let (M^{2n+1}, η) be a contact manifold without boundary and let Σ^n be a manifold, possibly with boundary. Let $\Sigma_0 \subset \Sigma$ be a compact smooth neighborhood retract (see Remark 2.2) and let $[\varphi]$ be the germ at Σ_0 of a Legendre immersion into M. The **semi-simplicial space of Legendre immersions** relative to $[\varphi]$, written $\mathcal{I}_\varphi(\Sigma, M)$, is the semi-simplicial complex whose simplices are smooth families $\psi : \Sigma \times \Delta^q \to M$ of Legendre immersions with $[\psi_t] = [\varphi]$ where $\psi_t \equiv \psi(-,t)$ and whose face and degeneracy maps are induced by the face and degeneracy maps of the standard simplex $\Delta^q \subseteq \mathbf{R}^{q+1}$. Note that if K is a semi-simplicial complex then a semi-simplicial map from K to $\mathcal{I}_\varphi(\Sigma, M)$ is a simplex-wise smooth map, $\psi : \Sigma \times |K| \to M$ with $[\psi_s] = [\varphi]$ and $\psi_s^* \eta = 0$ for all $s \in |K|$. Similarly the space $\mathcal{C}_\varphi(T(\Sigma), H(M))$ of **ℓ-bundle maps relative to $[\varphi]$** is the semi-simplicial space whose simplices are smooth maps $\tilde{\psi} : T(\Sigma) \times \Delta^q \to H(M)$ with $\tilde{\psi}(\ ,t) = \tilde{\psi}_t$ an ℓ-bundle injection and with $[\tilde{\psi}_t] = [\varphi_x]$ for all $t \in \Delta^q$ together with the obvious face and degeneracy maps. If $\Sigma_0 = \emptyset$ we will simply write $\mathcal{I}(\Sigma, M)$ and $\mathcal{C}(T(M), H(M))$. Again a semi-simplicial map $K \to \mathcal{C}_\varphi(T(\Sigma), H(M)$ is a simplex-wise smooth map $\tilde{\psi} : T(\Sigma) \times |K| \to H(M)$ with $\tilde{\psi}_s$ an ℓ-bundle injection such that $[\tilde{\psi}_s] = [\varphi_*]$ for all s.

The map $\psi \to \psi_*$ which assigns to a Legendre immersion its derivative $\psi_* : T(\Sigma) \to H(M)$ induces an injection of semi-simplicial complexes

$$d : \mathcal{I}_\varphi(\Sigma, M) \hookrightarrow \mathcal{C}_\varphi(T(\Sigma), H(M)).$$

The first form of the classification theorem is the following theorem:

Theorem 6.2. *The map $d : \mathcal{I}_\varphi(\Sigma, M) \hookrightarrow \mathcal{C}_\varphi(T(\Sigma), H(M))$ is a homotopy equivalence.*

In the case where $\Sigma_0 \subset \Sigma$ is a compact embedded submanifold a slightly stronger version of Theorem 6.2 can be given. Let $\varphi : \Sigma_0 \to M$ be an immersion with $\varphi^* \eta = 0$ and let $\tilde{\varphi} : T(\Sigma)_{|\Sigma_0} \to H(M)$ be an ℓ-bundle isomorphism which extends $\varphi_* : T(\Sigma_0) \to H(M)$. Of course when $\dim \Sigma_0 = \dim \Sigma$ no extension is necessary. We seek conditions under which the map φ extends to a Legendre immersion of Σ into M whose derivative agrees with $\tilde{\varphi}$ on Σ_0.

Denote by $\mathcal{I}_{\tilde{\varphi}}(\Sigma, M)^0$, resp. $\mathcal{C}_{\tilde{\varphi}}(\Sigma, M)^0$, the semi-simplicial complex whose simplices are smooth maps $\psi : \Sigma \times \Delta^q \to M$, resp. $\tilde{\psi} : T(\Sigma) \times \Delta^q \to H(M)$, as above except that we only require that $\tilde{\psi}_{s*|\Sigma_0} = \tilde{\varphi}$, for all $s \in \Delta^q$—agreement on a neighborhood is not required here. Differentiation defines an inclusion

$$d^0 : \mathcal{I}_{\tilde{\varphi}}(\Sigma, M)^0 \hookrightarrow \mathcal{C}_{\tilde{\varphi}}(T(\Sigma), H(M))^0$$

and the following theorem holds.

Theorem 6.3. *The map $d^0 : \mathcal{I}_{\tilde{\varphi}}(\Sigma, M)^0 \hookrightarrow \mathcal{C}_{\tilde{\varphi}}(T(\Sigma), H(M))^0$ is a homotopy equivalence.*

To prove the classification theorems we will need the following lemma (see [HP], p. 80).

Lemma 6.4 (Haefliger–Poenaru). *Let $p_i : E_i \to B_i, i = 1, 2$ be two fibrations of semi-simplicial complexes with p_i surjective and let $f : E_1 \to E_2$ be a map of fibrations with associated map of base complexes $f_0 : B_1 \to B_2$. Then any two of the following imply the third:*

1. *f is a homotopy equivalence,*
2. *f_0 is a homotopy equivalence, and*
3. *the restriction of f to each fiber of E_1 is a homotopy equivalence with the corresponding fiber of E_2.*

Now let $\Sigma_0 \subset \Sigma$ and $\varphi : V \to M$, V a neighborhood of Σ_0, be as in Theorem 6.3 and suppose that $\Sigma' \subset \Sigma$ is a compact, n-dimensional, embedded submanifold with Σ_0 in its interior. The process of restriction to Σ' induces maps

$$r_{\mathcal{I}} : \mathcal{I}_\varphi(\Sigma, M) \to \mathcal{I}_\varphi(\Sigma', M)$$

and

$$r_{\mathcal{C}} : \mathcal{C}_\varphi(T(\Sigma), H(M)) \to \mathcal{C}_\varphi(T(\Sigma'), H(M)).$$

If $\Sigma_0 \subseteq \Sigma$ is a compact embedded submanifold and $\tilde{\varphi}$ an ℓ-bundle isomorphism as in Theorem 6.3, then there are corresponding maps ${r_{\mathcal{I}}}^0$ and ${r_{\mathcal{C}}}^0$ of the obvious spaces.

Proposition 6.5. *The maps $r_{\mathcal{I}}$, $r_{\mathcal{C}}$, ${r_{\mathcal{I}}}^0$ and ${r_{\mathcal{C}}}^0$ are all fibrations. Moreover, their images are path components of their respective spaces.*

Proof. To show that $r_{\mathcal{I}}$ is a fibration let K be a finite complex and let $\psi' : \Sigma' \times K \times [0,1] \to M$ be a piecewise smooth family of Legendre immersions with $\psi'_{s,t} = \psi'(\_, s, t)$ equal to φ in neighborhood of Σ_0 and with $\psi'_{s,0}$ extending to a piecewise smooth family $\psi_0 : \Sigma \times K \to M$ of Legendre immersions. We must extend ψ_0 to a family $\psi : \Sigma \times |K| \times [0,1] \to M$ such that the equality $\psi_{s,t} = \psi'_{s,t}$ holds on Σ'.

First observe that $K \times 0$ is a subcomplex of the cell complex $K \times [0,1]$ and apply Theorem 4.8 to obtain a family $\psi'' : V \times K \times [0,1] \to M$ of Legendre immersions of a neighborhood V of Σ' compatible with ψ_0 and extending ψ'. Now apply Theorem 5.1 to obtain the family ψ. Notice that this also shows that the image of $r_{\mathcal{I}}$ is a path component.

That $r_{\mathcal{C}}$ is a fibration is immediate from Lemma 2.5 and the fact that $\mathbf{C}$-bundle injections enjoy the homotopy extension property. (This follows from the observation that a $\mathbf{C}$-bundle injections $\Phi : T(\Sigma)_{|A} \to H(M), A \subseteq \Sigma$ corresponds to a section over A of the fiber bundle $E \to \Sigma \times M \to \Sigma$ where $E_{(p,q)}, p \in \Sigma, q \in M$, is the space of all complex vector space isomorphisms from $T(\Sigma)_p^{\mathbf{C}}$ to $H(M)_q$.) This also shows that the image of $r_{\mathcal{C}}$ is a path component.

The proofs that ${r_{\mathcal{I}}}^0$ and ${r_{\mathcal{C}}}^0$ are fibrations are entirely similar. □

The following two lemmas are special cases of the classification theorem.

Lemma 6.6. *Let $D^n \subseteq \mathbf{R}^n$ be the closed unit ball in $\mathbf{R}^n$ centered at 0. Then the map $d : \mathcal{I}(D^n, M) \to \mathcal{C}(T(D^n), H(M))$ is a homotopy equivalence.*

Proof. Let D^n_ε be the closed ball of radius ε centered at 0 and set $\mathcal{I}_0 = \lim\limits_{\varepsilon\to 0} \mathcal{I}(D^n_\varepsilon, M)$ and $\mathcal{C}_0 = \lim\limits_{\varepsilon\to 0} \mathcal{C}(T(D_\varepsilon), H(M))$. Then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{I}(D_n, M) & \xrightarrow{d} & \mathcal{C}(T(D^n), H(M)) \\ \downarrow r_{\mathcal{I}} & & \downarrow r_{\mathcal{C}} \\ \mathcal{I}_0 & \xrightarrow{d_0} & \mathcal{C}_0 \end{array}$$

with $r_{\mathcal{I}}$, $r_{\mathcal{C}}$ and d_0 defined in the obvious way. We will show that $r_{\mathcal{I}}$ and $r_{\mathcal{C}}$ are surjective fibrations, that d_0 is a homotopy equivalence and that the fibers of $r_{\mathcal{I}}$ and $r_{\mathcal{C}}$ are contractible. The result then follows from Lemma 6.4.

That the maps $r_{\mathcal{I}}$ and $r_{\mathcal{C}}$ are fibrations follows from Proposition 6.5 and the fact that the direct limit of a set of fibrations is a fibration.

To see that the maps $r_{\mathcal{I}}$ and $r_{\mathcal{C}}$ are surjective let $\tau_\varepsilon : D^n \to D_\varepsilon$ be a diffeomorphism with $\tau_\epsilon = \text{ id}$ on a neighborhood of 0. Let $\psi : D_\varepsilon \times \Delta^q \to M$ (resp. $\tilde{\psi} : T(D_\varepsilon) \times \Delta^q \to H(M)$) be a simplex of $\mathcal{I}_0$(resp. $\mathcal{C}_0$). Then $\psi' = \psi \circ (\tau_\varepsilon \times \mathrm{id}_{\Delta^q}) : D^n \times \Delta^q \to M$ (resp. $\tilde{\psi}' = \tilde{\psi} \circ (\tau_{\epsilon *} \times \mathrm{id}_{\Delta^q})$) is a simplex of $\mathcal{I}(D^n, M)$, (resp. $\mathcal{C}(T(D^n), H(M))$) with $\psi = r_{\mathcal{I}}(\psi')$, (resp. $\tilde{\psi} = r_{\mathcal{C}}(\tilde{\psi}')$).

To prove that the fibers of $r_{\mathcal{I}}$ are contractible, let K be a finite complex and let $\psi : D^n \times |K| \to M$ be a map from K into $\mathcal{I}(D^n, M)$ with $r_{\mathcal{I}} \circ \psi$ constant. Then there is an $\varepsilon > 0$ with $\psi_s : D_\varepsilon \to M$ independent of $s \in |K|$. Let $R : D^n \times [0,1] \to D^n$ be a family of diffeomorphisms of D^n with $R_t = R(\ , t)$ the identity on a neighborhood of $0, R_0 = \mathrm{id}_{D_n}$ and $R_1(D^n) \subset D_\varepsilon$. Then the map $\Psi : D^n \times |K| \times [0,1] \to M$ defined by $\Psi(x, s, t) = \psi(R(x,t), s)$ is a homotopy between $\psi : K \to \mathcal{I}(D^n, M)$ and a constant map with $r_{\mathcal{I}} \circ \Psi : K \times [0,1] \to \mathcal{I}_0$ constant. In a similar way one shows that the fibers of $r_{\mathcal{C}}$ are contractible.

Finally, to see that d_0 is a homotopy equivalence, let $\tilde{\psi} : T(D_\epsilon) \times |K| \to H(M)$ be a map from a finite complex K into $\mathcal{C}_0$. Since $T(D_\varepsilon) = D_\varepsilon \times \mathbf{R}^n$ we can define a homotopy $\tilde{\Psi} : T(D_\varepsilon) \times K \times [0,1] \to H(M)$ by $\tilde{\Psi}((p,v), s, t) = \tilde{\psi}((tp, v), s)$. Then $\tilde{\psi}_t = \tilde{\Psi}(-,-,t)$ satisfies $\tilde{\Psi}_1 = \tilde{\psi}$ and Ψ_0 sends $T(D_\varepsilon) \times \{s\}$ to the single fiber $H(M)_{\psi(0,s)}$ of $H(M) \to M$ where $\psi : D_\varepsilon \times K \to M$ is the map of base spaces associated to $\tilde{\psi}$. By Theorem 4.8 the map $\psi : \{0\} \times K \to M$ extends to a family of Legendre immersions $\psi' : D_{\varepsilon'} \times K \to M$ with $\psi''_{s*} : T(D_{\varepsilon'})_{|0} \times K \to H(M)$ equal to $\tilde{\psi}' : T(D_\varepsilon)_{|0} \times K \to H(M)$, $\varepsilon' \leq \varepsilon$. Thus there is a homotopy $\tilde{\Psi}' : T(D_{\varepsilon'}) \times K \times [0,1] \to H(M)$ defined by

$$\tilde{\Psi}'((p,v), s, t) = \tilde{\psi}''_{s*}(tp, v)$$

between $\tilde{\psi}' : K \to \mathcal{C}_0$ and $d_0 \circ \psi'' : K \to \mathcal{I}_0 \hookrightarrow \mathcal{C}_0$. □

Remark 6.7. Lemma 6.6 applies also to the manifolds $D^k \times D^{n-k}$ (the corners of $D^k \times D^{n-k}$ can be smoothed out by extending all maps to a neighborhood of $D^k \times D^{n-k}$ in $\mathbf{R}^n$).

We now prove a special case of Theorem 6.3.

Lemma 6.8. *Let $\varphi : A^k \times D^{n-k} \to M$ be a Legendre immersion, where A^k denotes a closed annular neighborhood of ∂D^k in D^k. Then the map*

$$d^0 : \mathcal{I}_{\varphi_*}(D^k \times D^{n-k}, M)^0 \to \mathcal{C}_{\varphi_*}(T(D^k \times D^{n-k}), H(M))^0$$

is a homotopy equivalence.

Proof. We begin by introducing some notation. Let D^k_+ and D^k_- denote the cover of ∂D^{k+1} by disks defined by

$$D^k_+ = \{(x^1, x^2, \dots, x^{k+1}) \in \partial D^{k+1} \mid x^{k+1} \geq -1/2\}$$

and

$$D^k_- = \{(x^1, x^2, \dots, x^{k+1}) \in \partial D^{k+1} \mid x^{k+1} \leq 1/2\}\,.$$

and let A^k_- denote the annular neighborhood of ∂D^k_- defined by

$$A^k_- = \{(x^1, x^2, \dots, x^{k+1}) \in \partial D^{k+1} \mid |x^{k+1}| \leq 1/2\}\,.$$

The proof proceeds by induction on k, and it based on the following commutative diagram:

$$\begin{array}{ccc} \mathcal{I}(\partial D^{k+1} \times D^{n-k}, M) & \xrightarrow{d_+} & \mathcal{C}(T(\partial D^{k+1} \times D^{n-k}), H(M)) \\ \downarrow r_{\mathcal{I}} & & \downarrow r_{\mathcal{C}} \\ \mathcal{I}(D^k_- \times D^{n-k}, M) & \xrightarrow{d} & \mathcal{C}(T(D^k_- \times D^{n-k}), H(M)) \\ \downarrow r_{\mathcal{I}} & & \downarrow r_{\mathcal{C}} \\ \mathcal{I}(A^k_- \times D^{n-k}, M) & \xrightarrow{d_-} & \mathcal{C}(T(A^k_- \times D^{n-k}), H(M)) \end{array} \tag{6.9}$$

Before we start the induction process, we make a two observations about the diagram.

1. First notice that all vertical maps are fibrations, and the fibers of the maps $r_{\mathcal{I}}$ and $r_{\mathcal{C}}$ in the lower rectangle are of the form $\mathcal{I}_{\varphi_*}(D^k_- \times D^{n-k}, M)^0$ and $\mathcal{C}_{\varphi_*}(T(D^k_- \times D^{n-k}), H(M))$ for all k. Similarly, the fibers of the maps $r_{\mathcal{I}}$ and $r_{\mathcal{C}}$ in the upper rectangle are of the form $\mathcal{I}_{\varphi_*}(D^k_+ \times D^{n-k}, M)^0$ and $\mathcal{C}_{\varphi_*}(T(D^k_+ \times D^{n-k}), H(M))$ for all k.
2. Also notice that because $A^k_- \times D^{n-k}$ is diffeomorphic to $\partial D^k \times D^{n-(k-1)}$, the bottom row of the diagram is of the form

$$\mathcal{I}(\partial D^k_- \times D^{n-(k-1)}, M) \xrightarrow{d_+} \mathcal{C}(T(\partial D^k_- \times D^{n-(k-1)}), H(M))$$

3. By Lemma 6.6 and Remark 6.7, the middle arrow is always a homotopy equivalence.
4. Although the maps $r_{\mathcal{I}}$ and $r_{\mathcal{C}}$ are not surjective, their images are path components (see Proposition 6.5). It follows that any horizontal arrows in Diagram 6.9 which is a homotopy equivalence induces a homotopy equivalence between the image of $r_{\mathcal{I}}$ and the image of $r_{\mathcal{C}}$.

We now begin the induction process. Consider first the case $k = 1$. Because A^1_- is diffeomorphic to D^1 we can apply Lemma 6.6 to conclude that the map $d_- : \mathcal{I}(A^1_- \times D^{n-1}, M) \to \mathcal{C}(T(A^1_- \times D^{n-1}), H(M))$ in the bottom row is a homotopy equivalence. Next apply Lemma 6.4 to conclude that d induces a homotopy equivalence of the fibers:

$$d : \mathcal{I}_{\varphi_*}(D^1_- \times D^{n-1}, M)^0 \to \mathcal{C}_{\varphi_*}(T(D^1_- \times D^{n-1}), H(M))^0\,.$$

(Although $r_{\mathcal{I}}$ and $r_{\mathcal{C}}$ are not surjective $d_- : \text{Image}(r_{\mathcal{I}}) \to \text{Image}(r_{\mathcal{C}})$ is a homotopy equivalence, hence Lemma 6.4 still applies. A similar remark applies to future applications of Lemma 6.4).

Inductively assume that the theorem holds for $k \leq k_0$ and that d_- is a homotopy equivalence (and therefore a homotopy equivalence when restricted to the image of

$r_{\mathcal{I}}$). Set $k = k_0 + 1$ and apply Lemma 6.4 to the lower square of diagram 6.9 to prove the theorem for $k = k_0 + 1$.

Now consider the upper square. Since we now know that the fibers of $r_{\mathcal{I}}$ and $r_{\mathcal{C}}$ are homotopy equivalent, via d_+, and since d is a homotopy equivalence it follows from 6.4 that d_+ is a homotopy equivalence. Finally, identify d_+ with d_- (with $k = k_0 + 1$) to complete the induction step. □

Let $\Sigma_0 \subset \Sigma$ be a smooth neighborhood retract then using the flow of the vector field grad(f) one can prove the following lemma.

Lemma 6.10. *Let V_ε be neighborhood of Σ_0 such that there are maps $\tau^k : V_1 \times [0,1] \to V_1, k = 1,2,3,\dots$ with $\tau^k_t = \tau^k(\ ,t) : V_1 \to V_1$ the following conditions hold*

1. $\tau^k_0 = \mathrm{id}_{V_1}$
2. $\tau^k_1(V_1) \subseteq V_{1/k}$
3. $\tau^k_t : V_1 \to \tau^k_t(V_1)$ *is a diffeomorphism*
4. τ^k_t *is the identity on a neighborhood of* Σ_0

for all $k = 1,2,3,\dots$ and $t \in [0,1]$.

The next lemma is an immediate consequence of Lemma 6.10

Lemma 6.11. *Let $\Sigma_0 \subset \Sigma$ be a compact smooth neighborhood retract and let $\varphi : U \to M$ be a Legendre immersion on an open neighborhood of Σ_0. If V is a tubular neighborhood of Σ_0 then the spaces $\mathcal{I}_\varphi(V,M)$ and $\mathcal{C}_\varphi(T(V),H(M))$ are contractible.*

If in addition (i) Σ_0 is an embedded submanifold, (ii) $\psi : \Sigma_0 \to M$ is an immersion with $\psi^*\eta = 0$, and (iii) $\tilde{\psi} : T(\Sigma)_{|\Sigma_0} \to H(M)$ is an ℓ-bundle isomorphism extending $\psi_* : T(\Sigma_0) \to H(M)$, then $\mathcal{I}_{\tilde{\psi}}(V,M)^0$ and $\mathcal{C}_{\tilde{\psi}}(T(V),H(M))^0$ are contractible.

Proof. Let K be a finite complex and let $\rho : V \times K \to M$ be a map from K into $\mathcal{I}_\varphi(V,M)$. Then $\rho_s = \rho(-,s)$ agrees with φ on a neighborhood $V_{1/k}$ for some integer k and all $s \in K$ and the map $\Psi : V \times K \times [0,1] \to M$ defined by $\Psi(p,s,t) = \rho(\tau^k_t(p),s)$ is a homotopy from ρ to a constant map $\rho' : K \to \mathcal{I}_\varphi(V,M)$. Hence $\mathcal{I}_\varphi(V,M)$ is contractible. The proof that $\mathcal{C}_\varphi(T(V),H(M)$ is contractible is similar.

To prove that $\mathcal{I}_{\tilde{\psi}}(V,M)^0$ is contractible let K be a finite complex and let $\varphi : V \times |K| \to M$ represent a map from K into $\mathcal{I}_{\tilde{\psi}}(V,M)^0$. If we can construct a homotopy of φ to a map φ' with $\varphi'(-,s) : U \to M$ independent of s for U a neighborhood of Σ_0 in V we will be done by the first part of the proof. We will only sketch the construction of the homotopy.

Let $\varphi_0 = \varphi(-,s_0) : V \to M$ for $s_0 \in |K|$ a fixed point in $|K|$. By Proposition 4.6(1) there is a contact immersion $\Phi : N \to M$ for N a neighborhood of Σ_0 in $J(\Sigma)$ extending φ_0. Further, there is a neighborhood $V' \subset V$ of Σ_0 with the property that $\varphi_s = \varphi(\ ,s)$, for $s \in |K|$ have representations $\varphi_s = \Phi \circ j(f_s) \circ \tau_s : V' \to M$ as in 4.6(3). (Because $\varphi_{s*} = \tilde{\psi}$ on Σ_0 and $|K|$ is compact the neighborhood of s_0 in $|K|$ can be taken to be all of $|K|$.) Because $\tau_s = \mathrm{id}$ on Σ_0 and $|K|$ is compact it is possible to construct a homotopy of τ_s to a family $\tau'_s = \mathrm{id}$ on a neighborhood of Σ_0 and $\tau'_s = \tau_s$ outside of a compact neighborhood of Σ_0 (This can be proved using the classification theorem for smooth immersions, for example.) Therefore after shrinking V' if necessary, we may assume that $\tau_s = \mathrm{id}$ on V',. Finally, we construct the required homotopy to a map φ' by constructing a family of η-vector fields with support on a neighborhood of Σ_0 in N. Their flows yield the homotopy.

To show that $\mathcal{C}_{\tilde{\psi}}(T(V), H(M))$ is a contractible is much easier. Because V is a tubular neighborhood of Σ_0 there is a homotopy of bundle maps $H : T(V) \times [0,1] \to T(V)$ with the following properties:

1. $H(-,t) : T(V)_{|\Sigma_0} \to T(V)_{|\Sigma_0}$ is the identity,
2. $H(\ ,1) : T(V) \to T(V)$ is the identity, and
3. $H(TV, 0) \subseteq (TV)_{|\Sigma_0}$.

Let $\tilde{\varphi}_1 : T(V) \times |K| \to H(M)$ be a map from K into $\mathcal{C}_{\tilde{\psi}}(T(V), H(M))^0$. Then $\tilde{\psi} : T(V) \times |K| \times [0,1] \to H(M)$ with $\tilde{\varphi}(v,s,t) = \tilde{\varphi}_1(H(v,t), s)$ is homotopy from $\tilde{\psi}_1$ to a constant map. □

of Theorems 6.2 and 6.3. We will prove Theorem 6.2 only; the proof is formal and applies equally well to Theorem 6.3 with only slight changes of notation.

Begin by assuming that Σ is compact. We will do induction on the number of handles that must be attached to a tubular neighborhood V of Σ_0 to obtain Σ.

If no handles must be attached we are done by the preceding lemma.

Now suppose that $d' : \mathcal{I}_\varphi(\Sigma', M) \to \mathcal{C}_\varphi(T(\Sigma'), H(M))$ is a homotopy equivalence and that $\Sigma = \Sigma' \cup_{\varphi'} D^k \times D^{n-k}$. Then there is a map of fibrations (see Proposition 6.5)

$$\begin{array}{ccc} \mathcal{I}_\varphi(\Sigma, M) & \xrightarrow{d} & \mathcal{C}_\varphi(T(\Sigma), H(M)) \\ r_{\mathcal{I}} \downarrow & & \downarrow r_{\mathcal{C}} \\ \mathcal{I}_\varphi(\Sigma', M) & \xrightarrow{d'} & \mathcal{C}_\varphi(T(\Sigma'), H(M)) \end{array}$$

Although $r_{\mathcal{I}}$ and $r_{\mathcal{C}}$ are not surjective but they map onto connected components. Consequently $d' : \mathrm{Image}(r_{\mathcal{I}}) \to \mathrm{Image}(r_{\mathcal{C}})$ is a homotopy equivalence. By Lemma 6.8, d restricted to fibers is a homotopy equivalence. Therefore, d is a homotopy equivalence by Lemma 6.4.

If Σ is not compact the result follows by taking limits over compact submanifolds of Σ. □

Remark 6.12. Parts (1) and (2) of Theorem 2.4 follow from Theorem 6.2. They are restatements of the fact that the map d induces isomorphisms $\pi_n(\mathcal{I}_\varphi(\Sigma, M)) \cong \pi_n(\mathcal{C}_\varphi(T(\Sigma), H(M)))$ and $\pi_n(\mathcal{I}_{\tilde{\varphi}}(\Sigma, M)^0) \cong \pi_n(\mathcal{C}\varphi(T(\Sigma), H(M)))$ for $n = 0$, 1.

Remark 6.13. It follows from Lemma 2.5 that everywhere above we are free to replace "ℓ-bundle injection" and "ℓ-homotopy" by "**C**-bundle injection" and "**C**-homotopy."

The following approximation theorem was used in Section 3 to prove a theorem of Weinstein.

Theorem 6.14. *Let $\Sigma_0 \subseteq \Sigma^n$ be a compact, smooth neighborhood retract and let $[\varphi_0]$ be the germ over Σ_0 of a Legendre immersion into the $(2n+1)$-dimensional contact manifold (M, η). Suppose that $\Phi : T(\Sigma) \to H(M)$ is a* **C***-bundle injection over the smooth map $\varphi : \Sigma \to M$ with $[\Phi] = [\varphi_{0*}]$ and choose $\varepsilon > 0$.*

1. *Then there is a Legendre immersion $\psi : \Sigma \to M)$ with $\sup_{p \in \Sigma} |\psi(p) - \varphi(p)| < \varepsilon$ and with ψ_* and Φ both* **C***-homotopic relative to $[\varphi_{0*}]$.*
2. *If ψ_0 and ψ_1 are two such Legendre immersions then there is an ℓ-homotopy of Legendre immersions $\psi_t : \Sigma \to M$ relative to $[\varphi_0]$ between them that satisfies*

the inequality $\sup_{p\in\Sigma}|\psi_t(p)-\varphi(p)|<\varepsilon$ *for all* $t\in[0,1]$*. Here* $|\ |$ *denotes distance relative to the Riemannian metric on* M*.*

Proof. A standard argument using a representation of Σ as an increasing union of compact manifolds with boundary shows that we need only consider the case where Σ is compact. Then extending Φ to the double of Σ allows for a further reduction to the case $\partial\Sigma=\emptyset$.

Choose a smooth triangulation of Σ so fine that if $\sigma\subseteq\Sigma$ is an open n-simplex then there is a geodesically convex ball $U'_\sigma\subset M$ of diameter at most ε such that $\varphi(\sigma)\subseteq U'_\sigma$. If $\tau=\tau_k$ is a k-simplex set $U_\tau=\bigcup_{\tau<\sigma}U'_\sigma$, where $\tau<\sigma$ means that τ is a simplex contained in $\overline{\sigma}$. Let Σ_k denote the union of all closed k-simplices.

Construct open covers of Σ_k inductively as follows. For each vertex $v\in\Sigma_0$ let $v\in Q_v\subset\subset W_v\subset\subset V_v$ be open sets with $\overline{V}_v$ compact and $\varphi(\overline{V}_v)\subseteq U_v$ and such that the sets V_v are pairwise disjoint. Let $Q_0=\cup_v Q_v$, $W_0=\cup_v W_v$ and $V_0=\cup_v V_v$. We can choose the cover so that the complement Σ_1-Q_0 consists of a collection of disjoint closed 1-disks, one contained in each 1-simplex of Σ. Inductively assume that open sets $Q_k\subset\subset W_k\subset\subset V_k$ covering Σ_k have been constructed, and that the complement $\Sigma_{k+1}-Q_k$ consists of a collection of disjoint, closed $(k+1)$-disks, one contained in each $(k+1)$ simplex of Σ. For each such disk, say $D'\subset\tau$, where τ is a $(k+1)$-simplex, choose open sets $D'\subset Q_\tau\subset\subset W_\tau\subset\subset V_\tau$ with $\overline{V}_v$ compact and $\varphi(\overline{V}_\tau)\subseteq U_\tau$. We may assume that the sets V_τ are pairwise disjoint. Now set $Q_{k+1}=Q_k\cup\cup_{\tau\in\Sigma_{k+1}}Q_\tau$, with similar definitions for W_{k+1} and V_{k+1}. By shrinking if necessary, we may assume that $\Sigma_{k+2}-V_{k+1}$ consists of a disjoint union of closed $(k+2)$-disks.

Proceed inductively to construct ψ as follows. Suppose that for each $k\le k_0-1$ there is a $\mathbf{C}$-homotopy $\Phi^k_t:T(\Sigma)\to H(M), 0\le t\le k+1$ relative to $[\varphi_0]$ with base maps $\varphi^k_t:T(\Sigma)\to M$ satisfying the conditions

1. For each $\sigma\subseteq\Sigma_k,\varphi^k_t(V_\sigma)\subseteq U_\sigma$;
2. $\varphi^*_{k+1}\eta=0$ and $\Phi^k_{k+1}={\varphi_{k+1}}^k{}_*$ on $\overline{Q}^k=\bigcup_{\sigma\subseteq\Sigma_k}\overline{Q}_\sigma$
3. $\Phi^k_t=\Phi$ on $\Sigma-W^k$,where $W^k=\bigcup_{\sigma\subseteq\Sigma_k}W_\sigma$. (For $k=-1$ set $\Phi^k_t=\Phi$.)

Define $\Phi^{k_0}_t:T(\Sigma)\to H(M)$ satisfying (1), (2) and (3) as follows. For $t\le k_0$ set $\Phi^{k_0}_t=\Phi^{k_0-1}_t$. For $t\ge k_0$ and σ an open k_0-simplex, define Φ_t on all of V_σ by using Theorem 2.4(1) to obtain a homotopy $\tilde{\Phi}_t:T(\Sigma)_{|V_\sigma}\to H(M)_{|U_\sigma}, k_0\le t\le k_0+1$ fixed on a neighborhood of $\overline{Q}^{k-1}\cup\Sigma_0$ with base maps $\tilde{\varphi}_t:V_\sigma\to U_\sigma$ satisfying the following conditions:

1. $\tilde{\varphi}_{k_0+1}$ is a Legendre immersion,
2. $(\tilde{\varphi}_{k_0+1})_*=\tilde{\Phi}_{k_0+1}$, and
3. $\tilde{\Phi}_{k_0}=\Phi^{k_0-1}_{k_0}|V_\sigma$.

Now set $\Phi^{k_0}_t=\tilde{\Phi}_t$ on $\overline{Q}_\sigma\cup\Sigma_0\cap V_\sigma$ and $\Phi^{k_0}_t=\Phi^{k_0-1}_t$ on $V_\sigma-W_\sigma$ and use the homotopy extension property to extend to a $\mathbf{C}$-homotopy $\Phi_t:T(\Sigma)_{|V_\sigma}\to H(M)_{|U_\sigma}$. Do this for every k_0-simplex and set $\Phi^{k_0}_t=\Phi^{k_0-1}_{k_0}$ on $\Sigma-W^k$. The required immersion is $\psi={\varphi_{n+1}}^n$.

The proof of part 2 is similar and is left to the reader. □

7. Transversality

In this section we prove a transversality theorem for Legendre immersions. (A similar transversality theorem for Lagrangian immersions also holds; but, because its proof is almost identical to the one presented here we leave it to the interested reader to fill in the details.) Our proof is a modification of Morlet's multijet transversality theorem for C^∞-maps as presented in [M]. For general information about jet spaces and related matters see [GG] or [H1].

Notation. Let (M^{2n+1}, η) be a contact manifold and let Σ^n be a smooth manifold. We will assume that $\partial M = \partial\Sigma = \emptyset$. Let $C^\infty(\Sigma, M)$ denote the space of smooth mappings from Σ into M equipped with the Whitney C^∞-topology, and let $\mathcal{L}^\infty(\Sigma, M) \subseteq C^\infty(\Sigma, M)$ be the subspace of Legendre immersions with the induced topology. Let $L^k(\Sigma, M) \subseteq J^k(\Sigma, M)$ be the subspace of k-jets of germs of Legendre immersions of open sets of Σ into M. It follows from Theorem 4.8 with K and Σ_0 points that $L^0(\Sigma, M) = J^0(\Sigma, M) \equiv \Sigma \times M$. For any space N let N^s denote its s-fold product and let $N^{(s)}$ denote its configuration space, i.e. the subspace of N^s defined by the condition $N^{(s)} = \{(p_1, p_2, \dots, p_s) \,|\, p_i \neq p_j, \text{ for } i \neq j\}$. Let $\alpha : (L^k(\Sigma, M))^s \to \Sigma^s$ be the projection map and set $L^k_s \equiv \alpha^{-1}(\Sigma^{(s)})$. Finally let $j^k(\varphi)_p \in J(\Sigma, M)$ denote the k-jet of a map $\varphi : U \to M$ at $p \in U$, and define a map

$$j^k_s : \mathcal{L}(\Sigma, M) \to C^\infty(\Sigma^{(s)}, L^k_s(\Sigma, M))$$

by the formula, $j^k_s(\varphi)(p_1, p_2, \dots, p_s) = (j^k(\varphi)_{p_1}, j^k(\varphi)_{p_2}, \dots, j^k(\varphi)_{p_s})$.

Theorem 7.1. *Let $W \subseteq L^k_s(\Sigma, M)$ be a submanifold. Then the set*

$$\mathcal{L}^\infty_W(\Sigma, M) = \{\varphi \in \mathcal{L}^\infty(\Sigma, M) \,|\, j^k_s(\varphi) \text{ is transverse to } W\}$$

is a residual subset of $\mathcal{L}^\infty(\Sigma, M)$.

To prove Theorem 7.1 we construct a large family of perturbations of Legendre immersions, which are parameterized by polynomial maps and are localized on a small neighborhood of a point of Σ. By "large" we mean that each k-jet can be realized as the jet of an element of the family. One corollary of this construction is that $L^k(\Sigma, M)$ is an embedded submanifold of $J^k(\Sigma, M)$.

We begin by forming a class of polynomials which generate diffeomorphisms of $\mathbf{R}^n$ with prescribed k-jets at the origin. Choose $p_0 \in \Sigma$ and let $\chi_1 : U \to \mathbf{R}^n$ be a chart with $\chi_1(p_0) = 0$. Let $\varphi : U \to M$ be a Legendre immersion, let $q_0 = \varphi(p_0)$ and let $\chi_2 : V \to \mathbf{R}^{2n+1}$ be a chart centered at q_0 (i.e. $\chi_2(q_0) = 0$) satisfying the two conditions:

1. $\chi_2^*(\eta_0) = \eta$, where $\eta = du - \sum_{i=1}^{n} y_i dx^i$ (see 3.1)
2. $\iota = \chi_2 \circ \varphi \circ \chi_1^{-1}$ is the map $x \mapsto (x, 0, 0)$.

We choose U and V so that $\chi_1(U) = D_2$ and $\chi_2(V) = D_2 \times D_\varepsilon \times (-\delta, \delta)$, where D_r denotes the open ball of radius r about the origin in $\mathbf{R}^n$ and where δ and ε are positive real numbers.

Let P^k_n, for $k > 0$, be the vector space of polynomial maps from $\mathbf{R}^n$ to $\mathbf{R}^n$ of degree at most k, let P^0_n be the space of translation maps and let P^k be the space of real-valued polynomial functions on $\mathbf{R}^n$ of degree at most k. Let

$$H : P^k_n \times [0,1] \longrightarrow P^k_n$$

be a smooth retraction of P^k_n onto $\mathrm{id}_{\mathbf{R}^n}$ and let $\rho : \mathbf{R}^n \to [0,1]$ be a smooth function with support in D_2 such that $\rho \equiv 1$ on a neighborhood of $\overline{D}_1$. Given a polynomial map, $b \in P^k_n$ define a smooth map $\tau_b : \mathbf{R}^n \to \mathbf{R}^n$ by the formula,

$$\tau_b(x) = H(b, \rho(x))(x)$$

Observe that $\tau_b = \mathrm{id}_{\mathbf{R}^n}$ on $\mathbf{R}^n - \mathrm{supp}(\rho)$ and that $\tau_b = b$ on D_1.

Lemma 7.2. *There is a neighborhood B' of $id_{\mathbf{R}^n} \in P^k_n$ such that τ_b is a diffeomorphism of $\mathbf{R}^n$ onto itself for all $b \in B'$.*

Proof. Since for $b \in P^k_n$ the map τ_b is the identity outside of the support of ρ, by adding the point at infinity we can interpret τ_b as a smooth family of maps of the n-sphere, S^n, onto itself with $\tau_{\mathrm{id}} = \mathrm{id}_{S^n}$. Observe that the map $b \mapsto \tau_b$ is continuous (this follows from the definition of the Whitney C^∞-topology on $C^\infty(S^n, S^n)$). Note that the set of diffeomorphisms of the *compact* manifold S^n is open in $C^\infty(S^n, S^n)$ by Proposition 5.8, page 61 of [GG]. The neighborhood B' is therefore the inverse image of the set of diffeomorphisms of S^n. □

Now let $b \in P^{k+1}$ be a polynomial of degree $k+1$ and let $f_b : \mathbf{R}^n \to \mathbf{R}$ be the function defined by the formula $f_b(x) = \rho(x)b(x)$, where ρ is the bump function defined above. There is an open neighborhood B'' of the zero function in P^k with the property that for all $b \in B''$ the inclusion

$$j(f_b)(\mathbf{R}^n) \subseteq \mathbf{R}^n \times D_\varepsilon \times (-\delta, \delta)$$

holds, where $j(f_b) : \mathbf{R}^n \to \mathbf{R}^{2n+1}$ is the map defined by formula (3.2). Set $B \equiv B' \times B''$ and for $b = (b_1, b_2) \in B$ define a Legendre immersion $\varphi_b : U \to M$ by the formula

$$\varphi_b(p) = \begin{cases} \chi_2^{-1} \circ j(f_{b_2} \circ \tau_{b_1} \circ \chi_1(p) & \text{for } p \in U \\ \varphi(p) & \text{for } p \in U - \chi_1^{-1}(\mathrm{supp}\,\rho). \end{cases} \tag{7.3}$$

Notices that $\varphi_0 = \varphi$. The family φ_b is "large" is the following sense.

Lemma 7.4. *Let $U_1 = \chi_1^{-1}(D_1)$. Then there is a neighborhood $B_0 \subseteq B$ of $(\mathrm{id}, 0) \in P^k_n \times P^{k+1}$ such that the map*

$$\Phi : U_1 \times B_0 \longrightarrow J^k(\Sigma, M)$$

$$(p, b) \mapsto j^k(\varphi_b)_p$$

is a smooth embedding whose image is an open neighborhood of $j^k(\varphi)(U_1)$ in $L^k(\Sigma, M)$. In particular, $L^k(\Sigma, M)$ is an embedded submanifold of $J^k(\Sigma, M)$.

Proof. The map is clearly smooth. We next show that it is an immersion. Work in the local coordinates defined by the maps χ_1 and χ_2. We claim that the derivative Φ_* is injective at $(p, (\mathrm{id}_{\mathbf{R}^n}, 0)) \in U \times B$ for all $p \in \overline{U}_1$. Without loss of generality we need only show that this is the case for $p = 0$.

Choose $x_0 \in D_1$, $b_0 = (b_1, b_2) \in B$ and set

$$I = \varphi_*(x_0, (b_1, b_2) = \frac{d}{dt}_{|t=0} \Phi(t\,x_0, \mathrm{id} + t\,(b_1 - \mathrm{id}), t\,b_2).$$

We will interpret I as an element of $\mathbf{R}^n \times P^k(n, 2n+1)$, where $P^k(n,m)$ denotes the space of polynomials from $\mathbf{R}^n$ to $\mathbf{R}^m$ of degree at most k. (We have made use of the factorization $\mathbf{R}^{2n+1} = \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ and of the identification $T(J^k(\Sigma, M))_{j^k(\varphi)_{p_0}} = R^n \times P^k(n, 2n+1)$.) Suppose that $I = 0$. Then the computation,

$$\begin{aligned} I &= \left.\frac{d}{dt}\right|_{t=0} \{ j^k(\mathrm{id} + t\,(b_1 - \mathrm{id}))_{t x_0}, j^k(d(t\, b_2) \circ (\mathrm{id} + t\,(b_1 - \mathrm{id}))_{t\, x_0}, \\ &\qquad j^k(t\, b_2 \circ (\mathrm{id} + t\,(b_1 - \mathrm{id})))_{t\, x_0} \} \\ &= (x_0, j^k(b_1 - \mathrm{id})_0, j^k(db_2 \circ \mathrm{id})_0, j^k(b_2 \circ \mathrm{id})_0) \end{aligned}$$

shows that $x_0 = 0$, $b_1 = \mathrm{id}$, and $j^k(db_2)_0 = j^k(b_2)_0 = 0$. The last two conditions show that $b_2 = 0$. Hence Φ_* is injective, showing that Φ is an immersion.

By construction there is an inclusion $\Phi(U_1 \times B_0) \subseteq L^k(\Sigma, M)$ and $j^k(\varphi)(U_1)$ lies in the image of Φ. It remains to show that Φ covers an open set in $L^k(\Sigma, M)$ containing $j^k(\varphi)(U_1)$. The result holds for $k = 0$ because $J^0(\Sigma, M) = L^0(\Sigma, M) \equiv \Sigma \times M$.

Assume that $k > 0$ and choose a neighborhood C of $(\mathrm{id}_{\mathbf{R}^n}, 0, 0) \in P^k(n, 2n+1) = P^k_n \times P^k_n \times P^k$ so small that the map

$$\tilde{\chi} : D_1 \times C \to J^k(\Sigma, M) \ : \ (x, b) \mapsto j^k(\chi_2^{-1} \circ b \circ \chi_1)_{\chi_1^{-1}(x)}$$

is a diffeomorphism onto a neighborhood of $j^k(\psi)_p$ in $J^k(\Sigma, M)$. By Lemma 7.2 we can choose C so small that each polynomial $b \in C$ is of the form $(b_1, b_2 \circ b_1, b_3)$ where $b_1 \in B'$. Suppose that $b_1 : D_1 \to D_2$ is a diffeomorphism onto an open set in D_2.

Now suppose that $j^k(\psi)_p$ lies in the image of $\tilde{\chi}$ for $p \in U_1$ and that ψ is a Legendre immersion. We claim that $j^k(\psi)_p$ lies in the image of Φ. Let $\psi' = \chi_2 \circ \psi \circ \chi_1^{-1} : U' \to \mathbf{R}^{2n+1}$ for $U' \subseteq D_1$ a neighborhood of $x_0 = \chi_1(p)$. Then ψ' is of the form $\psi' = j(f) \circ \tau$ where $\tau : U'' \to D_2$ is a diffeomorphism into D_2, U'' is a neighborhood of x_0 and $f : \tau(U'') \to \mathbf{R}$ is a smooth function. Let $b_1' \in B'$ be chosen with $j^k(\tau)_{x_0} = j^k(b_1')_{x_0}$ and let $b_2' \in p^{k+1}$ be such that $j^{k+1}(b_2')_{\tau(x_0)} = j^{k+1}(f)_{\tau(x_0)}$. Now set $b_1 = b_1'$, $b_2 = db_2'$ and let b_3 be the k^{th} order part of b_2'. Then $j^k(\psi)_p = \tilde{\chi}(x_0, b_1, b_2 \circ b_1, b_3)$. By shrinking C we can arrange for (b_1', b_2') to be contained in B_0 showing that a neighborhood of $j^k(\varphi)(U_1)$ in $L^k(\Sigma, M)$ lies in the image of Φ as was to be shown. $\square$

of Theorem 7.1. Consider a point $z = ((j(\psi_1)_{p_1}, j(\psi_2)_{p_2}, \dots, j(\psi_s)_{p_s}) \in W$, where ψ_i, $i = 1, \dots, s$ are Legendre immersions of neighborhoods of p_i, $i = 1, \dots, s$ in Σ. About each point p_i chose a chart $U_i \xrightarrow{\cong} \mathbf{R}^n$ centered at p_i with $U_i \cap U_j = \emptyset$ for $i \neq j$, let $V_i \subset U_i$ be the inverse image of the unit ball in $\mathbf{R}^n$, let $\alpha_i : L^k_s(\Sigma, M) \to \Sigma$ be the i-th component of the projection map $\alpha : L^k_s(\Sigma, M) \to \Sigma^{(s)}$ and let W_z be a neighborhood of z in W with compact closure satisfying the condition $\alpha_i(\overline{W}_z) \subseteq V_i$. Choose a countable subset W_r, $r \in \mathbf{Z}$ covering W and set

$$\mathcal{L}_r \equiv \{ f \in \mathcal{L}^\infty(\Sigma, M) \,|\, j^k_s \text{ is transversal to } W \text{ on } \overline{W}_r \}.$$

We claim that $\mathcal{L}_r$ is open and dense.

Recall that a residual set is the intersection of a countable number of open dense sets and note that

$$\mathcal{L}^\infty_W(\Sigma, M) = \bigcap_{r=1}^\infty \mathcal{L}_r.$$

Since $\overline{W}_r$ is compact, $\mathcal{L}_r$ is open.

To show that $\mathcal{L}_r$ are all dense, we choose a Legendre immersion $\psi : \Sigma \to M$ and show that ψ is a limit point of $\mathcal{L}_r$ for each r. Fix r and recall that there is a point $z \in L^k_s(\Sigma, M)$ with $W_r = W_z$. Let $\rho : \mathbf{R}_n \to [0,1]$ be a function with support contained in D_2 and with $\rho \equiv 1$ on a neighborhood of $\overline{D}_1$. Because $U_i \cap U_j = \emptyset$ for $i \neq j$ we can use the construction in Lemma 7.4 on each of the sets U_i to obtain a smooth map $\Psi : \Sigma \times B \to M$, B a neighborhood of $((\mathrm{id}_{\mathbf{R}^n}, 0), \dots, (\mathrm{id}_{\mathbf{R}^n}, 0))$ in the s-fold product $(P^k_n \times P^{k+1})^s$, such that $\psi_b \equiv \Psi(\ ,b)$ is a Legendre immersion for all $b \in B$ and such that the map

$$\Phi : \begin{cases} \Sigma^{(s)} \times B \to L^k_s(\Sigma, M) \\ (p, b) \mapsto j^k_s(\psi_b)_p \end{cases}$$

is a submersion (and therefore transverse to W on $\overline{W}_r$). It follows from Lemma 3.2 of [M] that there is a dense subset $B' \subseteq B$ with $\psi_b \in \mathcal{L}_r$ for all $b \in B'$. But, $\psi = \psi_{((\mathrm{id}_{\mathbf{R}^n},0),\dots,(\mathrm{id}_{\mathbf{R}^n},0))}$ and hence a limit point of $\mathcal{L}_r$. □

We now give some easy corollaries of the transversality theorem. They are well known results which can also be proved directly using the Darboux Theorem.

Corollary 7.5. *When $\partial\Sigma = \emptyset$, the set of injective, Legendre immersions is a residual set in $\mathcal{L}^\infty(\Sigma, M)$.*

Proof. The equality $L^0(\Sigma, M) = \Sigma \times M$ implies the inequality $L^0_2(\Sigma, M) = \Sigma^{(2)} \times M^2$. Let

$$W = \{(q_1, q_2, p_1, p_2) \mid q_1 \neq q_2\} \subseteq L^0_2(\Sigma, M).$$

Since W is a submanifold of codimension $2n+1$, if ψ is a Legendre immersion with $j^0_2(\psi)$ transversal to W then $j^0_2(\psi)(\Sigma) \cap W = \emptyset$ and ψ is injective. The result now follows from the transversality theorem. □

Theorem 7.6. *Let Σ^n be a compact manifold (possibly with boundary) and let (M^{2n+1}, η) be a contact manifold without boundary. The space of Legendre embeddings of Σ into M is open and dense in $\mathcal{L}^\infty(\Sigma, M)$. Moreover, if $\psi : \Sigma \to M$ is a Legendre immersion then arbitrarily near ψ in $\mathcal{L}^\infty(\Sigma, M)$ are Legendre embeddings which are ℓ-regularly homotopic to ψ.*

Proof. We cannot apply Corollary 7.5 directly because Σ may have a boundary. To get around this problem let Σ' be a collaring of Σ. then by 4.6(2)) the continuous map induced by restriction, $\mathcal{L}^\infty(\Sigma', M) \to \mathcal{L}^\infty(\Sigma, M)$ is surjective. Since injective immersions are dense in $\mathcal{L}^\infty(\Sigma', M)$ they are dense in $\mathcal{L}^\infty(\Sigma, M)$. Since Σ is compact every injective immersion is an embedding. Moreover, the set of embeddings is open in $\mathcal{L}^\infty(\Sigma, M)$ because Σ is compact. □

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