Approximation Properties of

Subdivision Surfaces

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A dissertation submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Washington

2001

Program Authorized to Offer Degree: Mathematics

University of Washington

Graduate School

This is to certify that I have examined this copy of a doctoral dissertation by

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Abstract

Approximation Properties of
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Splines are piecewise polynomial functions defined on a domain in Euclidean space. Because they are easily computed and have high-order approximation power, they are useful for modeling surfaces. Modeling a complex surface with splines typically requires a number of spline patches, which must be smoothly joined, making splines cumbersome to use.

Subdivision schemes generalize splines to domains of arbitrary topology. Thus, subdivision functions can be used to model complex surfaces without the need to join patches. Like splines, subdivision schemes have a multiresolution structure (i.e, a nested sequence of function spaces) associated to subdivisions of the domain. This thesis shows that a particular class of subdivision functions also have high-order approximation power. Although only one subdivision scheme, Loop's, is analyzed, the approach appears to be more general.

The main result is an approximation theorem in Sobolev spaces H^s of functions with square integrable derivatives up to order s. It is shown that each function f in H^r, $r \leq 3$ can be approximated in H^s, $s < r$, with error on the order of $\lambda^{(r-s-\epsilon)k}$, where k is the multiresolution level of the approximation, λ is a number between $1/2$ and $5/8$, associated to the valences of the vertices of the domain complex, and $\epsilon > 0$ is arbitrary.

This approximation theorem provides a theoretical foundation for various applications of subdivision schemes, such as the solution of thin shell problems in elasticity.

TABLE OF CONTENTS

LIST OF FIGURES

iv

ACKNOWLEDGMENTS

I would like to thank Tom Duchamp for his help and encouragement over the years, and for his careful reading of my thesis. His enthusiasm for the field of geometric modeling is contagious. I would also like to thank my wife Tonia Kersulis for her support, encouragement, and sacrifices during my time as a graduate student.

Chapter 1

INTRODUCTION

Geometric modeling is concerned with the mathematical representation of shapes on a computer. Applications include computer-aided design, engineering and manufacturing, as well as computer graphics and animation. A geometric model of a surface is an approximation to some ideal surface. The ideal surface could be a physical object such as a persons hand, a mathematical object such as the solution to a differential equation, or a concept such as an artist's idea for the shape of an animated character.

We consider two traditional types of surface models: a polygonal mesh and a polynomial spline. A polygonal mesh (mathematically a polyhedron) is a simple and flexible type of model, but there are two problems. First, a polygonal mesh is not smooth. A non-smooth model may not be a satisfactory approximation to a smooth surface. Second, polygonal meshes have poor approximation properties (as discussed in Section 1.2). So accurate approximation of a curved surface requires a mesh with many faces.

Modeling with polynomial splines solves the smoothness and approximation problems, at least locally. A class of splines can be chosen that has an arbitrary degree of smoothness, and a correspondingly high-order of approximation. However, spline surfaces are inherently parameterized surface representations, and it is difficult to construct a parameterization for a complicated surface. So polynomial splines are used locally to represent a region of a surface called a "surface patch", and a global surface is typically composed of multiple patches. Figure 1.1 shows an example of a complicated surface modeled with spline patches. The use of patches introduces a new problem. It is difficult to ensure the desired smoothness conditions across adjoining patches.

Subdivision surfaces are a hybrid of polygonal meshes and polynomial splines. They are
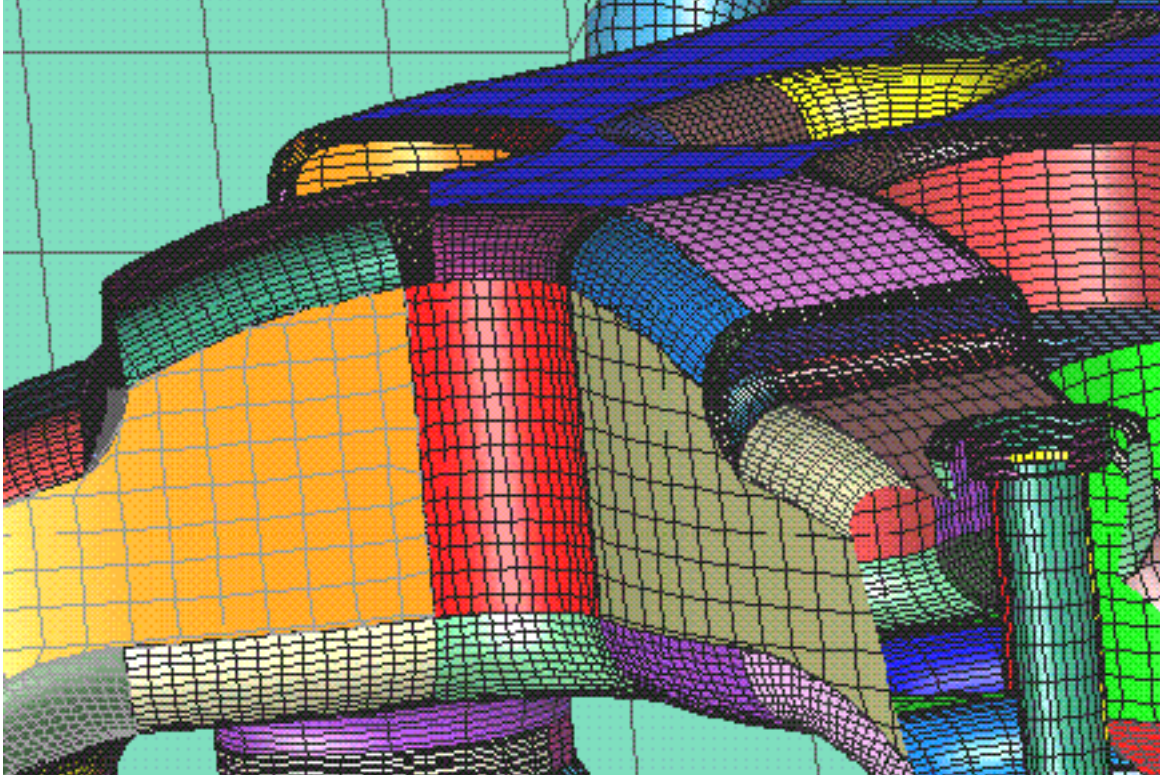
Figure 1.1: Modeling with polynomial spline patches. Each colored patch is a polynomial spline.

flexible, easily allowing for the modeling of surfaces of arbitrary topology. Also subdivision surfaces are smooth. For example, Loop's subdivision scheme (as described in Section 1.3) generates a specific class of subdivision surfaces. Theses surfaces are C^2, except at a finite set of points at which they are C^1. In this thesis we examine the global approximation properties of subdivision surfaces, specifically those generated by Loop's subdivision scheme.

Subdivision schemes such as Loop's are based on subdivision properties of polynomial splines. These subdivision rules are then extended to a more general setting. In the following sections of the Introduction we show how polynomial splines are defined by a subdivision process, introduce Loop's subdivision scheme, and review some approximation results for meshes and splines.

1.1 Subdivision Properties of Splines

A polynomial spline can be represented as the limit of a subdivision process. A uniform cubic spline is a C^2 function that is a piecewise polynomial of degree three, and the polynomial domains are of uniform size. In particular, consider the cubic spline ϕ shown in Figure 1.2(a). The spline ϕ is a polynomial on each interval between integers, it vanishes outside $[-2, 2]$, and is scaled so that $\phi(0) + \phi(-1) + \phi(1) = 1$. The set of integer translates of this function $\{\phi(\cdot - j) : j \in \mathbb{Z}\}$ is a basis for the space of all cubic splines that are polynomial on integer intervals.

In Figure 1.2(b), we show a typical cubic spline that is polynomial on integer intervals and a graph of its coefficients with respect to the basis. Notice that the piecewise linear curve connecting the coefficients roughly approximates the spline. It is called the *control net*.

PSfrag replacements

(a) Basis function

(b) Spline and control net
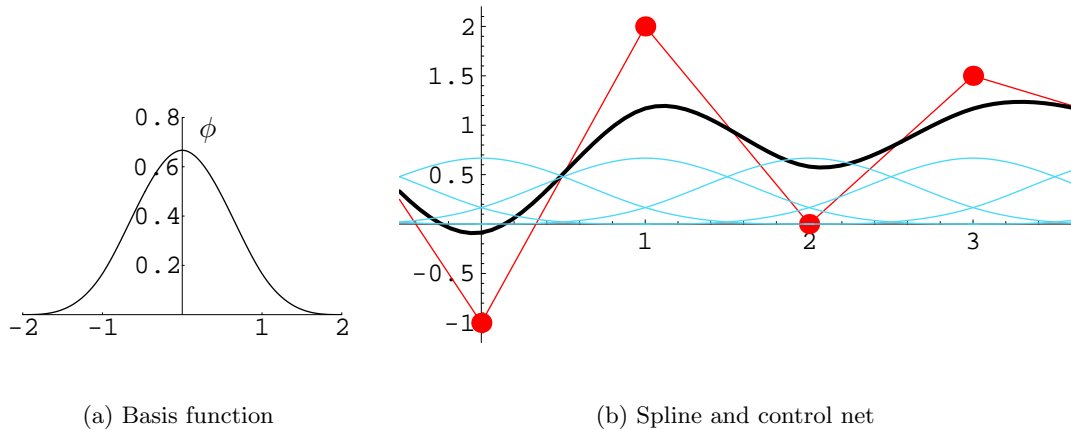
Figure 1.2: (a) Uniform cubic splines are spanned by translates of a single basis function. (b) The translated basis functions, their coefficients, and the resulting spline.

Let $\{\mathcal{S}^k\}$ be a sequence of function spaces defined by the dilates and translates of ϕ. Specifically, let $\mathcal{S}^k$ be spanned by $\{\phi(2^k(\cdot - j)) : j \in 2^{-k}\mathbb{Z}\}$. The space $\mathcal{S}^k$ consists of all cubic splines that are polynomial between consecutive points of $2^{-k}\mathbb{Z}$. Clearly our sequence of spaces is nested with $\mathcal{S}^k \subset \mathcal{S}^{k+1}$. Any spline $f \in \mathcal{S}^k$ is determined by a control net a^k at

level k, that is, $f(x) = \sum a_j^k \phi(2^k(x-j))$ summing over $j \in 2^{-k}\mathbb{Z}$. Since $f \in \mathcal{S}^{k+1}$ it also has a control net a^{k+1} at level $k+1$.

The map taking a^k into a^{k+1} is called the Lane-Riesenfeld subdivision scheme. The algorithm for cubic spline subdivision is as follows: At a vertex of the level k control net, we take a weighted average of this vertex value with the neighbors on either side to produce the corresponding vertex at level $k+1$. The weights used for this average are $\frac{3}{4}$ for the central vertex and $\frac{1}{8}$ for each neighbor, as represented by the vertex mask shown in Figure 1.3(a). Also, we average adjacent vertices of the level k control net to get the new midpoint vertices of the level $k+1$ control net. This averaging is represented by the edge mask shown in Figure 1.3(b). Figure 1.3(c) shows an example of the subdivision procedure. The successively refined control nets converge in the limit to the spline function. We can also form parameterized curves in $\mathbb{R}^3$ by taking the spline coefficients to be in $\mathbb{R}^3$.

1/8   3/4   1/8

(a) Vertex mask

1/2   1/2

(b) Edge mask

(c) Refined control nets
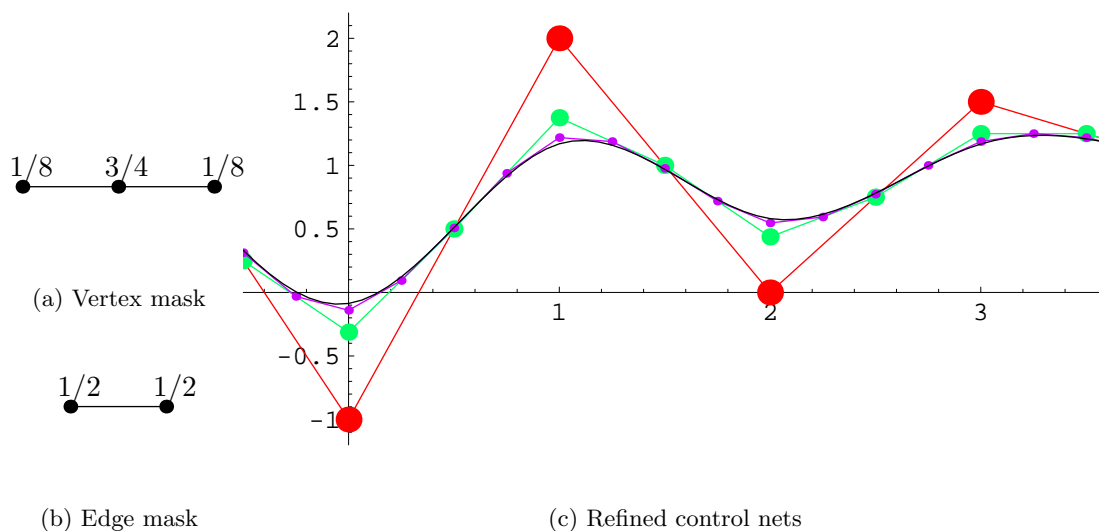
Figure 1.3: Refined control nets converge to the spline.

Quartic triangular splines are bivariate polynomial splines. They are C^2 functions on $\mathbb{R}^2$ that are piecewise degree-four polynomials. The pieces are defined by a regular triangular grid as in Figure 1.4(a). As in the case of the cubic splines, the spline space we are interested
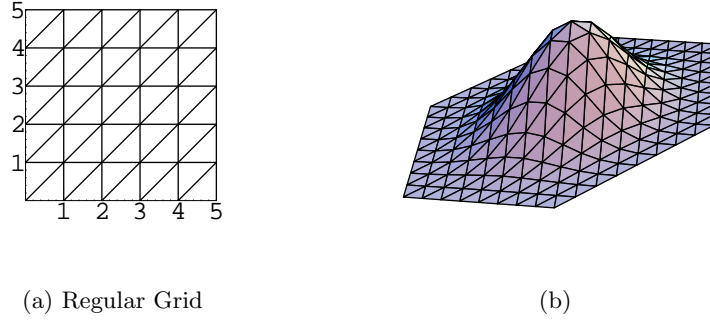
(a) Regular Grid   (b)

Figure 1.4: (a) A regular triangular grid in the plane. (b) The quartic triangular spline basis function.

in is spanned by translates of a single basis function ϕ, which is shown in Figure 1.4(b). We define nested spaces of quartic triangular splines by

$$\mathcal{S}^k = \operatorname{span}\left\{\phi\left(2^k(\cdot - j)\right) : j \in 2^{-k}\mathbb{Z}^2\right\}. \tag{1.1}$$

The control net for a quartic spline $f(x) = \sum a_j \phi(2^k(x-j)) \in \mathcal{S}^k$ at level k is the coefficient map $j \mapsto a_j$ for $j \in 2^{-k}\mathbb{Z}^2$. The subdivision procedure, transforming a control net at level k into a control net at level $k+1$, is given by a vertex mask and an edge mask as shown in Figures 1.5(a) and 1.5(b).

1.2   Approximation Results for Meshes and Splines

We now review the approximation power of mesh and spline geometric modeling techniques. A surface can locally be parameterized so the problem of locally approximating a surface can be cast as a problem of approximating a function, namely, the parameterization. Given a function f to be approximated and a space $\mathcal{S}$ of functions from which to choose an approximation, we seek a bound on the minimum approximation error

$$\operatorname{dist}(f, \mathcal{S}) = \min_{g \in \mathcal{S}} \|f - g\|. \tag{1.2}$$

(a) Valence 6 Vertex.   (b) Edge.   (c) Extraordinary vertex.
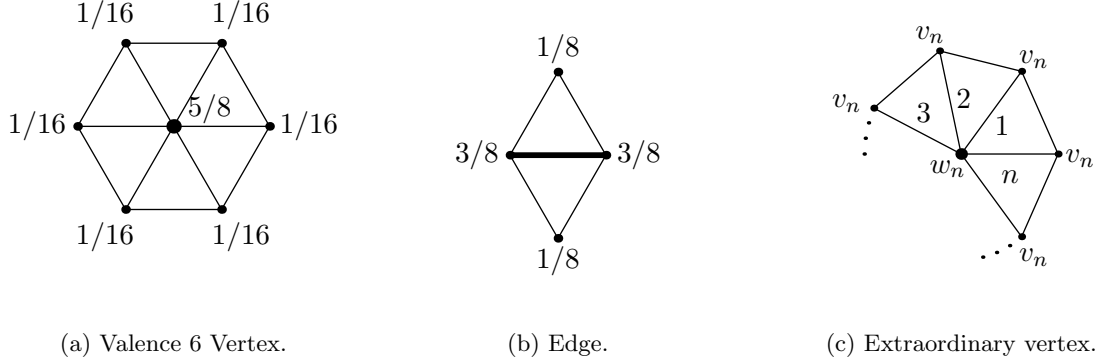
Figure 1.5: (a)-(b) Subdivision masks for quartic triangular splines. (c) For Loop's subdivision we add the extraordinary vertex masks.

We will use L^2-Sobolev norms to measure the approximation error. Recall, given a function $f : \Omega \to \mathbb{R}$ on a domain in $\mathbb{R}^2$, the H^s-Sobolev norm of f is defined by

$$\|f\|_{H^s(\Omega)} = \left(\sum_{|\alpha| \leq s} \|D^\alpha f\|_{L^2(\Omega)}^2 \right)^{1/2},$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index of non-negative integers, $|\alpha| = \alpha_1 + \alpha_2$, and $D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$. The Sobolev norm is the natural norm for the analysis of the finite element method. In Section 6.3 we apply our approximation result to analyze the finite element method using subdivision functions.

To analyze local approximation by a triangular mesh, we view the surface locally as a graph, a special form of a parameterized surface. Specifically, let $p \in M$ be a point on a smooth surface in $\mathbb{R}^3$, and let $T_p M$ be the tangent plane to M at p. By an orthogonal change of coordinates, e.g., a rotation, we can consider $T_p M$ to be the xy-plane. There is a neighborhood $N \subset M$ containing p such that N is the graph of a function $z = f(x, y)$ on a neighborhood of p in the xy-plane. Now suppose K is a triangular mesh that is a sufficiently good approximation to M. Then on a neighborhood of p on the xy-plane, the triangular mesh K is the graph of a piecewise linear function $z = \widetilde{f}(x, y)$. So we can measure the local approximation error of K by the norm $\|f - \widetilde{f}\|$. Conversely, the graph of any piecewise linear function defined near p on $T_p M$ is a local triangular mesh. So we have reduced the

problem of locally approximating a surface by a triangular mesh to a problem of function approximation by a piecewise linear function.

We review approximation results for piecewise linear functions. Given a triangulation K of a polygonal region $\Omega \subset \mathbb{R}^2$, let $\mathsf{PL}(K)$ denote the space of piecewise linear functions on Ω that are linear on each face of K. We define the chunkiness parameter $\gamma(T)$ of a triangle T to be the ratio of the radius of the inscribed ball of T to the radius of the circumscribed ball of T. The following approximation theorem for piecewise linear functions can be found, for example, in [3].

Theorem 1. *Given a maximum chunkiness Γ and a polygonal domain $\Omega \subset \mathbb{R}^2$, there is a constant $C = C(\Omega, \Gamma)$, such that for any triangulation K of Ω satisfying $\gamma(T) < \Gamma$ for each triangle T in K, we have the following bound on the approximation error for any function $f \in H^2(\Omega)$:*

$$\mathrm{dist}(f, \mathsf{PL}(K))_{H^s(\Omega)} \leq C \max_{T \in K} \mathrm{diam}(T)^{2-s} \|f\|_{H^2(\Omega)} \tag{1.3}$$

for $s = 0$ or 1.

Next we consider local approximation of a surface by quartic triangular splines. We assume the surface is locally parameterized, either as a graph as above or by some other parameterization. Thus, again, local surface approximation is reduced to function approximation. Approximation theorems for splines were first proved by deBoor and Fix [5], by constructing a quasi-interpolant. Approximation results in Sobolev spaces for box splines, of which quartic triangular splines are an example, were proved by Kowalski [12]. In particular, using the nested spaces of quartic triangular splines in (1.1) we have

$$\mathrm{dist}(f, \mathcal{S}^k)_{H^s} \leq C(\frac{1}{2})^{(4-s)k} \|f\|_{H^4} \qquad \text{for } s = 0, \ldots, 3, \tag{1.4}$$

where the constant C is independent of both f and k.

From (1.3) we say that the approximation power of piecewise linear functions is order 2, while from (1.4) the approximation power of quartic splines is order 4.

1.3 Loop's Subdivision Scheme

We now define Loop's subdivision scheme. Loop's subdivision functions are a generalization of quartic triangular splines, in which the domain can be any triangulated surface instead of just a regular triangulated domain of the plane. Given a triangulated surface K a control net on K is an assignment of a control point in $\mathbb{R}^m$ to each vertex of K. A subdivision function is defined as the limit of a sequence of control nets. A subdivision function is determined by two pieces of information: (i) a triangulated surface K^0, which is the domain and (ii) an initial control net on K^0. In the most common situation K^0 is a triangulated surface in $\mathbb{R}^3$ and also acts as the initial control net with control points in $\mathbb{R}^3$. In this case the image of the subdivision function into $\mathbb{R}^3$ is a surface called a *subdivision surface*.

At each level k of subdivision we construct a refined triangulation of K^0, denoted K^k, by introducing a vertex at the midpoint of each edge in K^{k-1} and splitting each triangle of K^{k-1} into 4 sub-triangles. The control net at level k is a control net on K^k. Each control point at level k is computed by averaging the neighbors of the level $k-1$ control net. The vertex and edge masks used for quartic triangular splines are also used for Loop's subdivision scheme, and in addition there is a vertex mask for *extraordinary vertices*, those with valence other than 6. The vertex and edge masks are shown in Figure 1.5. The formula for the weights w_n and v_n for extraordinary vertices of valence n can be found in Section 2.1.4.

Figure 1.6 shows an example of (a) an initial control net, which acts also as the domain K^0, (b)-(c) the control net after one and two levels of subdivision, and (d) the limit surface, that is, the image of the subdivision function.

The subdivision procedure converges to a continuous limit function. The process is local, in the sense that, the limit function restricted to a triangle of K^k only depends on the control net in a nearby neighborhood. So, if this neighborhood has the connectivity of a regular grid, then the subdivision function will be a quartic polynomial on that triangle. Since all the vertices introduced by subdivision have valence 6, any point, other than an extraordinary vertex, is eventually in a regular neighborhood. Therefore, other than at an extraordinary vertex, every point has a neighborhood on which Loop's subdivision functions are quartic splines.

(a) Initial Mesh   (b) Level 1   (c) Level 2   (d) Limit Surface
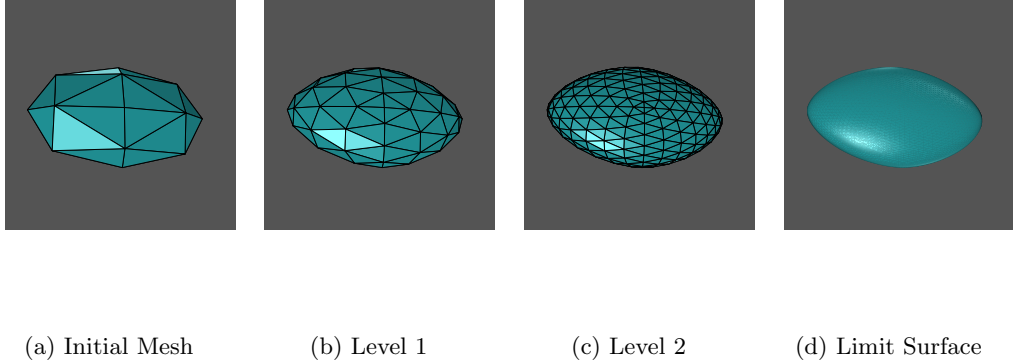
Figure 1.6: Loop's Subdivision procedure.

In this thesis we extend the approximation result for quartic triangular splines to Loop's subdivision functions. Although complete results are presented only for Loop's subdivision scheme, much of the theory is applicable to other stationary subdivision schemes that generate polynomial splines on regular grids. Denis Zorin [24] has reported similar results.

The main approximation theorem we prove for Loop's subdivision functions is analogous to (1.4). For any triangulated surface K, let $\mathcal{S}(K)$ denote the space of Loop's subdivision functions on K that are determined by some control net on K. Since K^k is a refinement of the triangulated surface K, we have a nested sequence of function spaces $\mathcal{S}(K^k) \subset \mathcal{S}(K^{k+1})$. In our theory, the role of the base $\frac{1}{2}$ which appears in (1.4) is played by the subdominant eigenvalues of the subdivision matrix, defined in Section 2.4. Let $\lambda_{\max} = \lambda_{\max}(K)$ be the largest subdominant eigenvalue of a subdivision matrix whose valence is represented by a vertex of K or $\frac{1}{2}$ whichever is greater.

Theorem 2. *Let $\mathcal{S}(K^k)$ be the space of Loop's subdivision functions on the k-times subdivided complex K^k. For integers $0 \leq s < r \leq 3$ and any $\epsilon > 0$ we have the following bound on the minimal $H^s(K)$-approximation error of a function $f \in H^r(K)$:*

$$\operatorname{dist}(f, \mathcal{S}(K^k))_{H^s(K)} \leq C_\epsilon \lambda_{\max}^{(r-s-\epsilon)k} \|f\|_{H^r(K)} \,,$$

where the constant $C_\epsilon = C(\epsilon, K)$ is independent of k and f.

Cases of particular interest follow:

Corollary 3. *Given a smooth function $f \in H^3(K)$ we have*

$$\text{dist}(f, \mathcal{S}(K^k))_{L^2(K)} = O(\lambda^{(3-\epsilon)k}) \tag{1.5}$$

for any $\epsilon > 0$, where $\lambda = \lambda_{\max}(K)$.

Corollary 4. *The space of Loop's subdivision functions at all levels $\bigcup_0^\infty \mathcal{S}(K^k)$ is dense in $H^2(K)$.*

Theorem 2 shows that the approximation power of Loop's subdivision functions is order 3. The behavior of the subdivision functions near the extraordinary vertex causes the deterioration of the approximation order.

An overview of the thesis is as follows. In Chapter 2 we review the fundamentals of subdivision surfaces and define differentiation on a simplicial surface. In Chapter 3 we show how the main theorem, Theorem 2, can be reduced to an approximation theorem on a neighborhood of an extraordinary vertex. We develop a general theory of approximation on these neighborhoods in Chapter 4 . The theory is based on a generalization of the idea of a quasi-interpolant. In Chapter 5 we construct a quasi-interpolant for Loop's scheme, completing the proof of the main theorem. Finally, in Chapter 6, we provide a summary of the thesis, consider possible extensions this result, and discuss applications to the finite element method.

Chapter 2

FUNDAMENTALS OF SUBDIVISION SURFACES

In this chapter we describe the fundamentals of subdivision surfaces. First we discuss triangulated surfaces in terms of the abstraction of a simplicial complex. The domains of the control nets for a subdivision surface and the domains of the subdivision function are both defined in terms of simplicial complexes. We define stationary subdivision schemes. Loop's scheme and its generalizations are particular examples. We define subdivision functions as the limit of the subdivision operation. Then we introduce the subdivision map and the characteristic map, which are key concepts for analyzing a subdivision surface near an extraordinary vertex.

To make sense of the main theorem we need a differentiable structure on a triangulated surface. We do this by introducing two atlases of coordinate charts on a triangulated surface. We introduce an atlas of affine coordinates, covering a triangulated surface except at the extraordinary vertices, and we introduce an atlas of characteristic coordinate charts which cover the entire triangulated surface.

2.1 Stationary Subdivision on Simplicial Complexes

2.1.1 Simplicial Surfaces

We review the necessary theory of simplicial complexes. Most of this material is based on Spanier [19]. A simplicial complex K is a set $\mathsf{Vertex}(K)$ called the *vertices* and a collection of finite, nonempty subsets of $\mathsf{Vertex}(K)$ called *simplices*, such that, (i) every set consisting of exactly one vertex is a simplex of K, and (ii) every non-empty subset of a simplex of K is also a simplex of K. A simplex s of K containing exactly $q+1$ vertices is called a *q-simplex*. Since the vertices of K can be identified with the 0-simplices of K, we consider K to be just the collection of simplices. The *star* of a simplex s is the collection of all simplices $\sigma \in K$

such that $s \subset \sigma$. We assume that all of our simplicial complexes are *star-finite*, that is, the star of each simplex is finite.

A *simplicial map* $\phi : K_1 \to K_2$ is a map $\phi : \mathsf{Vertex}(K_1) \to \mathsf{Vertex}(K_2)$ such that for every simplex $s \in K_1$, the image $\phi(s)$ is a simplex in K_2. If ϕ is one-to-one, then the map is an *embedding*, and if additionally it is onto $\mathsf{Vertex}(K_2)$ and ϕ^{-1} is also a simplicial map, then ϕ is a simplicial *isomorphism*.

A *subcomplex* L of a simplicial complex K, denoted $L \subset K$, is a subset of K that is a simplicial complex. Given any simplex $s \in K$, let $\bar{s} \subset K$ denote the subcomplex containing s and all of its subsets. For any subcomplex $L \subset K$ we define a subcomplex, called the *1-neighborhood* of L in K, denoted $N_1(L, K)$, by

$$N_1(L, K) = \bigcup \bar{s}, \qquad \text{where } s \in K \text{ and } \bar{s} \cap L \neq \emptyset.$$

Successively larger neighborhoods are defined recursively by $N_j(L, K) = N_1(N_{j-1}(L, K))$. If the ambient complex K is clear from context we simplify the notation to $N_1(L)$. For any subcomplex $L \subset K$ we define a subcomplex $K \setminus L$ by

$$K \setminus L = \bigcup \bar{s}, \qquad \text{where } s \in K \text{ and } s \notin L. \tag{2.1}$$

Notice, this is not the same as the set difference $\{s \in K : s \notin L\}$, which is in general is not a subcomplex.

Given a simplicial complex K, the *topological realization* of K, denoted $|K|$, is the set of all functions $\alpha : \mathsf{Vertex}(K) \to [0, 1]$ such that (i) the set of vertices $\{v \in K : \alpha_v \neq 0\}$ is a simplex of K, and (ii) $\sum \alpha_v = 1$. The topology of $|K|$ is given by the metric

$$d(\alpha, \beta) = \left(\sum_v (\alpha_v - \beta_v)^2 \right)^{1/2} \qquad \text{for } \alpha, \beta \in |K|.$$

For a simplex $s \in K$, the *closed simplex* $|s| \subset |K|$ is given by

$$|s| = \{\alpha \in |K| : \alpha_v = 0 \text{ for all } v \notin s\}.$$

If s is a q-simplex, then $|s|$ is homeomorphic (and in fact isometric) to the standard q-simplex $\{x \in \mathbb{R}^{q+1} : 0 \leq x_i \leq 1, \text{and } \sum x_i = 1\}$. We identify each vertex $v \in K$ with its

characteristic function $v \in |K|$, or more precisely, v_w is one if $w = v$ and otherwise zero. Then any point $\alpha \in |K|$ can be represented as a sum

$$\alpha = \sum_{v \in K} \alpha_v v\,,$$

where the sum is a convex combination of vertices in a simplex of K.

Next we define piecewise linear maps on a simplicial complex. Let X be either $\mathbb{R}^n$ for some n, or the topological realization of some complex. A *piecewise linear map $f : |K| \to X$* is given by

$$f(\alpha) = \sum_{v \in K} \alpha_v f(v) \qquad \text{for all } \alpha \in |K|.$$

For example, a simplicial map $\phi : K_1 \to K_2$ induces a piecewise linear map $|\phi| : |K_1| \to |K_2|$ given by

$$|\phi|(\alpha) = \sum_{v \in K} \alpha_v \phi(v).$$

An injective piecewise linear map from $|K|$ into $\mathbb{R}^n$ for some n is called a *geometric realization* of K.

Notice that a piecewise linear map is completely determined by its values on $\mathsf{Vertex}(K)$. Both piecewise linear functions and subdivision functions are determined by functions on $\mathsf{Vertex}(K)$, which we call *control nets*. The space of all control nets on K is denoted by

$$\mathsf{CN}(K) = \{u : \mathsf{Vertex}(K) \to \mathbb{R}\}\,.$$

A simplicial map $\phi : K_1 \to K_2$ induces a *pull-back $\phi^* : \mathsf{CN}(K_2) \to \mathsf{CN}(K_1)$* by $(\phi^*u)_v = u_{\phi(v)}$. The pull-back is used more generally for functions on arbitrary spaces.

Now, we focus on 2-dimensional simplicial complexes. A *simplicial surface* is a complex whose topological realization is a surface, possibly with boundary. A *simplicial sub-surface* is a subcomplex of a simplicial surface that is a simplicial surface itself. On a simplicial surface we call the 1-simplices *edges* and the 2-simplices *faces*. The collections of all these simplices on a given simplicial surface K are denoted by $\mathsf{Edge}(K)$ and $\mathsf{Face}(K)$. The *valence* of a vertex is the number of edges incident to the vertex. On a simplicial surface K without boundary the valence of every vertex is at least three, and any two vertices of the same valence have isomorphic stars.
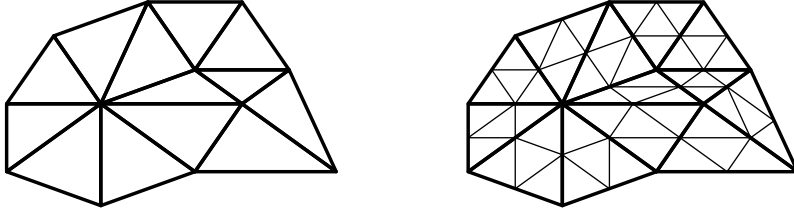
Figure 2.1: A simplicial surface K and its subdivided complex $D(K)$.

A *subdivision* of a complex K is a simplicial complex K' such that: (i) the vertices of K' are points of $|K|$, (ii) if s' is a simplex of K' there is a simplex s of K such that $s' \subset |s|$ (that is, s' is a finite subset of $|s|$), and (iii) the piecewise linear map $\iota : |K'| \to |K|$ induced by the identification of vertices in K' with points in $|K|$ is a homeomorphism. If K' is a subdivision of K, we identify $|K'|$ with $|K|$ by the piecewise linear homeomorphism ι in condition (iii). So if K'' is a subdivision of K', then K'' is also a subdivision of K.

For a simplicial surface K we define the "4 to 1" subdivision of K, denoted by $D(K)$. The vertices of $D(K)$ are of two types. First, for each vertex $v \in K$ we define $D(v) = v \in |K|$ to be a vertex of $D(K)$. We call these *V-type* vertices. Secondly, for each edge $e \in K$, let $D(e) \in |K|$ be the midpoint of $|e|$. Then let $D(e)$ be a vertex of $D(K)$, which we call an *E-type* vertex. So we have

$$\mathsf{Vertex}(D(K)) = D(\mathsf{Vertex}(K)) \cup D(\mathsf{Edge}(K)) \,.$$

Each face of K is split into four faces as shown in Figure 2.1. It is easy to check that $D(K)$ is a subdivision of K.

By repeating the process, we obtain a sequence of subdivided complexes $K = K^0, K^1,$ $K^2, \ldots$, where $K^{k+1} = D(K^k)$. For each subcomplex $L \subset K$ and $k \geq 0$ we have that L^k is a subcomplex of K^k, i.e., $L^k \subset K^k$. More generally, if $\rho : K_1 \to K_2$ is a simplicial map between simplicial surfaces then there is an induced map $\rho : K_1^k \to K_2^k$.

2.1.2 Subdivision Schemes

A *subdivision scheme* is a linear map $\mathcal{S} : \mathsf{CN}(K) \to \mathsf{CN}(D(K))$ for each simplicial surface K without boundary. Since $\mathcal{S}$ is linear we can represent it on a fixed complex K by $a :$

$D(K) \times K \to \mathbb{R}$, where

$$(\mathcal{S}u)_v = \sum_{w \in K} a_{vw} u_w \qquad \text{for all } v \in D(K). \qquad (2.2)$$

Here, and often in what follows, we abuse notation and write $v \in D(K)$ as shorthand for $v \in \mathsf{Vertex}(D(K))$ when the intended meaning is clear from context. Given $u \in \mathsf{CN}(K)$, we form a sequence $u = u^0, u^1, u^2, \ldots$ of control nets on refined complexes by $u^j = \mathcal{S}^j u \in \mathsf{CN}(K^j)$.

Let K be a simplicial surface without boundary. For any integer $m > 0$ and any vertex v in the subdivided complex $D(K)$, we define a subcomplex $U_m(v, K) \subset K$ of the undivided complex K, called the *undivided* $m-neighborhood$, by

$$U_m(v, K) = \left\{ s \in K : |s| \subset |N_m(v, D(K))| \right\}. \qquad (2.3)$$

Figure 2.2 shows some typical undivided m-neighborhoods for the two different types of vertices and various values of m.

Definition 5. *We say a subdivision scheme is* **local** *if there is an integer $m_w > 0$ called the* **mask width** *such that for any pair K and $\widetilde{K}$ of simplicial surfaces without boundary, any vertex $v \in D(K)$, and any embedding $\rho : U_{m_w}(v, K) \to \widetilde{K}$, we have*

$$a_{vw} = \begin{cases} a_{\rho(v)\rho(w)} & \text{for all } w \in U_{m_w}(v, K), \\ 0 & \text{otherwise .} \end{cases}$$

A local subdivision scheme of mask width m_w is given by a collection of *vertex masks* and *edge masks*. These are small simplicial surfaces with weights assigned to the vertices. To give a complete definition of a local subdivision scheme with mask width m_w, there must be a mask isomorphic to any possible undivided m_w-neighborhood $U_{m_w}(v, K)$, as in the examples of Figure 2.2.

Definition 6. *A subdivision scheme is* **affine** *if for every complex K and vertex $v \in D(K)$ we have $\sum_{w \in K} a_{vw} = 1$. A local and affine subdivision scheme is called* **stationary**.

For any subcomplex $L \subset K$, we call the set $N_{m_w-1}(L)$ the *control set* of L since, as the next proposition shows, for any control net $u \in \mathsf{CN}(K)$, it is only u on the control set

m	V − type		E − type	
1	•		──	
2				
3				
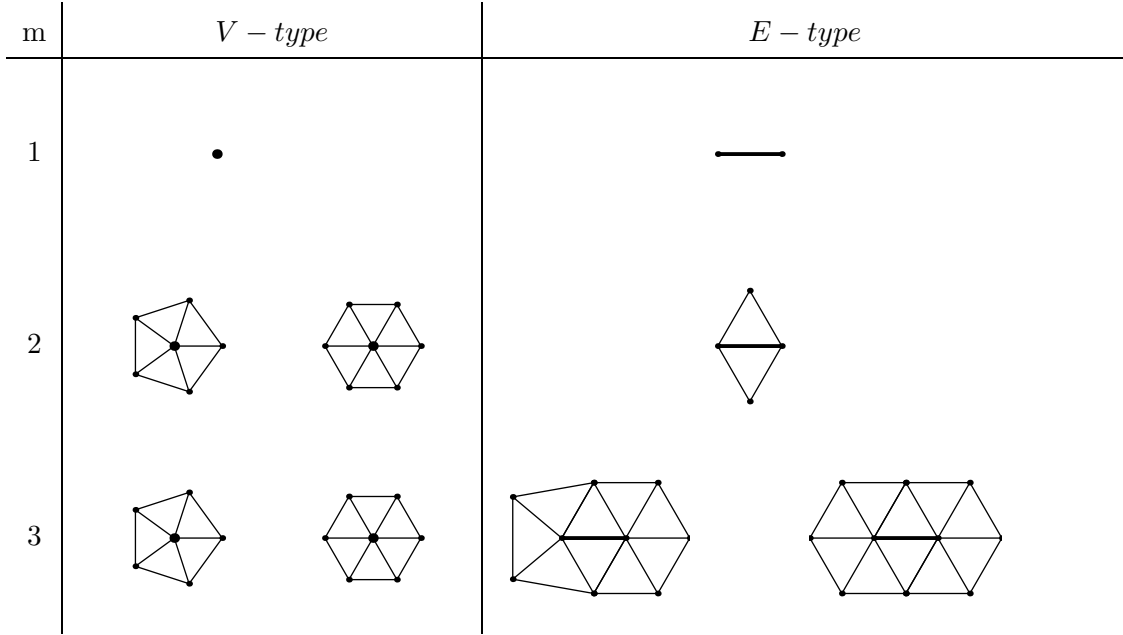
Figure 2.2: Some typical undivided m-neighborhoods $U_m(v, K)$ are shown for the two types of vertices v in the subdivided complex $D(K)$.

of L which affects the subdivided control nets $u^j = \mathcal{S}^j u$ on L^j. Moreover, the subdivided control nets u^j on L^j are independent of the complex K outside of $N_{m_w-1}(L)$. We state this formally by considering a pair of simplicial surfaces K and $\widetilde{K}$, where $N_{m_w-1}(L)$ is isomorphic to a subcomplex in $\widetilde{K}$.

Proposition 7. *Suppose the subdivision scheme $\mathcal{S}$ is local with mask width m_w. Let K and $\widetilde{K}$ be a pair of simplicial surfaces without boundaries, L a subcomplex of K, and ρ an embedding $\rho : N_{m_w-1}(L) \to \widetilde{K}$. Then for each $u \in \mathsf{CN}(K)$ and $\widetilde{u} \in \mathsf{CN}(\widetilde{K})$ such that $u_v = \widetilde{u}_{\rho(v)}$ for each $v \in N_{m_w-1}(L)$, we have*

$$u_v^j = \widetilde{u}_{\rho(v)}^j \qquad \text{for all } v \in L^j \text{ and } j \geq 0,$$

where $u^j = \mathcal{S}^j u$ and $\tilde{u}^j = \mathcal{S}^j \tilde{u}$.

Proof. We prove the equality on a larger domain

$$u_v^j = \widetilde{u}_{\rho(v)}^j \qquad \text{for all } v \in N_{m_w-1}(L^j) \text{ and } j \geq 0. \qquad (2.4)$$

The proof proceeds by induction. The base step is given in the hypothesis. To prove the inductive step, suppose (2.4) holds for some fixed j. Let $v \in N_{m_w-1}(L^{j+1})$, then by (2.2) we have

$$u_v^{j+1} = \sum_{w \in K^j} a_{vw} u_w^j. \tag{2.5}$$

By locality, the sum need only be taken over the vertices $w \in U_{m_w}(v, K^j)$. Since $v \in N_{m_w-1}(L^{j+1})$ and $w \in N_{m_w}(v, K^{j+1})$ by (2.3) we conclude $w \in N_{2m_w-1}(L^{j+1})$, but since $w \in \mathsf{Vertex}(K^j)$ we get a stronger result, namely $w \in N_{m_w-1}(L^j)$. So we have shown that the stencil $U_{m_w}(v, K^j)$ is contained in $N_{m_w-1}(L^j)$, and hence there is an isomorphic stencil $U_{m_w}(\rho(v), \widetilde{K}^j)$. So by locality, the masks are equivalent $a_{vw} = a_{\rho(v)\rho(w)}$. Also the inductive hypotheses shows $u_w^j = \widetilde{u}_{\rho(w)}^j$, so (2.5) gives us

$$u_v^{j+1} = \sum_{U_{m_w}(v, K^j)} a_{vw} \widetilde{u}_{\rho(w)}^j = \sum_{U_{m_w}(\rho(v), \widetilde{K}^j)} a_{\rho(v)\widetilde{w}} \widetilde{u}_{\widetilde{w}}^j = \widetilde{u}_{\rho(v)}^{j+1}.$$

$\square$

2.1.3 Subdivision Functions

A subdivision scheme is said to be *convergent* if for any simplicial surface K and control net $u \in \mathsf{CN}(K)$ there exists a continuous function $\mathcal{S}^\infty u \in C(|K|)$ such that

$$\sup_{v \in K^j} |(\mathcal{S}^j u)_v - \mathcal{S}^\infty u(v)| \longrightarrow 0, \quad \text{as } j \to \infty.$$

We call the limit function $\mathcal{S}^\infty u$ a *subdivision function*. The subdivision limit operator $\mathcal{S}^\infty : \mathsf{CN}(K) \to C(|K|)$ is linear since $\mathcal{S}$ is linear. For a given complex K, let $\mathcal{S}(K) = \mathcal{S}^\infty(\mathsf{CN}(K))$ be the space of subdivision functions on K. A natural basis of $\mathsf{CN}(K)$ is $\{\delta_v : v \in \mathsf{Vertex}(K)\}$, where δ_v is one at v and zero at all other vertices. Then the *subdivision basis functions* $\phi_v = \mathcal{S}^\infty \delta_v$ for $v \in \mathsf{Vertex}(K)$ form a basis for $\mathcal{S}(K)$.

Proposition 7 can immediately be extended to apply to subdivision functions, and we also give the following corollary.

Proposition 8. *Suppose $\mathcal{S}$ is a convergent, local subdivision scheme with mask width m_w. Let K and $\widetilde{K}$ be a pair of simplicial surfaces without boundaries, L a subcomplex of K, and*

ρ an embedding $\rho : N_{m_w-1}(L) \to \widetilde{K}$. Then for each $\widetilde{u} \in \mathsf{CN}(\widetilde{K})$ we have

$$\mathcal{S}^\infty(\rho^*\widetilde{u}) = \rho^*\mathcal{S}^\infty\widetilde{u} \ \text{on } |L|.$$

Corollary 9. *Let $\mathcal{S}$ be a convergent, local subdivision scheme with mask width m_w. Then for any simplicial surface K and any vertex $v \in K$, the support of the basis function ϕ_v is contained in the simplicial neighborhood $|N_{m_w}(v, K)|$.*

Of course we may apply the subdivision process starting on a subdivided complex. Let K be a simplicial surface and $j \geq 0$, then for any $u \in \mathsf{CN}(K^j)$ we get a subdivision function $\mathcal{S}^\infty u \in \mathcal{S}(K^j) \subset C(|K|)$. We denote a basis function at the j-th level of refinement by ϕ_v^j. Namely, for any $v \in \mathsf{Vertex}(K^j)$, let $\phi_v^j = \mathcal{S}^\infty\delta_v^j$, where $\delta_v^j \in \mathsf{CN}(K^j)$ is one at v and zero otherwise. Then for any $u \in \mathsf{CN}(K)$ and $j \geq 0$ we have

$$\mathcal{S}^\infty u = \sum_{v \in K} u_v \phi_v = \sum_{w \in K^j} (\mathcal{S}^j u)_w \phi_w^j. \tag{2.6}$$

So clearly, $\mathcal{S}(K) \subset \mathcal{S}(K^j)$.

Applying formula (2.6) with $u = \delta_v$ gives a so-called refinement relation for each basis function ϕ_v. More precisely, a *refinement relation* is a finite expansion of the form

$$\phi_v = \sum_{w \in K^{j-1}} a_w \phi_w^j.$$

2.1.4 Examples of Subdivision Schemes

Example: Piecewise Linear Subdivision. The simplest stationary subdivision scheme has mask width 1. There is a single vertex mask and a single edge mask as shown in the $m = 1$ line of Figure 2.2. The single vertex in the vertex mask has weight 1, and the weights of the edge mask are 1/2 at each vertex. It is easy to see that, given a control net $u \in \mathsf{CN}(K)$, the scheme converges to the piecewise linear function determined by the control net.

Example: Loop's Subdivision Scheme. Here we have a mask width of 2. There is a vertex mask for each valence, and an edge mask. These masks are shown in Figure 1.5, where the weights on the vertex masks are given by w_n for the central vertex and v_n for the adjacent vertices,

$$w_n = \frac{3}{8} + \frac{\left(3 + 2\cos(\frac{2\pi}{n})\right)^2}{64} \tag{2.7}$$

and

$$v_n = \frac{1 - w_n}{n} \, . \tag{2.8}$$

As described in the Introduction, this scheme converges to quartic triangular splines on a regular grid. More generally, Loop's scheme is convergent.

Example: Generalized Loop's Subdivision Scheme. This is a family of stationary subdivision schemes determined by three parameters. The mask width is 3. As in the case of Loop's scheme, there is a vertex mask for each valence. The mask weight for the central vertex of valence n is given by the parameter w_n. The weight assigned to the adjacent vertices in the vertex mask is given by (2.8), to ensure the scheme is affine. For a mask width of 3, an undivided neighborhood for E-type vertex $D(e)$ is determined by the valences n_1 and n_2 of the two vertices in e. To keep the edge masks simple we use the 2 masks shown in Figure 2.3. The mask in Figure 2.3(a) is used when exactly one of the vertices has valence 6 and the other vertex has valence $n \neq 6$. The mask weights a_n and b_n are parameters, and $c_n = (1 - a_n - b_n)/2$ to ensure that the scheme is affine. The edge mask in Figure 2.3(b) is used for all the other edges.
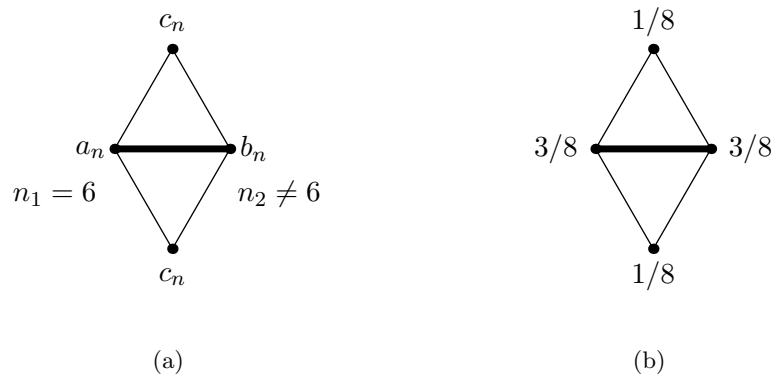
Figure 2.3: Edge masks for generalized Loop's subdivision scheme. The edge mask selection depends on the valences n_1 and n_2 of the incident vertices. Mask (a) is used when exactly one of these vertices is extraordinary, mask (b) is used otherwise.

| (a) | (b) | (c) | (d) |
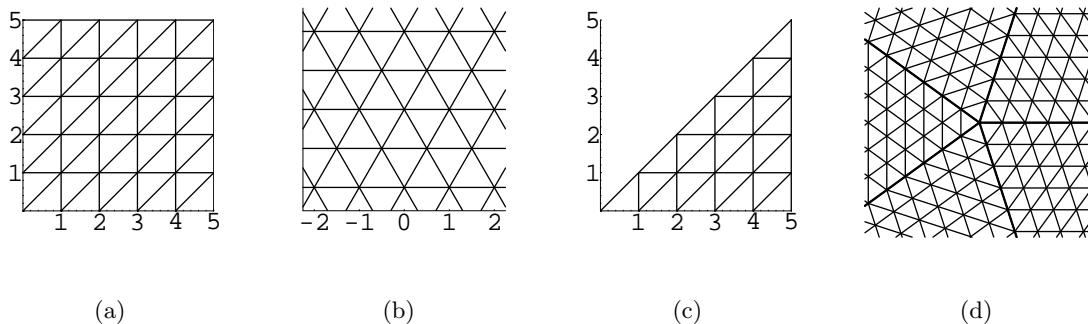
Figure 2.4: (a) and (b) The regular complex K_6 with 2 different regular realizations in $\mathbb{R}^2$: (a) The standard realization, and (b) the equilateral realization. (c) A wedge W. (d) The n-regular complex K_n, constructed from n copies of W.

2.2 Affine Coordinates

An atlas of coordinate charts is defined on $|K|$ with an isolated set of points deleted. On a simplicial surface without boundary K, the deleted points are the vertices of valence other than 6. We call these *extraordinary* vertices, and denote the set of all such vertices in K by $\mathsf{Ext}(K)$.

The *regular complex*, which we denote by K_6, is an infinite simplicial surface with

$$\mathsf{Vertex}(K_6) = \mathbb{Z}^2,$$

$$\mathsf{Edge}(K_6) = \big\{ \{j, j+e_1\}, \{j, j+e_2\}, \{j, j+e_1+e_2\} : j \in \mathbb{Z}^2 \big\},$$

and

$$\mathsf{Face}(K_6) = \big\{ \{j, j+e_1, j+e_1+e_2\}, \{j, j+e_2, j+e_1+e_2\} : j \in \mathbb{Z}^2 \big\},$$

where e_1 and e_2 are the standard basis vectors in $\mathbb{Z}^2$. The given vertex locations define a geometric realization of K_6 in $\mathbb{R}^2$ (shown in Figure 2.4(a)), which we call the *standard* realization. Other *regular realizations* of K_6 are given by applying an invertible linear transformation to the vertices of K_6. In particular, we have the *equilateral realization*, shown in Figure 2.4(b), where all edges have unit length.

We now construct an atlas of coordinate charts on $|K| \setminus \mathsf{Ext}(K)$. Let $\iota : L \to K_6$ be an embedding of a simplicial sub-surface $L \subset K$ into the regular complex, and let $\psi : |K_6| \to \mathbb{R}^2$ be a regular realization of K_6. Then the composition

$$x : |L|^\circ \xrightarrow{|\iota|} |K_6| \xrightarrow{\psi} \mathbb{R}^2 \tag{2.9}$$

is a coordinate chart on the coordinate neighborhood $|L|^\circ$, where $|L|^\circ$ denotes the interior of $|L|$. The collection of all such charts forms a C^∞-atlas on $|K| \setminus \mathsf{Ext}(K)$, called the *affine atlas*. Moreover, the transition functions are affine maps, i.e., linear maps composed with translations. If a function $f \in C(K)$ (i.e., $C(|K|)$ but we drop the $|\cdot|$ to simplify notation) is class C^r with respect to this atlas, we say it is C^r-*away-from-extraordinary-vertices* or $f \in C^r(|K| \setminus \mathsf{Ext}(K))$.

An affine coordinate system (U, x) in which the coordinate images of faces in K are equilateral triangles with unit length sides is called an *equilateral affine coordinate system*. For any affine coordinate neighborhood U there are equilateral affine coordinate charts on U.

The composition of a polynomial with an affine map is again a polynomial. So the property that a function is a polynomial on an open neighborhood U of $|K| \setminus \mathsf{Ext}(K)$ is well defined. If (U, x) is an affine coordinate system, and a function $f \in C(K)$ is polynomial in x on $x(U)$ then in any other affine coordinate chart (V, x') with $U \cap V \neq \emptyset$ the function f will also be polynomial in x' on $x'(U \cap V)$. However, given a neighborhood U and a function f on U that is affine polynomial on U, it is not necessarily true that there is an extension to all of $|K| \setminus \mathsf{Ext}(K)$ that is globally polynomial.

2.3 The n−Regular Complex

All the new vertices introduced by "4-to-1" subdivision are valence 6, while the original vertices correspond to vertices in the subdivided complex of the same valence. So for any vertex v of valence $n \neq 6$ in a simplicial surface K, the neighborhoods $N_{2^k-1}(v, K^k)$ of v in the subdivided complexes have a single vertex of valence n, and all the other vertices have valence 6. The n-regular complex is an infinite simplicial complex with a single vertex of valence n surrounded by valence 6 vertices.

Define a wedge $W \subset K_6$ by

$$W = \{s \in K_6 : j_1 \geq 0 \text{ and } 0 \leq j_2 \leq j_1 \text{ for all } (j_1, j_2) \in s\}, \qquad (2.10)$$

as shown in Figure 2.4(c). For each valence $n \geq 3$, we define the $n-regular\ complex\ K_n$ as follows: Let W_i for $i \in \mathbb{Z}_n$, denote n disjoint copies of W, and denote the vertices of W_i by (i, j) for $i \in \mathbb{Z}_n$ and $j \in \mathsf{Vertex}(W)$, where $\mathbb{Z}_n$ is the group of integers modulo n. We then identify vertices by the equivalence relation

$$(i, (j, 0)) \equiv (i - 1, (j, j)) \quad \text{for all } j \geq 0 \text{ and } i \in \mathbb{Z}_n \,,$$

the subtraction being taken modulo n. The resulting complex K_n is a simplicial surface without boundary (as in Figure 2.4(d)). The central vertex, denoted by v_0, has valence n and all other vertices have valence 6.

The simplicial automorphisms of K_n are generated by a rotation ρ

$$\rho : (i, j) \mapsto (i + 1, j) \qquad (2.11)$$

and a reflection f

$$f : (i, j) \mapsto (-i, \tilde{f}(j)), \qquad (2.12)$$

where $\tilde{f}$ is the reflection on W. Actually, for $n = 6$ the group generated by ρ and f is a proper subgroup of the automorphism group, namely the isotropy subgroup of v_0 (i.e., the automorphisms that leave v_0 fixed). However, we call the group generated by ρ and f the "automorphism group" of K_n.

The $n-$regular complex is self similar under subdivision. We first define the contraction map on the wedge W. The vertices of W are points of $\mathbb{R}^2$ that define a geometric realization of W, and therefore we have a geometric realization of $D(W)$. The map $c : \mathsf{Vertex}(W) \to \mathsf{Vertex}(D(W))$ given, in terms of this realization, by

$$c(x) = x/2$$

is a simplicial isomorphism. Applying c to each wedge W_i of K_n extends c to a simplicial isomorphism

$$c : K_n \to D(K_n),$$

which we call the *contraction* map.

Contraction induces a piecewise linear homeomorphism $|c| : |K_n| \rightarrow |D(K_n)|$, which when composed with the identification induced by subdivision $\iota : |D(K_n)| \rightarrow |K_n|$, results in a piecewise linear homeomorphism on $|K_n|$

$$c : |K_n| \rightarrow |K_n| . \tag{2.13}$$

In an abuse of notation the map (2.13) is also denoted by c.

The contraction map (2.13) generates an additive action of the group $\mathbb{Z}$ on $|K_n|$. Similarly, the contraction map c and the rotation ρ generate an action of the group $\mathbb{Z} \times \mathbb{Z}_n$ on $|K_n|$. Given a group G acting on a $|K_n| \setminus v_0$, for $g \in G$, let ϕ_g be the corresponding transformation of $|K_n| \setminus v_0$, for instance, $\phi_j = c^j$ or $\phi_{j,k} = c^j \circ \rho^k$ in our two examples. A *fundamental region*, with respect to the action, is an open, connected region $\Omega \subset |K_n|$ such that (i) $\phi_g \Omega \cap \Omega = \emptyset$ for all $g \in G$ other than the identity, and (ii) the union $\cup \overline{\phi_g \Omega}$ over all $g \in G$ is all of $|K_n| \setminus v_0$,where $\overline{A}$ denotes the closure of A. For example, consider the action of $\mathbb{Z}$ on $|K|$ generated by c, and define the annular simplicial surface $\Omega_0 \subset K_n$ by

$$\Omega_0 = N_2(v_0, K_n) \setminus N_1(v_0, K_n), \tag{2.14}$$

using definition (2.1). Then the region $|\Omega_0|^\circ$, shown in Figure 2.5, is a fundamental region with respect to the action. Let $\Omega_j = c^j \Omega_0 \subset K_n^j$ for $j \geq 0$, and for $j < 0$ define the subcomplex $\Omega_j \subset K_n$ by $|\Omega_j| = (c^j)^{-1}(|\Omega_0|)$. For each face $T \in K_n^k$, we define an integer j_T called the *annulus index* of T as follows: If $T \notin N_1(v_0, K_n^k)$, we let j_T be the unique integer satisfying $|T| \subset |\Omega_j|$, and if $T \in N_1(v_0, K_n^k)$, we let $j_T = k$. Notice we have the identity $j_T \leq k$.

2.4   The Subdivision Map and the Characteristic Map

The subdivision map represents the operation of stationary subdivision near an extraordinary vertex. It has been central in the analysis of subdivision schemes. Let $\mathcal{S}$ be a local subdivision scheme with mask width m_w. When applying subdivision on the $n-$regular complex K_n, the control net on $N_m(v_0, K_n)$ determines the control net on $N_{2m-m_w+1}(v_0, K_n^1)$. When $m \geq m_w$ there is a simplicial embedding of $N_m(v_0, K_n)$ into $N_{2m-m_w+1}(v_0, K_n^1)$ that
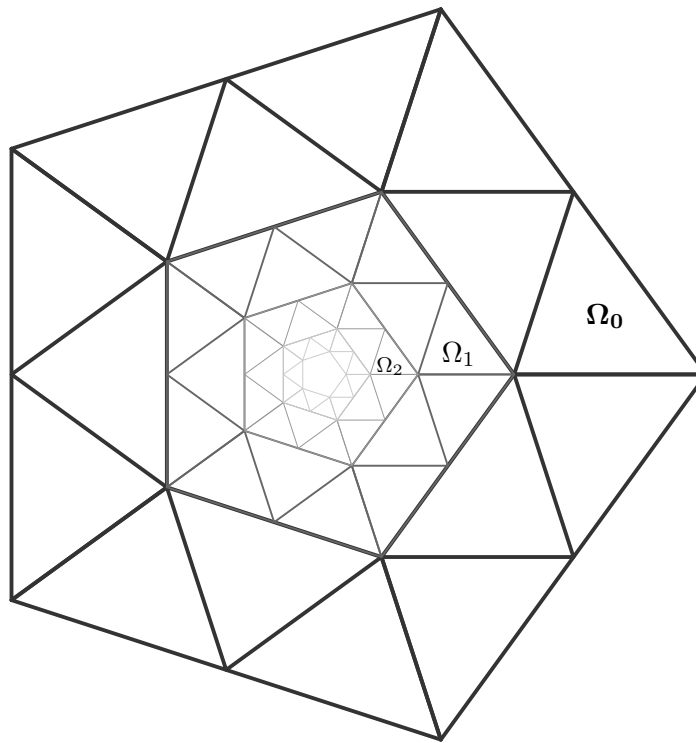
Figure 2.5: $K_n \setminus v_0$ is decomposed into annuli. The outer complex is $\Omega_0 \subset K_n$ and its interior $|\Omega_0|^\circ$ is a fundamental annulus.

fixes v_0. Thus, the control nets over the m−neighborhoods at successively finer levels, $N_m(v_0, K_n^k)$ for $k = 0, \ldots$, are obtained by repeatedly applying a fixed linear transformation to the initial control net on $N_m(v_0, K_n)$. This linear transformation is the subdivision map.

For any $m \geq m_w$ we define the *subdivision map* $S_{n,m}$ as the linear transformation

$$S_{n,m} = c^* \mathcal{S} : \mathsf{CN}(N_m(v_0, K_n)) \to \mathsf{CN}(N_m(v_0, K_n)).$$

Usually we are concerned only with the special case S_{n,m_w}, which we will write more simply as S_n. In Section 2.4.1, we show how $S_{n,m}$ is completely determined by S_n. For a fixed scheme and valence n, we denote the distinct eigenvalues of S_n by $\lambda_0, \ldots, \lambda_N$, ordered so that $|\lambda_0| \geq |\lambda_1| \geq \cdots \geq |\lambda_N|$.

Let u be an eigenvector of S_n with eigenvalue λ_j, and let $\phi = \mathcal{S}^\infty u$ be the corresponding limit function defined on $|N_1(v_0, K_n)|$. We derive an identity for ϕ we call the *contraction identity*. Applying $\mathcal{S}^\infty$ to the eigenvalue identity $\lambda_j u = S_n u$, we get

$$\lambda_j \mathcal{S}^\infty u \;=\; \mathcal{S}^\infty S_n u \;=\; \mathcal{S}^\infty c^* \mathcal{S} u.$$

After applying Proposition 8, this becomes $\lambda_j \mathcal{S}^\infty u = c^* \mathcal{S}^\infty u$, which can be expressed in terms of ϕ as

$$\phi(c(z)) = \lambda_j \phi(z) \qquad \text{for all } z \in |N_1(v_0, K_n)|. \tag{2.15}$$

The derivation of the proceeding paragraph can be extended to generalized eigenvectors of S_n. Let $J = U^{-1} S_n U$ be the Jordan normal form of S_n, with diagonal elements ordered by non-increasing magnitude, i.e., so that $|J_{jj}| \geq |J_{kk}|$ for all $j \leq k$. The columns of U, $(u_0, u_1, \ldots, u_N) = U$, are generalized eigenvectors of S_n. Let $\Phi = (\phi_0, \phi_1, \ldots, \phi_N) = \mathcal{S}^\infty U$ denote the row vector of limit functions, each entry given by $\phi_i = \mathcal{S}^\infty u_i$. Applying the limit operator to the identity $UJ = S_n U$, we compute the following:

$$\mathcal{S}^\infty (UJ) = \mathcal{S}^\infty (S_n U) = \mathcal{S}^\infty c^* \mathcal{S} U = c^* \mathcal{S}^\infty U.$$

Expressing this in terms of Φ we have

$$\Phi(c(z)) = \Phi(z) \cdot J \qquad \text{for all } z \in |N_1(v_0, K_n)|. \tag{2.16}$$

The components of Φ form a basis of the space of subdivision functions on $|N_1(v_0, K_n)|$. Namely, given $u \in \mathsf{CN}(N_{m_w}(v_0, K_n))$ defining the subdivision function $f = \mathsf{S}^\infty u$, we have

$$f = \mathsf{S}^\infty u = \mathsf{S}^\infty U U^{-1} u = \Phi \cdot A, \tag{2.17}$$

where $A = U^{-1}u$.

If S is a convergent stationary subdivision scheme, then for every valence n, the *dominant eigenvalue* λ_0 is one. It is algebraically simple, and all other eigenvalues have magnitudes strictly less than one. The *sub-dominant eigenvalue* λ_1 determines the C^1-smoothness of the subdivision functions. This was first made precise in the work of in Reif [16]. The key tool that he introduced was the characteristic map.

Definition 10. *Suppose S is a stationary subdivision scheme. For a fixed valence n, suppose the distinct eigenvalues of the subdivision map $\lambda_0, \lambda_1, \ldots, \lambda_N$, ordered by non-increasing magnitude, satisfy the following conditions:*

(i) The dominant eigenvalue λ_0 is one, and is an algebraically simple eigenvalue.

(ii) The sub-dominant eigenvalue λ_1 is real and positive, and is of geometric and algebraic multiplicity 2.

(iii) The other eigenvalues, λ_j for $j > 1$, are of magnitude strictly less than λ_1.

Let $u_1, u_2 \in CN(N_{m_w}(v_0, K_n))$ be linearly independent λ_1-eigenvectors of S_n. Then the $\mathbb{R}^2$-valued control net $u = (u_1, u_2)$ defines a continuous map

$$\mathsf{S}^\infty u : |N_1(v_0, K_n)| \to \mathbb{R}^2,$$

called a **characteristic map**.

This definition is a slight modification from that found in Reif [16]. Zorin extended the definition considerably in [25] and [23], relaxing the condition on the algebraic and geometric multiplicities of the sub-dominant eigenvalue.

An eigen-analysis of the subdivision map can be made using the discrete Fourier transform. This technique shows that Loop's scheme has characteristic maps for all valences, see Loop [13], Schwietzer [18], or Zorin [25].

2.4.1 Characteristic Maps on $|K_n|$

We will show how an eigen-structure of the smallest subdivision matrix S_{n,m_w} determines the eigen-structure of all larger subdivision matrices $S_{n,m}$. Applying this result, we extend the characteristic maps to all of $|K_n|$.

Fix the valence n and, to simplify notation, let $N_m = N_m(v_0, K_n)$. For any $m' > m \geq m_w$ we can decompose $\mathsf{CN}(N_{m'})$ into $\mathsf{CN}(N_m) \oplus V_{m',m}$, where

$$V_{m',m} = \{u \in \mathsf{CN}(N_{m'}) : u_v = 0 \text{ for all } v \in N_m\}. \tag{2.18}$$

With respect to this decomposition $S_{m'}$, has the following block form:

$$S_{m'} = \begin{bmatrix} S_m & 0 \\ * & * \end{bmatrix}. \tag{2.19}$$

So restricting an eigenvector of $S_{m'}$ to N_m gives an eigenvector of S_m.

For any $m > m_w$, notice that V_{m,m_w} is a space of "nilpotent" vectors. That is, for any $v \in V_{m,m_w}$ the iterates $S_m^k v$ are zero for large enough k. This follows since

$$S_m V_{m,s} \subset V_{m,2s-m_w+1}, \tag{2.20}$$

so the region of zeros keeps growing as one repeatedly applies S_m. Recall that a square matrix N is nilpotent if $N^k = 0$ for some k.

Theorem 11. *For any $m > m_w$, let $J = U^{-1} S_n U$ be the Jordan normal form of the subdivision matrix S_n. Then*

$$S_{n,m} = \widetilde{U} \begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} \widetilde{U}^{-1}, \qquad \text{where} \quad \widetilde{U} = \begin{bmatrix} U & 0 \\ * & I \end{bmatrix},$$

and where N is nilpotent and I is an identity matrix.

Some implications of this theorem for any $m > m_w$ follow:

(i) The non-zero spectrum of $S_{n,m}$ coincides with the non-zero spectrum of S_n.

(ii) The algebraic and geometric multiplicities of any non-zero eigenvalue in the spectrum of $S_{n,m}$ are independent of m.

(iii) An eigenvector $u \in \mathsf{CN}(N_{m_w})$ with non-zero eigenvalue λ has a unique extension to a λ-eigenvector $\widetilde{u} \in \mathsf{CN}(N_m)$ of $S_{n,m}$. Indeed if $\widetilde{u}$ and $\widetilde{\widetilde{u}} \in \mathsf{CN}(N_m)$ are both λ-eigenvectors extending u, then $\widetilde{u} - \widetilde{\widetilde{u}}$ is a λ-eigenvector, but $\widetilde{u} - \widetilde{\widetilde{u}} \in V_{m,m_w}$, so by Theorem 11 it is nilpotent.

As a consequence of (iii), each eigenvector can be extended uniquely to all of K_n. In particular, we can extend the characteristic maps.

Definition 10$'$. *Suppose for a fixed valence n, the subdivision scheme satisfies the conditions of Definition 10. Let $u = (u_1, u_2)$ be a pair of linearly independent λ_1-eigenvectors of S_n. Then by comment (iii) there are unique extensions $\widetilde{u} = (\widetilde{u_1}, \widetilde{u_2})$ to control nets on all of K_n. We call the generated subdivision function*

$$S^\infty \widetilde{u} : |K_n| \to \mathbb{R}^2$$

a **characteristic map.**

Proof of Theorem 11. From (2.20) we see that V_{m,m_w} is an S_m-invariant and nilpotent subspace. So we can block diagonalize $S_{n,m}$ via a similarity transformation as follows:

$$S_{n,m} = \widetilde{U} \begin{bmatrix} * & 0 \\ * & N \end{bmatrix} \widetilde{U}^{-1}, \quad \text{where} \quad \widetilde{U} = \begin{bmatrix} * & 0 \\ * & I \end{bmatrix}$$

and N is nilpotent.

Next we show that given a cycle of k generalized $\lambda-$eigenvectors, $u_1, u_2, \ldots, u_k \in \mathsf{CN}(N_{m_w})$, corresponding to a $(k \times k)$-Jordan block of S_n, we can construct extensions $\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_k \in \mathsf{CN}(N_m)$ that from a cycle of generalized $\lambda-$eigenvectors of $S_{n,m}$. This will complete the proof.

A cycle of generalized eigenvectors u_i of S_{n,m_w} satisfy

$$S_{n,m_w} u_i = \lambda u_i + u_{i-1} \quad \text{for } i = 1, \ldots, k,$$

with $u_0 = 0$. We define a doubly indexed sequence of control nets $u_i^j \in \mathsf{CN}(N_{m_j})$, for $i = 0, \ldots, k$ and $j \geq 0$. The sizes of the expanding neighborhoods are given by the recurrence

$m_{j+1} = 2m_j - m_w + 1$ with $m_0 = m_w$. The control nets are defined inductively, with base definitions $u_i^0 = u_i$ and $u_0^j \equiv 0$, by the rule

$$u_i^{j+1} = \tfrac{1}{\lambda}\left(c^* \mathcal{S} u_i^j - u_{i-1}^{j+1}\right) \quad \text{for } i = 1, \ldots, k. \tag{2.21}$$

For each $j > 0$ we prove the following statements:

$$u_i^j \equiv u_i^{j-1} \qquad \text{on } N_{m_{j-1}}, \tag{2.22}$$

and

$$S_{m_j} u_i^j = \lambda u_i^j + u_{i-1}^j \qquad \text{for } i = 1, \ldots, k. \tag{2.23}$$

We prove (2.22) and (2.23) by induction on j, assuming they are true for some fixed value of j and proving they are true at $j + 1$.

Proof of (2.22).

We prove (2.22) at $j + 1$ inductively on i. The base step is trivial, $u_0^{j+1} \equiv u_0^j \equiv 0$. Assuming (2.22) holds for a fixed $i - 1$ and $j + 1$, we must show that it holds also for i and $j + 1$. From (2.23) we have the following identities on N_{m_j}

$$\lambda u_i^j + u_{i-1}^j = c^* \mathcal{S} u_i^j$$
$$u_i^j = \tfrac{1}{\lambda}\left(c^* \mathcal{S} u_i^j - u_{i-1}^{j+1}\right) \qquad \text{by (2.22) at } i-1 \text{ and } j+1$$
$$= u_i^{j+1} \qquad \text{by (2.21).}$$

Proof of (2.23).

Fix $i = 1, \ldots k$. From (2.23) at i and j, we have $c^* \mathcal{S} u_i^j = \lambda u_i^j + u_{i-1}^j$ on N_{m_j}. Applying subdivision and contraction yields

$$c^* \mathcal{S}(c^* \mathcal{S} u_i^j) = \lambda(c^* \mathcal{S} u_i^j) + c^* \mathcal{S} u_{i-1}^j \quad \text{on } N_{m_{j+1}}.$$

From (2.21) $c^* \mathcal{S} u_i^j = \lambda u_i^{j+1} + u_{i-1}^{j+1}$ on $N_{m_{j+1}}$, so substituting this into the parenthesized terms above yields

$$c^* \mathcal{S}(\lambda u_i^{j+1} + u_{i-1}^{j+1}) = \lambda(\lambda u_i^{j+1} + u_{i-1}^{j+1}) + c^* \mathcal{S} u_{i-1}^j \quad \text{on } N_{m_{j+1}}$$
$$c^* \mathcal{S} u_i^{j+1} = \lambda u_i^{j+1} + u_{i-1}^{j+1} + \tfrac{1}{\lambda} c^* \mathcal{S}\left(u_{i-1}^j - u_{i-1}^{j+1}\right).$$

By (2.22), the parenthesized term above is 0 on N_{m_j}, and therefore $c^* \mathcal{S} \left(u_{i-1}^j - u_{i-1}^{j+1} \right) \equiv$ 0 on $N_{m_{j+1}}$ proving (2.23).

$\square$

2.4.2 Equivariant Characteristic Maps

The n-regular complex is symmetric, and in this section we show that there are special characteristic maps which share these symmetries.

The rotation ρ (2.11) and reflection f (2.12) on K_n generate a symmetry group $\langle \rho, f \rangle$ of automorphisms of K_n. An isomorphic group $\langle \widetilde{\rho}, \widetilde{f} \rangle$ acting on $\mathbb{R}^2$ is generated by a clockwise rotation $\widetilde{\rho}$ through an angle $2\pi/n$, and a reflection $\widetilde{f}(x, y) = (x, -y)$. The correspondence between generators induces a group isomorphism $g \mapsto \widetilde{g}$ for any $g \in \langle \rho, f \rangle$.

Definition 12. *A characteristic map* $\chi : |N_1(v_0, K_n)| \to \mathbb{R}^2$ *is* **equivariant** *if the following diagram commutes for all* $g \in \langle \rho, f \rangle$.

$$
\begin{array}{ccc}
|N_1(v_0, K_n)| & \xrightarrow{\chi} & \mathbb{R}^2 \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \widetilde{g}} \\
|N_1(v_0, K_n)| & \xrightarrow{\chi} & \mathbb{R}^2
\end{array}
\tag{2.24}
$$

Using a discrete Fourier transform, Peters and Reif [14] show that if the characteristic maps are injective, then equivariant characteristic maps exist. Figure 2.6 shows an equivariant control net on $N_2(v_0, K_n)$ for Loop's subdivision scheme. It generates an equivariant characteristic map.

2.5 Subdivision Smooth Structure

Reif introduced the concept of the characteristic map and proved the following key theorem in [16].

Theorem 13 (Reif). *If all characteristic maps are regular and injective then subdivision surfaces are C^1-surface for almost every control net.*
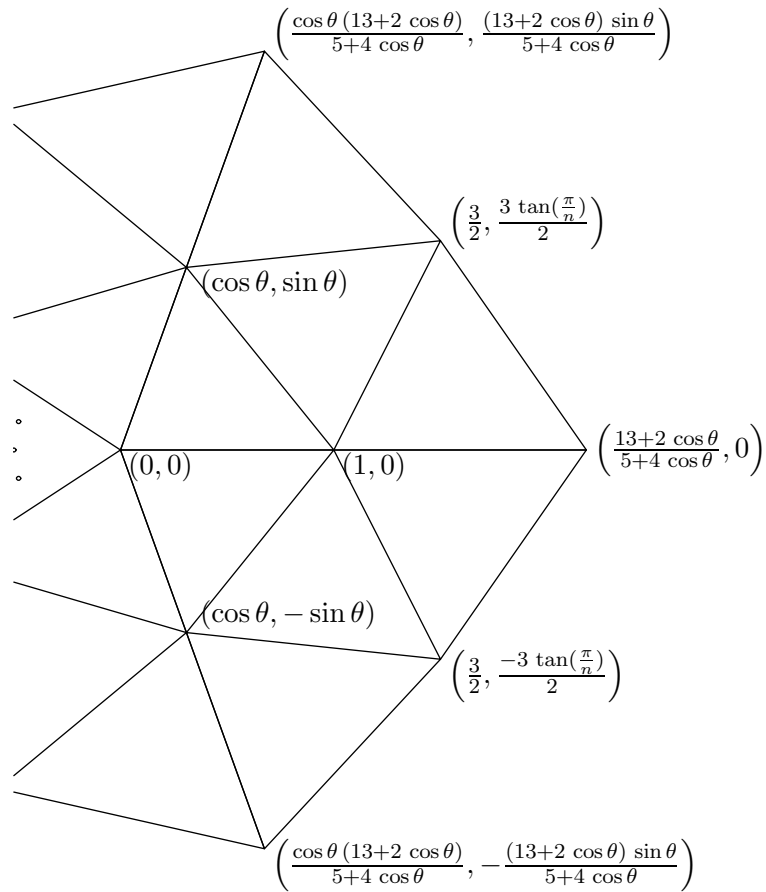
Figure 2.6: A Loop subdivision control net for an equivariant characteristic map on K_n. Not all control values are shown, but the control net itself is equivariant, so the remaining control values can be computed by applying a rotation of angle $\theta = \frac{2\pi}{n}$.

We develop a parallel theory in this section. Suppose S is a stationary subdivision scheme (i) that generates C^r-subdivision functions away-from-extraordinary-vertices, (ii) that has characteristic maps for all valences, and (iii) whose characteristic maps are injective and regular on $|K| \setminus \mathsf{Ext}(K)$. In Proposition 15 we show that the characteristic maps can be used to define a C^r-atlas on any simplicial complex $|K|$ without boundary. We then show in Proposition 17 that the subdivision functions are of class $C^1(|K|)$ with respect to this atlas. From Propositions 15 and 17 and the basic theory of differentiable manifolds (e.g. [2]), we get the following corollary.

Corollary 14. *Given an $\mathbb{R}^3$ valued control net on a simplicial surface without boundary, if the generated subdivision function is injective and everywhere of full rank then its image is a C^1-surface embedded in $\mathbb{R}^3$.*

Loop's subdivision functions are C^2-away-from-extraordinary-vertices, and characteristic maps exist for all valences. In his thesis [25], Zorin showed that the characteristic maps are regular and injective for all valences. Thus Loop's subdivision scheme determines a C^2-atlas on any simplicial surface $|K|$ without boundary, and Loop's subdivision functions are C^1.

Suppose S is a stationary subdivision scheme such that for every valence there are injective characteristic maps. For each vertex v in a simplicial surface K without boundary, we define a *characteristic coordinate system* on $|K|$ centered at v. Recall, a coordinate system on an n-dimensional manifold M is a pair (U, ϕ), where U is an open subset of M and $\phi : U \to \mathbb{R}^n$ is continuous and injective. We construct a coordinate chart on $|N_1(v, K)|^\circ$. Let ι_v be an identification between $N_1(v, K)$ and $N_1(v_0, K_n)$, where n is the valence of v, and let $\chi : |N_1(v_0, K_n)| \to \mathbb{R}^2$ be a characteristic map. The composition $\psi_v = \chi \circ i_v : |N_1(v)|^\circ \to \mathbb{R}^2$ is a *characteristic coordinate chart*,

$$\psi_v : |N_1(v)|^\circ \xrightarrow{\iota_v} |N_1(v_0, K_n)| \xrightarrow{\chi} \mathbb{R}^2. \tag{2.25}$$

We compute the characteristic coordinate representation of the contraction map, which we use often in the following chapters. We define a contraction map near a vertex v, denoted

$c_v : |N_1(v)| \to |N_1(v)|$, by $c_v = \iota_v^{-1} \circ c \circ \iota_v$. In characteristic coordinates (2.25) we have

$$c_v(y) = \psi_v \circ c_v \circ \psi_v^{-1} = \chi \circ c \circ \chi^{-1}(y) = \lambda_1 \chi(\chi^{-1}(y)) \qquad \text{by (2.15)}$$

$$= \lambda_1 y. \tag{2.26}$$

Recall a C^r-*atlas* on a manifold M is a collection $\{(U_\alpha, \psi_\alpha)\}$ of coordinate systems on M such that the coordinate neighborhoods $\{U_\alpha\}$ cover M, and for any pair (U_1, ψ_1) and (U_2, ψ_2) of coordinate systems, the *transition function* $\psi_1 \circ \psi_2^{-1} : \psi_2(U_1 \cap U_2) \to \psi_1(U_1 \cap U_2)$ is a C^r-diffeomorphism. A characteristic map is *regular* on $|K| \setminus \mathsf{Ext}(K)$ if its Jacobian is non-singular when expressed locally in affine coordinates.

Proposition 15. *Suppose $\mathcal{S}$ is a stationary subdivision scheme such that: (i) subdivision functions are C^r-away-from-extraordinary-vertices and (ii) there are characteristic maps which are injective and regular on $|K| \setminus \mathsf{Ext}(K)$. Then for any simplicial surface K without boundary, the collection of characteristic charts $\{(|N_1(v)|^\circ, \psi_v)\}$ for $v \in \mathsf{Vertex}(K)$ is a C^r-atlas on $|K|$. Furthermore, the atlas is C^r-compatible with the affine coordinate charts (2.9) and with the subdivision smooth structure of $D(K)$.*

Definition 16. *A stationary subdivision scheme $\mathcal{S}$ satisfying the conditions of Proposition 15 is called a $\mathbf{C^r}$-subdivision scheme, and the C^r-atlas of characteristic charts $\{(|N_1(v)|^\circ, \psi_v)\}$ on a simplicial surface K without boundary is called the subdivision smooth structure on $|K|$.*

Notice the subdivision surfaces of a C^r-subdivision scheme are not necessarily C^r-surfaces.

Proof of Proposition 15. The neighborhoods $|N_1(v)|^\circ$ clearly cover $|K|$. Fix a vertex $v \in K$ and a characteristic chart $\psi_v = \chi \circ \iota_v$. We need to show that the transition functions

$$\tau = \psi_v \circ \psi^{-1} : \psi(U \cap |N_1(v)|^\circ) \to \mathbb{R}^2 \tag{2.27}$$

are C^r with C^r inverses for various charts $\psi : U \to \mathbb{R}^2$. We consider four different kinds of charts:

(i) Characteristic chart centered at v. Suppose $\widetilde{\psi} = \widetilde{\chi} \circ \iota_v$ is another characteristic chart centered at v using a different characteristic map. Let u and $\widetilde{u}$ be the λ_1-eigenvectors of S_n that generate the characteristic maps, i.e., $\chi = \mathbb{S}^\infty u$ and $\widetilde{\chi} = \mathbb{S}^\infty \widetilde{u}$. Since the λ_1-eigenspace of S_n is 2-dimensional, $u = \widetilde{u}A$ for some non-singular 2×2 matrix A. Thus $\chi = \mathbb{S}^\infty u = \mathbb{S}^\infty \widetilde{u} \cdot A = \widetilde{\chi} \cdot A$, and so

$$\begin{aligned}
\tau(z) = \psi_v \circ \widetilde{\psi}^{-1}(z) = \chi \circ \widetilde{\chi}^{-1}(z) \\
= \widetilde{\chi}(\widetilde{\chi}^{-1}(z)) \cdot A \\
= z \cdot A.
\end{aligned}$$

This shows that the transition function τ is linear and therefore smooth.

Now consider the other case, where $\widetilde{\psi}_v = \chi \circ \widetilde{\iota}_v$ is composed of an identification $\widetilde{\iota}_v$, possibly different than ι_v, and the characteristic map χ. Let $\sigma = \iota_v \circ \widetilde{\iota}_v^{-1}$, and extend it to an isomorphism on all of K_n. We reduce this case to that of the previous paragraph by writing $\widetilde{\psi} = \widetilde{\chi} \circ \iota_v$, with $\widetilde{\chi} = \chi \circ \sigma$. Then by Proposition 8 we have

$$\widetilde{\chi} = \sigma^* \mathbb{S}^\infty u = \mathbb{S}^\infty \sigma^* u. \tag{2.28}$$

A simple calculation shows that $\sigma^* u$ is a λ_1-eigenvector S_n, and therefore (2.28) is a characteristic map.

(ii) Affine chart. Since characteristic charts are defined by subdivision, they are C^r on $|K| \backslash \mathsf{Ext}(K)$. Since the characteristic maps, expressed in affine coordinates, are regular on $|K| \setminus \mathsf{Ext}(K)$, their inverses are C^r by the inverse function theorem.

(iii) Characteristic chart centered at a vertex adjacent to v. Let $e = (v, w) \in \mathsf{Edge}(K)$, let $(|N_1(w)|^\circ, \psi_w)$ be a characteristic coordinate chart centered at w, and let $\tau = \psi_v \circ \psi_w^{-1}$ be the transition function. There is an affine coordinate system $(|N_1(v) \cap N_1(w)|^\circ, \widetilde{\psi})$, and we can write τ as a composition

$$\tau = (\psi_v \circ \widetilde{\psi}^{-1}) \circ (\psi_w \circ \widetilde{\psi}^{-1})^{-1}.$$

The operands of the composition are C^r as shown in (ii) above, so τ is also C^r.

(iv) Characteristic chart of $D(K)$. It suffices to consider a specific $\psi_{D(v)}$. The identification ι_v induces an isomorphism $\iota_{D(v)} : N_1(v, D(K)) \to N_1(v_0, D(K_n))$. Hence we consider the characteristic chart $\psi_{D(v)} = \chi \circ c^{-1} \circ \iota_{D(v)}$. Then we have

$$\psi_v \circ \psi_{D(v)}^{-1} = \chi \circ c^{-1} \circ \chi^{-1}. \tag{2.29}$$

As in the computation of (2.26) we have $\psi_v \circ \psi_{D(v)}^{-1}(z) = \lambda_1^{-1} z$.

$\square$

Proposition 17. *Suppose $\mathbb{S}$ is C^1-subdivision scheme. Then the subdivision functions are C^1 with respect to the subdivision smooth structure.*

Proof. Let K be a simplicial surface without boundary and let $f \in \mathbb{S}(K)$. Clearly $f \in C^1(K \setminus \mathsf{Ext}(K))$, so we need only analyze f near extraordinary vertices. Let $v \in \mathsf{Ext}(K)$ be a vertex of valence n. Let $J = U^{-1} S_n U$ be the Jordan form of S_n with distinct eigenvalues $1 < \lambda_1 < |\lambda_2| \le \cdots \le |\lambda_N|$ along the diagonal in non-increasing order of magnitude, and let $\Phi = (1, \phi_1, \phi_2, \ldots, \phi_N) = \mathbb{S}^\infty U$, a basis of subdivision functions on $N_1(v_0, K_n)$. We apply subdivision k_0 times so that $N_{m_w}(v, K^{k_0})$ is isomorphic to $N_{m_w}(v_0, K_n)$. We express f on $|N_1(v, K^{k_0})|^\circ$ in terms of the characteristic coordinate $y = (\phi_1, \phi_2)$,

$$f(y) = \Phi(y) \cdot A \tag{2.30}$$

for some coefficient vector A. Differentiating (2.30) with respect to a coordinate variable y_i, and evaluating at $c^k(y)$ we get

$$\frac{\partial f}{\partial y_i}(c^k(y)) = \frac{\partial \Phi}{\partial y_i}(c^k(y)) \cdot A. \tag{2.31}$$

To continue, we derive a contraction identity for $\frac{\partial \Phi}{\partial y_i}$ by differentiating the contraction identity (2.16) and using (2.26) to get

$$\frac{\partial \Phi}{\partial y_i}(c(y)) = \lambda_1^{-1} \frac{\partial \Phi}{\partial y_i}(y) \cdot J. \tag{2.32}$$

Substituting this into (2.31) we get

$$\frac{\partial f}{\partial y_i}(c^k(y)) = \lambda_1^{-k} \frac{\partial \Phi}{\partial y_i}(y) \cdot J^k A. \tag{2.33}$$

Next we split off the first few components of Φ, J and A. Let $\Phi_3(y)$ be all but the first three components of Φ, i.e., $\Phi(y) = \big(1, y_1, y_2, \Phi_3(y)\big)$, similarly let $A = (a_0, a_1, a_2, A_3)^T$. Let J_3 be the minor of J corresponding to all but the first 3 rows and columns, i.e., $J = \mathrm{diag}(1, \lambda_1, \lambda_1, J_3)$. We can then write the partial derivatives of Φ as

$$\frac{\partial \Phi}{\partial y_i} = (0, \delta_{i,1}, \delta_{i,2}, \frac{\partial \Phi_3}{\partial y_i}),$$

and substituting this into (2.33) we get

$$\frac{\partial f}{\partial y_i}(c^k(y)) = a_i + \lambda_1^{-k} \frac{\partial \Phi_3}{\partial y_i}(y) \cdot J_3^k A_3 . \tag{2.34}$$

We will show that in the limit as $y \to 0$, we have $\frac{\partial f}{\partial y_i}(y) = a_i$. Consider a fundamental annulus Ω_0 as in (2.14)

$$\left\| \frac{\partial f}{\partial y_i}(\cdot) - a_i \right\|_{L^\infty(c^k \Omega_0)} = \left\| \frac{\partial f}{\partial y_i}(c^k(\cdot)) - a_i \right\|_{L^\infty(\Omega_0)}$$

$$= \lambda_1^{-k} \left\| \frac{\partial \Phi_3}{\partial y_i}(\cdot) \cdot J_3^k A_3 \right\|_{L^\infty(\Omega_0)} \qquad \text{by (2.34)}.$$

Now $\frac{\partial \Phi_3}{\partial y_i}$ is continuous over Ω_0, so

$$\left\| \frac{\partial f}{\partial y_i}(\cdot) - a_i \right\|_{L^\infty(\Omega_k)} \leq C \lambda_1^{-k} \| J_3^k A_3 \|. \tag{2.35}$$

The spectral radius of J_3 is $|\lambda_2| < \lambda_1$, so there is a matrix norm such that $\|J_3\| < \lambda_1$, and hence

$$\left\| \frac{\partial f}{\partial y_i} - a_i \right\|_{L^\infty(\Omega_k)} \leq C \left(\frac{\|J_3\|}{\lambda_1} \right)^k \| A_3 \| . \tag{2.36}$$

In particular, the left-hand side converges to 0 as $k \to \infty$ so f is C^1 at v. $\qquad \square$

2.6   A Little More Smoothness

In Section 2.5 we constructed a C^r-smooth structure on a simplicial surface $|K|$ using the characteristic maps of the subdivision scheme. Now we squeeze a little more smoothness out of this construction. This allows us to define a proper subset of $C^r(K)$, containing "smother" functions, although this set is not $C^{r+1}(K)$. The additional smoothness is characterized by the property that the order r derivatives are locally Lipschitz.

Recall that a map $f : U \to \mathbb{R}^m$ on a domain $U \subset \mathbb{R}^n$ is *Lipschitz* if there exists a constant C such that

$$\|f(x) - f(y)\| \leq C\|x - y\| \qquad \text{for all } x, y \in U.$$

We say f is *locally Lipschitz* if there is a neighborhood U_x of each point $x \in U$ such that f is Lipschitz on U_x. We denote, by $C_{\text{loc}}^{r,1}(U)$, the class of functions $f \in C^r(U)$ such that $D^\alpha f$ is locally Lipschitz for all $|\alpha| = r$. (The "1" in the notation reflects the fact that Lipschitz continuous functions are Hölder continuous with Hölder exponent 1.) Some properties of locally Lipschitz functions which follow immediately from the definition are listed: (i) differentiable functions are locally Lipschitz, (ii) products and sums of locally Lipschitz functions are locally Lipschitz, and (iii) the composition of locally Lipschitz functions is locally Lipschitz.

We next give a detailed multi-index version of the chain rule and then show that the class $C_{\text{loc}}^{r,1}$ is closed under composition. We use standard multi-index notation. That is, $\alpha \in \mathbb{Z}_+^n$ is a multi-index of non-negative integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, and $x^\alpha = \prod_1^n x_i^{\alpha_i}$, $\alpha! = \prod_1^n \alpha_i!$, and $|\alpha| = \sum_1^n \alpha_i$. Partial derivatives are represented by D^α, where $D = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n})$. Also we write $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for each i, and define $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!}$ for $\beta \leq \alpha$.

Proposition 18 (Chain Rule). *Suppose $\tau : \Omega \to \Omega'$ is C^r-map from a domain Ω in $\mathbb{R}^n$ into a target Ω' in $\mathbb{R}^m$, and suppose $f \in C^r(\Omega')$. Then for any multi-index $\alpha \in \mathbb{Z}_+^n$, such that $1 \leq |\alpha| \leq r$ we have*

$$D^\alpha(f \circ \tau) = \sum_{1 \leq |\beta| \leq |\alpha|} (D^\beta f \circ \tau) \cdot M\tau_{\alpha\beta}, \tag{2.37}$$

where

$$M\tau_{\alpha\beta} = \sum c_{\alpha\beta}^{\gamma\mu} (D^{\gamma_1} \tau^{\mu_1}) \cdots (D^{\gamma_{|\beta|}} \tau^{\mu_{|\beta|}}), \tag{2.38}$$

the sum is taken over ordered sequences of multi-index pairs $(\gamma_i, \mu_i) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^m$ for $i = 1$ to $|\beta|$. The multi-index pairs satisfy

$$\gamma_1 + \cdots + \gamma_{|\beta|} = \alpha, \qquad \text{with } |\gamma_i| \geq 1,$$

and

$$\mu_1 + \cdots + \mu_{|\beta|} = \beta, \qquad \text{with } |\mu_i| = 1.$$

The constants are $c_{\gamma\mu}^{\alpha\beta}$ are unique positive integers that are independent of τ and f.

Lemma 19. *The composition of functions in $C_{\text{loc}}^{r,1}$ is in $C_{\text{loc}}^{r,1}$.*

Proof. Let $g : U \to V$ and $f : V \to W$ be elements in $C_{\text{loc}}^{r,1}$. Since $f \circ g$ is clearly in $C^r(U)$, it suffices to show that $D^\alpha(f \circ g)$ is locally Lipschitz for every multi-index $\alpha \in \mathbb{Z}_+^n$ such that $|\alpha| = r$. Applying the chain rule to the composition yields the expansion (2.37). The result then follows from properties (i), (ii) and (iii) above. $\qquad\square$

In Section 2.5 we defined a C^r-atlas on $|K|$ which determines a class of C^r-functions on the manifold $|K|$. Manifolds can be constructed with other types of structure, resulting in different classes of functions. The structure is not completely arbitrary but must have a basic set of properties: properties that any reasonable set of transition functions would have to satisfy. These properties are formalized in the definition of a pseudo-group of transformations. For references on this general construction of manifolds see either Chapter 3 of Thurston [22] or Chapter 1 of Kobayashi and Nomizu [11].

Definition 20. *A* **pseudo-group** *of transformations on $\mathbb{R}^n$ is a set $\mathcal{G}$ of homeomorphisms between open sets of $\mathbb{R}^n$ satisfying the following conditions:*

(i) *The domains of the elements $g \in \mathcal{G}$ cover $\mathbb{R}^n$.*

(ii) *The restriction of an element $g \in \mathcal{G}$ to an open subset of its domain is also in $\mathcal{G}$.*

(iii) *The composition $g_1 \circ g_2$ of two elements of $\mathcal{G}$, when defined, is in $\mathcal{G}$.*

(iv) *The inverse of an element in $\mathcal{G}$ is also in $\mathcal{G}$.*

(v) *If $g : U \to V$ is a homeomorphism between open sets of $\mathbb{R}^n$, and U is covered by an open collection U_α such that the restriction $g_{|U_\alpha} \in \mathcal{G}$, then $g \in \mathcal{G}$.*

A pair of coordinate charts (U_i, ψ_i) and (U_j, ψ_j) are said to be $\mathcal{G}$-*compatible* if the transition function $\tau_{ij} = \psi_i \circ \psi_j^{-1}$ is an element of $\mathcal{G}$. A $\mathcal{G}$-*manifold* is a topological space with an atlas of $\mathcal{G}$-compatible charts.

Let $\Gamma_{\text{loc}}^{r,1}(\mathbb{R}^n)$ denote the space of homeomorphisms $f : U \to V$ between domains in $\mathbb{R}^n$ such that $f \in C_{\text{loc}}^{r,1}(U)$ and $f^{-1} \in C_{\text{loc}}^{r,1}(V)$.

Proposition 21. *For $r \geq 1$ the collection $\Gamma_{loc}^{r,1}(\mathbb{R}^n)$ is a pseudo-group of transformations on $\mathbb{R}^n$.*

Proof. Condition (iii) of Definition 20 follows from Lemma 19, while all the other conditions follow trivially from the definition of $\Gamma_{\text{loc}}^{r,1}(\mathbb{R}^n)$. $\qquad\square$

Proposition 22. *In Proposition 15 we can replace C^r with $C_{\text{loc}}^{r,1}$.*

Definition 23. *A stationary subdivision scheme $\mathcal{S}$ satisfying the conditions of Proposition 22 is called a $\mathbf{C_{\text{loc}}^{r,1}}$-subdivision scheme.*

Examining the proof of Proposition 15, one sees that the same proof still works once we show that the Inverse Function Theorem holds for $C_{\text{loc}}^{r,1}$-maps.

Theorem 24 (Inverse Function Theorem). *Suppose $f : U \to V$ is a $C_{\text{loc}}^{r,1}(U)$ homeomorphism between domains in $\mathbb{R}^n$, with $r \geq 1$, and the Jacobian of f does not vanish on U. Then f^{-1} is in $C_{\text{loc}}^{r,1}(V)$.*

Proof. The inverse function theorem for C^r maps shows that $f^{-1} \in C^r(V)$, and that the Jacobian of f^{-1} is given by

$$Df^{-1}(y) = Df(f^{-1}(y))^{-1}.$$

Since the inverse function f^{-1} is in $C^r(V)$, the Jacobian Df is in $C_{\text{loc}}^{r-1,1}(U)$, and matrix inversion is smooth, we conclude that f^{-1} is in $C_{\text{loc}}^{r,1}(V)$ by Lemma 19. $\qquad\square$

Theorem 25. *Loop's subdivision scheme is a $C_{\text{loc}}^{2,1}$-subdivision scheme.*

Proof. We must show that the hypotheses of Proposition 22 are satisfied. That is, Loop's subdivision functions are $C_{\text{loc}}^{2,1}$-away-from-extraordinary-vertices, and that for all valences, there are characteristic maps which are injective and regular. Loop's subdivision functions are locally triangular quartic box splines away from an extraordinary vertex. Therefore they are C^2 away from an extraordinary vertex and piecewise C^∞. So they are $C_{\text{loc}}^{2,1}$-away-from-extraordinary-vertices.

We have already mentioned that characteristic maps exist for all valences and are injective and regular. □

2.7 Some Geometric Properties of Neighborhoods

This section contains some technical results, which we will need in Chapter 4, about the geometry of certain neighborhoods in the n-regular complex. For each valence n, we fix an equivariant characteristic map $y : |K_n| \to \mathbb{R}^2$, and use it as a global chart on K_n. Let λ be the sub-dominant eigenvalue of the subdivision map of valence n. We denote, by the subscript Λ, geometric properties of domains in K_n with respect to the euclidean metric in characteristic coordinates. For example, we define the *diameter* of a set $\Omega \subset |K_n|$ by $\text{diam}_\Lambda \Omega := \sup\{|y(p) - y(q)| : p, q \in \Omega\}$.

Recall, from Section 2.3, the definition of the annular simplicial complexes Ω_j and the annulus index j_T for a face $T \in K_n^k$. Let $W \subset K_n$ be a fixed wedge of K_n, let $W_j = W^j \cap \Omega_j \subset K_n^j$ for $j \geq 0$, and let $W_j = W \cap \Omega_j \subset K_n$ for $j < 0$.

Proposition 26. *For any fixed integer $d \geq 0$, there exists constants C_0 through C_3 such that for any $T \in \mathsf{Face}(K_n^k)$ and $k \geq 0$ we have*

$$C_0 \lambda^{2j_T} (\tfrac{1}{2})^{2(k-j_T)} \leq \text{area}_\Lambda |N_d T| \leq C_1 \lambda^{2j_T} (\tfrac{1}{2})^{2(k-j_T)} \tag{2.39}$$

and

$$C_2 \lambda^{j_T} (\tfrac{1}{2})^{k-j_T} \leq \text{diam}_\Lambda |N_d T| \leq C_3 \lambda^{j_T} (\tfrac{1}{2})^{k-j_T} . \tag{2.40}$$

Proof. We prove the proposition by successively reducing the collection of faces for which we must show that the bounds (2.39) and (2.40) are satisfied. Since the coordinate system is rotational equivariant we need only show (2.39) and (2.40) are satisfied for faces $T \in W^k$ and $k \geq 0$. The contraction map $c : W^k \to W^{k+1}$ is a simplicial isomorphism and a dilation in characteristic coordinates by (2.26). Therefore we get the identities

$$\text{area}_\Lambda |N_d(cT)| = \lambda^2 \, \text{area}_\Lambda |N_d T| \qquad \text{and} \qquad \text{diam}_\Lambda |N_d(cT)| = \lambda \, \text{diam}_\Lambda |N_d T| \tag{2.41}$$

for any face $T \in W^k$ and $k \geq 0$. Thus we need only show that (2.39) and (2.40) are satisfied for faces $T \in W$ and $k = 0$. Let k' be a positive integer such that $2^{k'} \geq d + 1$, and consider

the subcomplex $N_{2^{k'}}(v_0, W) \subset W$. Since there are only finitely many faces in $N_{2^{k'}}(v_0, W)$ we can choose constants C_0 through C_3 such that (2.39) and (2.40) are satisfied for any face $T \in N_{2^{k'}}(v_0, W)$. Thus it suffices to show that (2.39) and (2.40) are satisfied for faces $T \in W \setminus N_{2^{k'}}(v_0, W)$. For each face $T \in W \setminus N_{2^{k'}}(v_0, W)$ we have $j_T \leq -k'$ and thus $c^{-j_T} T \in \mathsf{Face}(W_0^{-j_T})$. Therefore by (2.41) it suffices to show (2.39) and (2.40) are satisfied for every face $T \in W_0^k$ and $k \geq k'$, that is, we must find constants C_0 through C_3 so that for any such T we have the inequalities

$$C_0 (\tfrac{1}{2})^{2k} \leq \mathrm{area}_\Lambda |N_d T| \leq C_1 (\tfrac{1}{2})^{2k} \tag{2.42}$$

and

$$C_2 (\tfrac{1}{2})^k \leq \mathrm{diam}_\Lambda |N_d T| \leq C_3 (\tfrac{1}{2})^k . \tag{2.43}$$

Notice that $|N_d W_0^{k'}|$ is an affine coordinate neighborhood, containing $|N_d T|$ for every $T \in W_0^k$ and $k \geq k'$. In affine coordinates all the neighborhoods $N_d T$ are similar, and in particular we have the identities $\mathrm{area}_A |N_d T| = C (\tfrac{1}{2})^{2k}$ and $\mathrm{diam}_A |N_d T| = C' (\tfrac{1}{2})^k$ where the subscript A is used to refer to computations in the affine coordinate system. Let $y = \tau(x)$ be the transition function from the affine coordinates to characteristic coordinates.

To show (2.42) we express $\mathrm{area}_\Lambda |N_d T|$ as an integral in x-coordinates by

$$\mathrm{area}_\Lambda |N_d T| = \int_{|N_d T|} \det \left| \frac{\partial \tau}{\partial x}(x) \right| dx .$$

Therefore for any $T \in W_0^k$ and $k \geq k'$ the following inequality holds

$$C (\tfrac{1}{2})^{2k} \min \det \left| \frac{\partial \tau}{\partial x}(x) \right| \leq \mathrm{area}_\Lambda |N_d T| \leq C (\tfrac{1}{2})^{2k} \max \det \left| \frac{\partial \tau}{\partial x}(x) \right|,$$

where the minimum and maximum are taken over $x \in |N_d(W_0^{k'})|$, proving (2.39).

The lower bound in (2.40) follows from (2.39). To see this consider a bounded domain $\Omega \subset \mathbb{R}^2$. Let $p, q \in \overline{\Omega}$ be a diameter of Ω, that is, $\mathrm{diam}\,\Omega = |p - q|$. Then Ω is contained the ball of radius $\mathrm{diam}\,\Omega$ centered at p. So we have a weak form of the diametric inequality,

$$\mathrm{area}\,\Omega \leq \pi (\mathrm{diam}\,\Omega)^2 .$$

Combining this with (2.39) yields the left side of (2.40) with $C_2 = \sqrt{C_0/\pi}$.

To prove the upper bound in (2.40) let $T \in W_0^k$ with $k \geq k'$ and let x_0 and x_1 be affine coordinates of a diameter, that is, $\mathrm{diam}_\Lambda(N_d T) = |\tau(x_0) - \tau(x_1)|$. Applying the mean value inequality we get

$$\mathrm{diam}_\Lambda(N_d T) = |\tau(x_1) - \tau(x_2)|$$
$$\leq \left\| \frac{\partial \tau}{\partial x} \right\| |x_1 - x_0| \leq C' \left\| \frac{\partial \tau}{\partial x} \right\| \left(\tfrac{1}{2}\right)^k, \tag{2.44}$$

where

$$\left\| \frac{\partial \tau}{\partial x} \right\| = \max_{|N_d(W_0^{k'})|} \left\| \frac{\partial \tau}{\partial x}(x) \right\|,$$

and $\|\frac{\partial \tau}{\partial x}(x)\|$ is the operator norm of the Jacobian matrix. This completes the proof of (2.43). $\qquad\square$

We will see in Chapter 4 that it is useful to work with a slightly larger neighborhoods of $N_d T$. For a given face $T \in K_n^k$, let $N_d^c T$ be the closed circumscribing ball of $N_d T$ with respect to characteristic coordinates. For a simplicial sub-surface $K \subset K_n^k$, we define

$$N_d^c K = \bigcup_{T \in \mathsf{Face}(K)} N_d^c T. \tag{2.45}$$

The diameters of $|N_d T|$ and $N_d^c T$ are related by

$$\mathrm{diam}_\Lambda |N_d T| \leq \mathrm{diam}_\Lambda N_d^c T \leq 2 \, \mathrm{diam}_\Lambda |N_d T|.$$

Therefore as a Corollary to Proposition 26 we have the following.

Corollary 27. *There exists constants C_0 and C_1 such that for any $T \in \mathsf{Face}(K_n^k)$ and $k \geq 0$ we have*

$$C_0 \lambda^{j_T} \left(\tfrac{1}{2}\right)^{k-j_T} \leq \mathrm{diam}_\Lambda N_d^c T \leq C_1 \lambda^{j_T} \left(\tfrac{1}{2}\right)^{k-j_T}.$$

If $|T| \subset N_2(v_0, K_n)$, then $j_T \geq 0$ and $k - j_T \geq 0$, so we get the following Corollary.

Corollary 28. *If $k \geq 0$ and $T \in \mathsf{Face}(K_n^k)$ and $|T| \subset |N_2(v_0, K_n)|$, then*

$$C_0 \lambda_{\min}^k \leq \mathrm{diam}_\Lambda N_d^c T \leq C_1 \lambda_{\max}^k,$$

where $\lambda_{\min} = \min\{\tfrac{1}{2}, \lambda\}$ and $\lambda_{\max} = \max\{\tfrac{1}{2}, \lambda\}$.

The same techniques that we used to bound the diameter and area of $N_d^c T$ in characteristic coordinates also apply to bound them in affine coordinates. In particular, we have the following lemma.

Lemma 29. *Suppose U is an affine coordinate neighborhood. Then there exists a constant C such that for k sufficiently large and $T \in \mathsf{Face}(\Omega_0^k)$, where $N_d^c T \subset U$, then*

$$\mathrm{diam}_A(N_d^c T) \leq C(\tfrac{1}{2})^k \quad and$$

$$\mathrm{area}_A(N_d^c T) \leq C(\tfrac{1}{2})^{2k},$$

where the A subscript indicates that computations are made with respect to affine coordinates.

For a fixed value of $d \geq 0$, the following proposition gives a bound on the number of times the neighborhoods $N_d^c T$ overlap.

Proposition 30. *There exists a constant C such that for any $k \geq 0$ and $y \in |K_n|$*

$$\eta^k(y) := \#\{T \in \mathsf{Face}(K_n^k) : y \in N_d^c T\} \leq C. \tag{2.46}$$

Proof. Since $c : K_n^k \rightarrow K_n^{k+1}$ is a simplicial isomorphism and a dilation in characteristic coordinates we have the following scaling relation

$$\eta^k(y) = \eta^{k+1}(cy). \tag{2.47}$$

So it suffices to prove $\eta^0(y) < C$ for all $y \in |K_n|$. We decompose this problem into two subproblems

(i) $\eta^0(y) < C$ for $y \in |N_1(v_0, K_n)|$, and

(ii) $\eta^0(y) < C$ for $y \in |K_n \setminus N_1(v_0, K_n)|$.

In case (i) we show the neighborhoods $N_d^c \Omega_j$ and $|N_1(v_0, K_n)|$ are disjoint for all j sufficiently negative, say $j \leq j'$. It then follows that $\eta^0(y) < \#\{T \in K_n : j_T > j'\}$. Let $d_0 = \mathrm{dist}(v_0, |\Omega_0|)$ be the Euclidean distance in characteristic coordinates from v_0 to the annulus Ω_0. A scaling argument shows $\mathrm{dist}(v_0, |\Omega_j|) = \lambda^j d_0$. To estimate $\mathrm{dist}(v_0, N_d^c \Omega_j)$ let

$p \in N_d^c T$ be a point which achieves the minimum distance, with T a face in $\Omega_j \subset K_n$. Then for any $t \in |T|$ we have by the triangle inequality

$$\mathrm{dist}(v_0, N_d^c \Omega_j) \geq |t| - |p - t|$$
$$\geq \mathrm{dist}(v_0, |\Omega_j|) - \mathrm{diam}_\Lambda N_d^c T$$
$$\geq d_0 \lambda^j - C_1 (2\lambda)^j \,,$$

where the last term of the last inequality comes from Corollary 27. As $j \to -\infty$ the first term dominates and approaches ∞. So for all j sufficiently negative say $j \leq j'$ we have that $\mathrm{dist}(v_0, N_d^c \Omega_j)$ exceeds $\mathrm{diam}_\Lambda |N_1(v_0, K_n)|$ proving

$$N_d^c \Omega_j \cap |N_1(v_0, K_n)| = \emptyset \,.$$

This completes the proof of case (i).

By (2.47), to prove case (ii) it is equivalent to prove

(ii') $\eta^k(y) < C$ for all $y \in |\Omega_0|$ and $k \geq 0$.

Let $\widetilde{\Omega}_0 = |N_1(v_0, K_n^1)| \cup |K_n \setminus N_4(v, K_n)| = \overline{\cup_{|j|>1} |\Omega_j|}$ be a central region of $|K_n|$ together with an annulus. Notice $|\Omega_0|$ and $\widetilde{\Omega}_0$ are disjoint, being separated by $|\Omega_1|$ and $|\Omega_{-1}|$. Let $k' \geq 0$ be such that
$$C_1 \lambda_{\max}^{k'} < \mathrm{dist}(|\Omega_0|, \widetilde{\Omega}_0) \,,$$

where C_1 is given by Corollary 27. We prove the following claim: For any face $T \in K_n^k$ such that $k \geq k'$ and $|j_T| > 1$ the neighborhoods $N_d^c T$ and $|\Omega_0|$ are disjoint. The claim follows from the estimate

$$\mathrm{dist}(|\Omega_0|, N_d^c T) \geq \mathrm{dist}(|\Omega_0|, \widetilde{\Omega}_0) - \mathrm{diam}_\Lambda N_d^c T$$
$$\geq C_1 \lambda_{\max}^{k'} - C_1 \lambda_{\max}^k > 0 \,.$$

We now show (ii'). For any $T \in K_n^k$ with $k > k'$ and $|j_T| \leq 1$ we get a lower bound on the area of $|T|$,

$$\mathrm{area}_\Lambda |T| \geq C_2 (\tfrac{1}{2})^{2k} \,, \tag{2.48}$$

by taking $d = 0$ in Proposition 26, and by Corollary 27 we have the bound

$$\text{diam}_\Lambda\, N_d^c T \le C_1 (\tfrac{1}{2})^k\,. \tag{2.49}$$

Fix $y_0 \in |\Omega_0|$ and $k > k'$. Let $B(y_0, C_1(\tfrac{1}{2})^k)$ be the ball of radius $C_1(\tfrac{1}{2})^k$ centered at y_0. Then for any $T \in \mathsf{Face}(K_n^k)$ such that $y_0 \in N_d^c T$, we have $|T| \subset |N_d T| \subset B(y_0, C_1(\tfrac{1}{2})^k)$, by (2.49). Then we have the following string of inequalities

$$\pi C_1^2 (\tfrac{1}{2})^{2k} = \text{area}_\Lambda\, B(y_0, C_1(\tfrac{1}{2})^k) \ge \sum_{|T| \subset B(y_0, C_1(\tfrac{1}{2})^k)} \text{area}_\Lambda |T|$$

$$\ge \#\{T \in \mathsf{Face}(K_n^k) : |T| \subset B(y_0, C_1(\tfrac{1}{2})^k)\}\ \cdot C_2 (\tfrac{1}{2})^{2k}\,.$$

The last inequality follows from (2.48). So

$$\eta^k(y_0) \le \#\{T \in \mathsf{Face}(K_n^k) : |T| \subset B(y_0, C_1(\tfrac{1}{2})^k)\} \le \pi \frac{C_1^2}{C_2}.$$

Since this bound is independent of k and y_0 we are done. $\qquad\square$

Next we define the final piece of geometric information that we need about the neighborhoods $N_d^c T$.

Definition 31. *A region $\Omega \subset \mathbb{R}^n$ is* **star-shaped** *with respect to a ball B if, for all $x \in \Omega$, the closed convex hull of $\{x\} \cup B$ is a subset of Ω. Suppose Ω is star-shaped with respect to some ball. Define $\rho_{max}(\Omega) = \sup\{\rho : \Omega$ is star-shaped with respect to a ball of radius $\rho\}$. Then the* **chunkiness-parameter** *of Ω, denoted by $\gamma(\Omega)$, is defined by*

$$\gamma(\Omega) = \frac{diam(\Omega)}{\rho_{max}(\Omega)}.$$

Observe that the ball, which has a chunkiness parameter of 2, has the smallest chunkiness-parameter of any domain.

Proposition 32. *There exist constants γ_0 and k', such that for any $k \ge k'$ and $T \in \mathsf{Face}(\Omega_0^k)$,*

$$\gamma_A(N_d^c T) \le \gamma_0\,.$$

The proof of this proposition depends on the following lemma.

Lemma 33. *Let $f : \Omega \to \Omega'$ be a C^2-diffeomorphism between bounded domains of $\mathbb{R}^2$. Then there is a constant R such that the image of any ball $B \subset \Omega$ of radius $r < R$ is convex. Moreover, there are constants C_1 and C_2 such that*

$$C_1 r \leq \mathit{inradius}\, f(B) \leq \mathit{circradius}\, f(B) \leq C_2 r,$$

for all such balls B, where inradius and circradius are the radii of inscribing and circumscribing balls of $f(B)$.

Proof of Proposition 32. By rotational symmetry, we need only consider faces in a single wedge $|W|$ of K_n. Let k' be such that $|N(\Omega_0 \cap W)^{k'}|$ is an affine coordinate neighborhood, and let x be an affine coordinate on it.

The transition map $x = \tau(y)$ from characteristic coordinates is C^2 on $N(\Omega_0 \cap W)^{k'}$. Let R be the constant associated with this map by Lemma 33. Choose $k'' > k'$ such that $\mathrm{diam}_\Lambda(N_d^c T) < R$ for all $k \geq k''$ and $T \in \mathsf{Face}(\Omega_0 \cap W)^k$. Then by Lemma 33, the neighborhood $N_d^c T$ is convex in affine coordinates and

$$\gamma_A(N_d^c T) \leq \frac{2\,\mathrm{circradius}\,\tau(N_d^c T)}{\mathrm{inradius}\,\tau(N_d^c T)} \leq \frac{2C_2}{C_1}.$$

$\square$

Proof of Lemma 33. To prove the existence of R, consider applying an invertible linear map J to a circle of radius r. The image is an ellipse with principle radii $\sigma_{\min}(J)r$ and $\sigma_{\max}(J)r$, where $0 < \sigma_{\min}(J) \leq \sigma_{\max}(J)$ are the singular values of J. Let

$$\sigma_{\max} = \max_{x \in N(\Omega \cap W)^{k'}} \sigma_{\max}(Df(x)),$$

where $Df(x)$ represents the Jacobian of f at x, and similarly define $\sigma_{\min}$. By expanding f as a linear map and a second-order error term, we see that, there is some $R > 0$ such that, for any ball $B \subset \Omega$ of radius $r < R$, we have

$$1/2\sigma_{\min}r \leq \mathrm{inradius}\, f(B) \leq \mathrm{circradius}\, f(B) \leq 2\sigma_{\max}r.$$

To see that the image of sufficiently small balls are convex, use the fact that a region with smooth boundary is convex if and only if the curvature of the boundary is non-negative.

$\square$

Chapter 3

GLOBAL APPROXIMATION IN SOBOLEV SPACES

In the previous chapter we defined a $C_{\text{loc}}^{R,1}$-subdivision scheme, and we showed that it induced a $C_{\text{loc}}^{R,1}$-atlas on any finite simplicial surface without boundary $|K|$. In this chapter we use the $C_{\text{loc}}^{R,1}$-atlas to define H^{R+1}-Sobolev spaces on $|K|$. Then we show that subdivision functions are in $H^2(K)$ given only that the the subdivision scheme is $C_{\text{loc}}^{1,1}$.

Also, we construct a family of linear maps

$$P^k : L^1(K) \to \mathcal{S}(K^k) \qquad \text{for all } k \geq 0, \tag{3.1}$$

and derive an asymptotic bound for the approximation error of the form

$$\|P^k f - f\|_{H^s(K)} = O(a^k) \tag{3.2}$$

for sufficiently smooth functions f. The global approximation operators P^k will be constructed from approximation operators acting on the n-regular neighborhoods. The final theorem of this chapter shows how bounds on the approximation errors in K_n can be used to derive bounds on the global approximation error.

We review Sobolev spaces on a domain of $\mathbb{R}^m$, in Section 3.1, then in Section 3.2 we extend the definition to compact simplicial surfaces. In the final section, we state and prove the global approximation theorem.

3.1 Sobolev Spaces in $\mathbb{R}^{\mathbf{n}}$

The functions in a Sobolev space have derivatives in L^p spaces. Let $\Omega \subset \mathbb{R}^n$ be a domain, i.e., an open set. The L^p norms for $1 \leq p \leq \infty$ on Ω are given by

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p dx \right)^{1/p} \qquad \text{and} \qquad \|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|.$$

We identify functions which are identical almost everywhere (with respect to Lebesgue measure), so that these are actually norms, and define $L^p(\Omega)$ to be the class of functions on Ω with finite norm. We say $f \in L^p_{\mathrm{loc}}(\Omega)$ if for each domain U compactly contained in Ω we have $f_{|U} \in L^p(U)$. The broadest class of functions we consider is $L^1_{\mathrm{loc}}(\Omega)$, which contains all other L^p spaces.

Sobolev functions possess weak derivatives. Given a function $f \in L^1_{\mathrm{loc}}(\Omega)$ and a multi-index α, we say $g \in L^1_{\mathrm{loc}}(\Omega)$ is a *weak derivative* of f if

$$\int_\Omega f(x)\, D^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega g(x)\phi(x)\, dx \qquad \text{for all } \phi \in C_0^\infty(\Omega),$$

and we write $D^\alpha f = g$. The zero subscript on a function space denotes the class of functions with compact support, e.g., the space $C_0^\infty(\Omega)$ consists of functions $f \in C^\infty(\Omega)$ such that $\mathrm{supp}\, f$ is compactly contained in Ω.

The following proposition gives a product rule for the product of a weakly differentiable function with a classically differentiable function.

Proposition 34. *Let u be a function, defined on a domain $\Omega \subset \mathbb{R}^n$, with weak derivatives $D^\alpha u$ for all $|\alpha| \leq k$, and let $\rho \in C^k(\Omega)$. Then $\rho \cdot u$ has weak derivatives of order k, and $D^\alpha(\rho u)$ is given by the product rule*

$$D^\alpha(\rho \cdot u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \rho \cdot D^{\alpha-\beta} u . \tag{3.3}$$

Proof. First consider the case $k = 1$ and $D^\alpha = \frac{\partial}{\partial x_j}$ for some $j = 1, \ldots, n$. For any $\phi \in C_0^\infty(\Omega)$ we have

$$\int \rho u \frac{\partial}{\partial x_j} \phi = \int u \left(\frac{\partial}{\partial x_j}(\rho \phi) - (\frac{\partial}{\partial x_j}\rho)\phi \right)$$

$$= -\int \left(\rho \frac{\partial}{\partial x_j} u + (\frac{\partial}{\partial x_j}\rho)u \right) \phi . \tag{3.4}$$

Now for arbitrary $k > 1$ and multi-index $|\alpha| \leq k$ we apply (3.4) a total of $|\alpha|$ times to show

$$\int \rho\, u\, D^\alpha \phi = (-1)^{|\alpha|} \int \left(\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \rho\, D^{\alpha-\beta} u \right) \phi .$$

$\square$

For any integer $s \geq 0$ and exponent $1 \leq p \leq \infty$, define the *Sobolev space* $H^{s,p}(\Omega)$ to be the class of functions $f \in L^p(\Omega)$ such that for any multi-index $|\alpha| \leq s$, the weak derivative $D^\alpha f$ exists and is in $L^p(\Omega)$. Define the $H^{s,p}(\Omega)$ semi-norms and norms by

$$|f|_{H^{s,p}(\Omega)} = \left(\sum_{|\alpha|=s} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{1/p} \quad \text{and} \quad \|f\|_{H^{s,p}(\Omega)} = \left(\sum_{m=0}^{s} |f|_{H^{m,p}(\Omega)}^p \right)^{1/p},$$

$$|f|_{H^{s,\infty}(\Omega)} = \max_{|\alpha|=s} \|D^\alpha f\|_{L^\infty(\Omega)} \quad \text{and} \quad \|f\|_{H^{s,\infty}(\Omega)} = \max_{0 \leq m \leq s} |f|_{H^{\infty,m}(\Omega)}.$$

We usually are concerned with the case when $p = 2$ and simply write $H^s(\Omega)$. We also will need the space $H^{s,p}_{\text{loc}}$ of locally $H^{s,p}$ functions and $H^{s,p}_0$ of functions in $H^{s,p}$ with compact support.

The interaction of Sobolev functions with $C^{r,1}_{\text{loc}}$-functions (C^r-functions with locally Lipschitz derivatives of order r) plays an important role, since this is the class of continuous functions defined on simplicial surfaces in Section 2.6. The next proposition can be found in Evans and Gariepy [8], Section 4.2.3.

Proposition 35. *The function $f : \Omega \to \mathbb{R}$ is locally Lipschitz if and only if $f \in H^{1,\infty}_{\text{loc}}(\Omega)$.*

Thus we have an identification of $C^{r,1}_{\text{loc}}(\Omega)$ with $H^{r+1,\infty}_{\text{loc}}(\Omega)$. The product rule and chain rule hold almost everywhere for these functions, as we show in the next three propositions.

Proposition 36. *Suppose $\tau : \Omega \to \Omega' \subset \mathbb{R}^m$ is a $C^{k,1}_{\text{loc}}(\Omega)$ map for some $k \geq 1$ and $f \in C^\infty(\Omega')$ then $f \circ \tau$ has weak derivatives of order $k + 1$ and satisfies the chain rule:*

$$D^\alpha(f \circ \tau) = \sum_{1 \leq |\beta| \leq |\alpha|} \left(D^\beta f \circ \tau \right) \cdot M\tau_{\alpha\beta} \tag{3.5}$$

where $M\tau_{\alpha\beta} \in L^\infty_{\text{loc}}(\Omega)$ is given by (2.38), as in the chain rule for smooth functions.

Proof. For any multi-index $|\alpha| = k$ the chain rule (3.5) applies by the classical rules of differentiation. We will be done if we can show an order 1 derivative of each term in the resulting series is given by an application of the product rule and chain rule. That is we must show that formally applying the product rule and chain rule to each term in the sum gives the weak derivative of that term. Notice, in the case $|\alpha| = k$, for any β we have that $D^\beta f \circ \tau \in C^k(\Omega)$, and $M\tau_{\alpha\beta} \in C^1(\Omega)$ except for the terms when $|\beta| = 1$ in which case by (2.38) we have $M\tau_{\alpha\beta} = D^\alpha \tau^\beta$, which is locally Lipschitz.

By Proposition 35 the term $D^\alpha \tau^\beta$ has weak derivatives of order 1. Therefore by Proposition 34 the term $(D^\beta f \circ \tau) \cdot D^\alpha \tau^\beta$ has weak derivatives of order 1 given by the product rule. $\square$

Proposition 37. *Let $\tau : \Omega \to \Omega'$ be a $C^{r,1}_{\mathrm{loc}}$-diffeomorphism between domains in $\mathbb{R}^n$ for some $r \geq 1$. For any $f \in H^{r+1}(\Omega')$ the composition $f \circ \tau$ has weak derivatives $D^\alpha f$ for all $|\alpha| \leq r + 1$, and the weak derivatives are given by the chain rule*

$$D^\alpha(f \circ \tau) = \sum_{1 \leq |\beta| \leq |\alpha|} (D^\beta f \circ \tau) \cdot M\tau_{\alpha\beta},$$

where $M\tau_{\alpha\beta}$ is given by (2.38), as in the chain rule for smooth functions.

Proof. For any $f \in H^{r+1}(\Omega')$ there is a sequence $\{f_n\}$ of functions in $C^\infty(\Omega')$ converging to f in the $H^{r+1}(\Omega')$-norm.

Using the change of coordinates $x = \tau(y)$ and that f_n converges to f in $L^2(\Omega')$ we compute as follows: For any $\phi \in C_0^\infty(\Omega)$ and any multi-index α such that $1 \leq |\alpha| \leq r + 1$ we have

$$\begin{aligned}
\int_\Omega (f \circ \tau)(y)\, D^\alpha \phi(y)\, dy &= \int_{\Omega'} f(x)\, D^\alpha \phi(\tau^{-1}x) \left| \det \frac{\partial \tau^{-1}}{\partial x} \right| dx \\
&= \lim_{n \to \infty} \int_{\Omega'} f_n(x)\, D^\alpha \phi(\tau^{-1}x) \left| \det \frac{\partial \tau^{-1}}{\partial x} \right| dx \\
&= \lim_{n \to \infty} \int_\Omega (f_n \circ \tau)(y)\, D^\alpha \phi(y)\, dy.
\end{aligned}$$

We now apply Proposition 36 and continue the computation using that $D^\beta f_n \to D^\beta f$ in $L^2(\Omega')$ for all $|\beta| \leq r + 1$

$$\begin{aligned}
\int_\Omega (f \circ \tau)(y)\, D^\alpha \phi(y)\, dy &= (-1)^{|\alpha|} \lim_{n \to \infty} \int_\Omega \sum_{1 \leq |\beta| \leq |\alpha|} D^\beta f_n(\tau y)\, M\tau_{\alpha\beta}(y)\, \phi(y)\, dy \\
&= (-1)^{|\alpha|} \lim_{n \to \infty} \int_{\Omega'} \sum_{1 \leq |\beta| \leq |\alpha|} D^\beta f_n(x)\, M\tau_{\alpha\beta}(\tau^{-1}x)\, \phi(\tau^{-1}x) \left| \det \frac{\partial \tau^{-1}}{\partial x} \right| dx \\
&= (-1)^{|\alpha|} \int_{\Omega'} \sum_{1 \leq |\beta| \leq |\alpha|} D^\beta f(x)\, M\tau_{\alpha\beta}(\tau^{-1}x)\, \phi(\tau^{-1}x) \left| \det \frac{\partial \tau^{-1}}{\partial x} \right| dx \\
&= (-1)^{|\alpha|} \int_\Omega \sum_{1 \leq |\beta| \leq |\alpha|} (D^\beta f \circ \tau)(y)\, M\tau_{\alpha\beta}(y)\, \phi(y)\, dy.
\end{aligned}$$

$\square$

Proposition 38. *Let $u \in C^{r,1}_{\mathrm{loc}}(\Omega)$ and $f \in H^{r+1}(\Omega)$ then $f \cdot u$ has weak derivatives $D^\alpha(f \cdot u)$ for all $|\alpha| \leq r+1$ and they are given by the product rule (3.3).*

Proof. Let $\{f_n\}$ be a sequence of functions in $C^\infty(\Omega)$ converging to f in the $H^{r+1}(\Omega)$-norm. Then $D^\beta f_n \to D^\beta f$ in $L^2(\Omega)$ for any $|\beta| \leq r+1$. So for any $|\alpha| \leq r+1$ and $\phi \in C^\infty(\Omega)$ we have

$$
\begin{aligned}
\int f\, u\, D^\alpha \phi &= \lim_{n \to \infty} \int f_n\, u\, D^\alpha \phi \\
&= \lim_{n \to \infty} \int \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta f_n\, D^{\alpha-\beta} u\, \phi \qquad \text{by Prop. 34} \\
&= \int \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta f\, D^{\alpha-\beta} u\, \phi \,.
\end{aligned}
$$

$\square$

To construct Sobolev spaces on surfaces we need two operations: multiplication and change of coordinates. The key propositions are given next. For cases where the multiplicand and change of coordinate functions are in C^{r+1}, these results are standard and can be found in Adams [1]. Here we use $C^{r,1}_{\mathrm{loc}}$ functions instead.

Proposition 39. *Given $\rho \in C^{r,1}_0(\Omega)$ there exists a constant C such that*

$$
\|\rho \cdot f\|_{H^{r+1}(\Omega)} \leq C\|f\|_{H^{r+1}(\Omega)} \qquad \text{for all } f \in H^{r+1}(\Omega). \tag{3.6}
$$

Proof. By the product rule, Proposition 38, we have

$$
D^\alpha(\rho \cdot f) = \sum_{\beta \leq \alpha} \binom{\beta}{\alpha} D^{\alpha-\beta} \rho \cdot D^\beta f \qquad \text{for any } |\alpha| \leq r+1.
$$

Applying Hölder's inequality, we see that each term is in $L^2(\Omega)$, and moreover

$$
\begin{aligned}
\|D^\alpha \rho \cdot f\|_{L^2(\Omega)} &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \|D^{\alpha-\beta} \rho\|_{L^\infty(\Omega)} \cdot \|D^\beta f\|_{L^2(\Omega)} \\
&\leq C\|\rho\|_{H^{r+1,\infty}(\Omega)} \sum_{\beta \leq \alpha} \|D^\beta f\|_{L^2(\Omega)} \\
&\leq C\|\rho\|_{H^{r+1,\infty}(\Omega)} \|f\|_{H^{r+1}(\Omega)}.
\end{aligned}
$$

Summing over $|\alpha| \leq r+1$, we get (3.6). $\square$

Now we examine a change of coordinate formula for Sobolev functions. Suppose $\tau : \Omega \to \Omega'$ is a $C^{r,1}(\overline{\Omega})$-diffeomorphism between domains in $\mathbb{R}^n$ with $r \geq 1$. That is, for all $|\alpha| \leq r$, the partial derivative $D^\alpha \tau$ is bounded and uniformly continuous on Ω, and for all $|\alpha| = r$, the partial derivative $D^\alpha \tau$ is Lipschitz continuous on Ω. Proposition 24 shows that τ^{-1} is also of class $C^{r,1}(\overline{\Omega'})$.

Proposition 40. *Suppose $\tau : \Omega \to \Omega'$ is a $C^{r,1}(\overline{\Omega})$-diffeomorphism between domains in $\mathbb{R}^n$. There exists a constant C such that for any domains $A \subset \Omega$ and $A' = \tau(A) \subset \Omega'$, we have*

$$1/C \, \|f \circ \tau\|_{H^{r+1}(A)} \leq \|f\|_{H^{r+1}(A')} \leq C \, \|f \circ \tau\|_{H^{r+1}(A)} \qquad \text{for all } f \in H^{r+1}(A'). \tag{3.7}$$

The constant $C = C(\Omega, \tau)$ is independent of A.

Proof. It suffices to prove the leftmost inequality of (3.7). Indeed the rightmost inequality follows by considering the $C^{r,1}(\overline{\Omega'})$-diffeomorphism $\tau^{-1} : \Omega' \to \Omega$.

We first consider the L^2-norm. For any $A \subset \Omega$ and $f \in L^2(A')$ we have

$$\|f \circ \tau\|_{L^2(A)}^2 = \int_A f(\tau y)^2 \, dy$$
$$= \int_{A'} f(x)^2 \left| \det \frac{\partial \tau^{-1}}{\partial x}(x) \right| dx$$

Since $\tau \in C^1(\overline{\Omega})$ there is a constant C such that

$$\left| \det \frac{\partial \tau^{-1}}{\partial x}(x) \right| \leq C \qquad \text{for all } x \in \Omega.$$

Thus we have the inequality

$$\|f \circ \tau\|_{L^2(A)} \leq C \|f\|_{L^2(A')}, \tag{3.8}$$

where the constant C is independent of A.

Now we consider Sobolev norms. For any $|\alpha| \leq r + 1$ and $f \in H^{r+1}(A)$ we have by Proposition 37 that the weak derivative $D^\alpha(f \circ \tau)$ is given by (3.5), where $M\tau_{\alpha\beta} \in L^\infty(\Omega)$.

Thus

$$\|D^\alpha (f \circ \tau)\|_{L^2(A)} \leq \sum_{1 \leq |\beta| \leq |\alpha|} \|(D^\beta f \circ \tau)(M\tau_{\alpha\beta})\|_{L^2(A)}$$

$$\leq \sum_{1 \leq |\beta| \leq |\alpha|} \|M\tau_{\alpha\beta}\|_{L^\infty(\Omega)} \|D^\beta f \circ \tau\|_{L^2(A)}$$

$$\leq C \sum_{1 \leq |\beta| \leq |\alpha|} \|D^\beta f\|_{L^2(A')} \leq C\|f\|_{H^{r+1}(A)}.$$

In the last line we have used (3.8). Notice the constant C is independent of A. Summing over all $|\alpha| \leq r + 1$ completes the proof. □

The following proposition gives an identity for the H^m-semi-norm of a dilated function.

Proposition 41. *Suppose Ω is a domain in $\mathbb{R}^2$. Let $c(x) = \lambda x$ be a dilation map on $\mathbb{R}^2$, with pull-back c^*. Then for any $f \in H^m(\Omega)$, we have*

$$|c^* f|_{H^m(c^{-1}\Omega)} = \lambda^{m-1} |f|_{H^m(\Omega)}.$$

Proof. Differentiating we get $D^\alpha c^* f = \lambda^{|\alpha|} c^* D^\alpha f$ for any multi-index α. So

$$|c^* f|^2_{H^m(c^{-1}\Omega)} = \sum_{|\alpha|=m} \int_{c^{-1}\Omega} (D^\alpha c^* f)^2 \, dx$$

$$= \lambda^{2m} \sum_{|\alpha|=m} \int_{c^{-1}\Omega} (D^\alpha f(\lambda x))^2 \, dx$$

$$= \lambda^{2(m-1)} \sum_{|\alpha|=m} \int_{\Omega} (D^\alpha f(x))^2 \, dx$$

$$= \lambda^{2(m-1)} |f|^2_{H^m(\Omega)}.$$

□

3.2 Sobolev Spaces on Simplicial Surfaces

We define a Sobolev norm on a finite simplicial surface K without boundary. Suppose $\mathcal{S}$ is a $C_{\text{loc}}^{R,1}$-subdivision scheme. We use an atlas of coordinate charts and a subordinate partition of unity to decompose a function into finitely many pieces, each compactly supported on

some n-regular complex. This construction follows the construction of Sobolev spaces on a compact manifold as found in Taylor [21].

The Sobolev norm on $|K|$ is defined in terms of an atlas and a partition of unity. The Sobolev norms on the n-regular complex $|K_n|$ is defined using a characteristic map χ_n as a global coordinate chart. That is, for every $\Omega \subset |K_n|$ we define

$$\|f\|_{H_\Lambda^s(\Omega)} = \|f \circ \chi_n^{-1}\|_{H^s(\chi_n(\Omega))}. \tag{3.9}$$

For each vertex $v \in K$ of valence n, we fix a characteristic coordinate chart $\psi_v = \chi_n \circ \iota_v : |N_1(v)|^\circ \to \mathbb{R}^2$, as in (2.25). Define a central subcomplex of K_n by

$$\widehat{K}_n = N_3(v_0, K_n^2) \qquad \text{for each valence } n, \tag{3.10}$$

and let $\widehat{U}_v = \iota_v^{-1}(|\widehat{K}_n|^\circ)$ be the corresponding open neighborhood of v. Let ρ_v be a $C_{\text{loc}}^{R,1}$-partition of unity, subordinate to the cover $\{\widehat{U}_v\}$, and define the *cut-off representative* of f near v to be $f_v \in C(K_n)$, given by

$$(\rho_v \cdot f) \circ \iota_v^{-1}, \tag{3.11}$$

where we extend by zero to the rest of K_n. Then we define the global Sobolev norm on $|K|$ for each non-negative integer $s \leq R + 1$ by

$$\|f\|_{H^s(K)} = \sum_{v \in K} \|f_v\|_{H_\Lambda^s(K_n)}. \tag{3.12}$$

Using Propositions 39 and 40, one can show that the definition of the norm is independent, up to equivalence, of the choice of atlas and subordinate partition of unity. That is, if $\{\widetilde{\psi}_\alpha, U_\alpha\}$ is another finite atlas and $\widetilde{\rho}_\alpha$ is a subordinate partition of unity, then the norm defined by

$$\|f\| = \sum_\alpha \|(\widetilde{\rho}_\alpha \cdot f) \circ \widetilde{\psi}_\alpha^{-1}\|_{H^s}$$

is equivalent to the $H^s(K)$ norm defined in (3.12).

We now extend Proposition 17, showing that subdivision functions are in $H^2(K)$.

Theorem 42. *Suppose $\mathcal{S}$ is a $C_{\text{loc}}^{1,1}$-subdivision scheme. Then all subdivision functions $\mathcal{S}(K)$ on K are in $H^2(K)$.*

Proof. Fix $u \in \mathsf{CN}(K)$ and let $f = \mathcal{S}^\infty u$. We want to show that $\|f \circ \iota_v^{-1}\|_{H^p_\Lambda(\widehat{K}_n)}$ is finite for each vertex $v \in K$ of valence n. It follows that $f \in H^2(K)$, since

$$\|f\|_{H^2(K)} = \sum_{v \in K} \|(\rho \cdot f) \circ \iota_v^{-1}\|_{H^2_\Lambda(K_n)} \leq C \sum_{v \in K} \|f \circ \iota_v^{-1}\|_{H^2_\Lambda(\widehat{K}_n)} < \infty \,,$$

by Proposition 39.

At a sufficiently fine level, $f \circ \iota_v^{-1}$ is a subdivision function on $\widehat{K}_n$. Let $k_0 \geq 3$ be such that $2^{k_0} - m_w + 1 \geq 3 \cdot 2^{k_0-2}$, where m_w is the mask width of the subdivision scheme. Let $\widetilde{u} = \iota_v^{-1*} u \in \mathsf{CN}(N_{2^{k_0}}(v_0, K_n^{k_0}))$, then by Proposition 8 we have

$$f \circ \iota_v^{-1} = \iota_v^{-1*} f = \iota_v^{-1*} \mathcal{S}^\infty u^{k_0} = \mathcal{S}^\infty \iota_v^{-1*} u^{k_0} = \mathcal{S}^\infty \widetilde{u} \qquad \text{on } |\widehat{K}_n|.$$

Let $\Omega_0 = N_{3 \cdot 2^{k_0-2}}(v_0, K_n^{k_0}) \setminus N_{3 \cdot 2^{k_0-3}}(v_0, K_n^{k_0})$ be a fundamental annulus in $K_n^{k_0}$. Let $\Omega_j = c^j \Omega_0 \subset K_n^{k_0+j}$, then the annular regions $\{|\Omega_j| : j \geq 0\}$ have disjoint interiors and cover $\widehat{K}_n \setminus \{v_0\}$. So we have

$$|f|^2_{H^m_\Lambda(\widehat{K}_n)} = \sum_{j=0}^\infty |f|^2_{H^m_\Lambda(\Omega_j)} \,. \tag{3.13}$$

A subdivision function on $N_{3 \cdot 2^{k_0-2}}(v_0, K_n^{k_0})$ is determined by a control net in

$$V = \mathsf{CN}(N_{3 \cdot 2^{k_0-2}+m_w-1}(v_0, K_n^{k_0})).$$

Let $S = S_{n, 3 \cdot 2^{k_0-2}+m_w-1}$ be the subdivision matrix acting on V, with distinct eigenvalues $1 = \lambda_0 > \lambda_1 > |\lambda_2| \geq \cdots \geq |\lambda_N|$.

For each $m = 0, 1$ or 2, we decompose V into a pair of S-invariant subspaces. For $m = 1$ or 2, let T_m be spanned by eigenvectors corresponding to the λ_0 through λ_{m-1} eigenvalues of S, and let $T_0 = \{0\}$ be the trivial subspace. Let V_m be spanned by generalized eigenvectors corresponding to the λ_m through λ_N eigenvalues of S. Then for each $m = 0, 1$ or 2 we have a decomposition of V

$$V = T_m \oplus V_m \tag{3.14}$$

and a corresponding decomposition of S

$$S = S_T \oplus S_m \,. \tag{3.15}$$

Furthermore, the spectral radius of S_m is given by

$$\rho(S_m) = |\lambda_m|. \tag{3.16}$$

From Theorem 11, it follows that: (i) The λ_0-eigenspace of S is 1-dimensional, containing constant control nets which generate constant valued subdivision functions, and (ii) The λ_1-eigenspace of S is a 2-dimensional space of control nets which generate components of a characteristic maps , i.e., they generate linear functions in characteristic coordinates. So $\mathcal{S}(T_m)$ are functions which are trivial in the $H^m_\Lambda(\widehat{K}_n)$-semi-norm; i.e.,

$$|\mathcal{S}^\infty \widetilde{u}|_{H^m_\Lambda(\widehat{K}_n)} = 0 \qquad \text{for } \widetilde{u} \in T_m \text{ and } m = 0, 1 \text{ or } 2. \tag{3.17}$$

Continuing, we prove the claim

$$|\mathcal{S}^\infty \widetilde{u}|_{H^m_\Lambda(\Omega_j)} = \lambda_1^{-(m-1)j} |\mathcal{S}^\infty S^j \widetilde{u}|_{H^m_\Lambda(\Omega_0)} \tag{3.18}$$

for any $\widetilde{u} \in V$, $j \geq 0$, and $m = 0, 1$ or 2. Since $S^j = (c^*\mathcal{S})^j = c^{*j}\mathcal{S}^j$, we have

$$
\begin{aligned}
|\mathcal{S}^\infty S^j \widetilde{u}|_{H^m_\Lambda(\Omega_0)} &= |\mathcal{S}^\infty c^{*j} \mathcal{S}^j \widetilde{u}|_{H^m_\Lambda(\Omega_0)} \\
&= |c^{*j} \mathcal{S}^\infty \widetilde{u}|_{H^m_\Lambda(\Omega_0)} && \text{by Prop. 8} \\
&= \lambda_1^{(m-1)j} |\mathcal{S}^\infty \widetilde{u}|_{H^m_\Lambda(c^j\Omega_0)} && \text{by Prop. 41,}
\end{aligned}
$$

proving (3.18).

Notice that by the smoothness hypothesis for the subdivision scheme, $\mathcal{S}^\infty \widetilde{u} \in C^{1,1}(\Omega_0) \subset H^2_\Lambda(\Omega_0)$ for any $\widetilde{u} \in V$. So for any norm on V, the H^m-operator norm for $\mathcal{S}^\infty$ satisfies

$$|\mathcal{S}^\infty \widetilde{u}|_{H^m_\Lambda(\Omega_0)} \leq |\mathcal{S}^\infty|_{H^m} \|\widetilde{u}\|. \tag{3.19}$$

We now combine (3.13), (3.18) and (3.19). For any $\widetilde{u} \in V$ and $m = 0, 1$ or 2, we apply the decomposition (3.14) and (3.15) to get $\widetilde{u} = \widetilde{u}_T + \widetilde{u}_m$ and $S^j \widetilde{u} = S_T^j \widetilde{u}_T + S_m^j \widetilde{u}_m$. By the triviality of T_m (3.17) we have

$$
\begin{aligned}
|\mathcal{S}^\infty S^j \widetilde{u}|_{H^m_\Lambda(\Omega_0)} &= |\mathcal{S}^\infty S_m^j \widetilde{u}_m|_{H^m_\Lambda(\Omega_0)} \\
&\leq |\mathcal{S}^\infty|_{H^m} \|S_m\|^j \|\widetilde{u}_m\|, \tag{3.20}
\end{aligned}
$$

where $\|S_m\|$ is the operator norm of S_m. Then applying (3.13) and (3.18) we have

$$
\begin{aligned}
|\mathcal{S}^\infty \widetilde{u}|^2_{H_\Lambda^m(N_{2L+2})} &= \sum_{j=0}^\infty |\mathcal{S}^\infty \widetilde{u}|^2_{H_\Lambda^m(\Omega_j)} \\
&= \sum_{j=0}^\infty \lambda_1^{-2(m-1)j} |\mathcal{S}^\infty S^j \widetilde{u}|^2_{H_\Lambda^m(\Omega_0)} \\
&\leq |\mathcal{S}^\infty|^2_{H^m} \|\widetilde{u}_m\|^2 \sum_{j=0}^\infty \left(\lambda_1^{-(m-1)} \|S_m\| \right)^{2j}.
\end{aligned}
\tag{3.21}
$$

For any $\epsilon > 0$ we can choose a vector norm on V such that $\|S_m\| \leq \rho(S_m) + \epsilon$. So by (3.16), the parenthesized factor in (3.21) satisfies the bound

$$
\lambda_1^{-(m-1)} \|S_m\| \leq \begin{cases} (1+\epsilon)\lambda_1 & m = 0 \\ (\lambda_1 + \epsilon) & m = 1 \\ (|\lambda_2| + \epsilon)/\lambda_1 & m = 2 \end{cases}
$$

for any $\epsilon > 0$. In particular, ϵ can be chosen so that these factors are always less than 1 since, $1 > \lambda_1 > |\lambda_2|$. Therefore the geometric series in (3.21) converges and $\mathcal{S}^\infty \widetilde{u} \in H_\Lambda^2(\widehat{K}_n)$. $\qquad\square$

3.3 A Global Approximation Theorem

Consider a family of linear operators

$$
\mathcal{Q} = \{Q^k : L^1_{\Lambda,\mathrm{loc}}(K_n) \to \mathcal{S}(K_n^k) \qquad \text{for } k \geq 0\}.
\tag{3.22}
$$

The family $\mathcal{Q}$ is said to be *local* if there is an integer d called the *support width* such that the following condition holds for every subcomplex $K \subset K_n^k$:

$$
\text{If } f \equiv 0 \text{ on } |N_d K|, \quad \text{then} \quad Q^k f \equiv 0 \text{ on } |K|.
\tag{3.23}
$$

The global approximation $P^k f$ is constructed from the cut-off representatives f_v as in (3.11) by

$$
P^k f = \sum_{v \in K} \iota_v^* Q^k f_v,
\tag{3.24}
$$

where $\iota_v^* Q^k f_v$ is extended by zero outside of $N_1(v, K)$.

The asymptotic bounds in (3.2) for the case of box splines on a Euclidean domain are $a^k = (1/2)^{pk}$. In our analysis, the role of the base $\frac{1}{2}$ is played by the sub-dominant eigenvalues of the subdivision matrix. Let $\lambda_{\max}(K_n)$ be the sub-dominant eigenvalue of the subdivision matrix S_n or $\frac{1}{2}$, whichever is greater. For the simplicial surface K, let

$$\lambda_{\max}(K) = \max\{\lambda_{\max}(K_n) : n \text{ is the valence of a vertex } v \text{ in } K\}. \tag{3.25}$$

Let N_1 be a neighborhood of $\widehat{K_n}$ in K_n, and suppose for each valence which is represented in K, we have a local estimate

$$\|Q^k f - f\|_{H^s_\Lambda(\widehat{K_n})} \leq C_\epsilon \lambda_{\max}^{(r-s-\epsilon)k} \|f\|_{H^r_\Lambda(N_1)} \qquad \text{for all } f \in H^r_\Lambda(N_1), \tag{3.26}$$

where $\lambda_{\max} = \lambda_{\max}(K_n)$.

Theorem 43. *Suppose for each valence n represented in the complex K, there is a family $\mathcal{Q}$ of local approximation operators as in (3.22), with support width d_n, that satisfy the estimate (3.26). Then the global approximation operator P^k given by (3.24) satisfies the estimate*

$$\|P^k f - f\|_{H^s(K)} \leq C_\epsilon \lambda_{\max}^{(r-s-\epsilon)k} \|f\|_{H^r(K)} \quad \text{for all } f \in H^r(K), \tag{3.27}$$

where $\lambda_{\max} = \lambda_{\max}(K)$ and k is sufficiently large. In particular, (3.27) is satisfied for all $k \geq k_0$, where k_0 satisfies $2^{k_0-2} - m_w - d + 1 \geq 0$ and $d = \max\{d_n : v \text{ is of valence } n \text{ for some } v \in K\}$.

Proof. Fix $f \in H^r(K)$, then by the definition of $H^r(K)$ we have $f_v \in H^r_\Lambda(K_n)$ for each vertex $v \in K$ and $\text{supp } f_v \subset |\widehat{K_n}|$.

We first show that the pull back of a local approximation $\iota_v^* Q^k f_v$ is a subdivision function on K^k. (Recall, we extend the function by zero outside $|N_1(v, K)|$). By linearity, this shows that the global approximation $P^k f$ given by (3.24) is a subdivision function. Let $u \in \mathsf{CN}(K_n)$ be such that $Q^k f_v = \mathcal{S}^\infty u$. We show that $\iota_v^* Q^k f_v = \mathcal{S}^\infty \iota^* u$ on all of $|K|$, where we extend the control net $\iota^* u$ by zero so that $\iota^* u \in \mathsf{CN}(K^k)$. Now

$$\text{supp } f_v \subset |\widehat{K_n}| = |N_{3 \cdot 2^{k-2}}(v_0, K_n^k)|.$$

Hence by locality of the local approximation,

$$\text{supp } u \subset N_{3 \cdot 2^{k-2}+d-m_w}(v_0, K_n^k).$$

Therefore

$$\text{supp } \mathcal{S}^\infty \iota^* u = |N_{3 \cdot 2^{k-2}+d}(v, K^k)|. \tag{3.28}$$

By Proposition 8 we have

$$\iota^* \mathcal{S}^\infty u = \mathcal{S}^\infty \iota^* u \qquad \text{on } N_{2^k - m_w + 1}(v, K^k).$$

Finally, we see that the hypothesis on k_0 implies that there is an inclusion

$$N_{3 \cdot 2^{k-2}+d}(v, K^k) \subset N_{2^k - m_w + 1}(v, K^k) \qquad \text{for all } k \geq k_0.$$

Thus, the functions $i_v^* Q^k f_v$ and $\mathcal{S}^\infty \iota^* u$ are identical on their supports, and hence identical on all of $|K|$.

We now prove the global error estimate (3.27). By definition (3.12) of the Sobolev norm,

$$\|P^k f - f\|_{H^s(K)} = \sum_{v \in K} \|\rho_v \cdot (P^k f - f) \circ \iota_v^{-1}\|_{H^s_\Lambda(K_n)}. \tag{3.29}$$

By (3.24) and the partition of unity, we can write

$$P^k f - f = \sum_{w \in K} (Q^k f_w - f_w) \circ \iota_w.$$

Thus, substituting into (3.29) and using the triangle inequality we get

$$\|P^k f - f\|_{H^s(K)} \leq \sum_{v,w \in K} \|\rho_v \cdot (Q^k f_w - f_w) \circ \iota_w \circ \iota_v^{-1}\|_{H^s_\Lambda(K_n)}$$

$$\leq \sum_{v \in K} \|\rho_v \cdot (Q^k f_v - f_v)\|_{H^s_\Lambda(K_n)} +$$

$$\sum_{(v,w) \in K} \|\rho_v \cdot (Q^k f_w - f_w) \circ \iota_w \circ \iota_v^{-1}\|_{H^s_\Lambda(K_n)}. \tag{3.30}$$

Because $\rho_v \cdot (Q^k f_w - f_w)$ is supported in $|N_1(v)|^\circ \cap |N_1(w)|^\circ$, applying Proposition 40 yields the inequality

$$\|\rho_v \cdot (Q^k f_w - f_w) \circ \iota_w \circ \iota_v^{-1}\|_{H^s_\Lambda(K_n)} \leq C_{vw} \|\rho_v \cdot (Q^k f_w - f_w)\|_{H^s_\Lambda(K_n)}. \tag{3.31}$$

Applying Proposition 39 to the inequality (3.31) then gives

$$\|\rho_v \cdot (Q^k f_v - f_v)\|_{H^s_\Lambda(K_n)} \leq \|Q^k f_v - f_v\|_{H^s_\Lambda(\widehat{K_n})}. \tag{3.32}$$

Substituting (3.31) and (3.32) into (3.30) yields the inequality

$$\|P^k f - f\|_{H^s(K)} \leq C \sum_{v \in K} \|Q^k f_v - f_v\|_{H^s_\Lambda(\widehat{K_n})},$$

which, together with the local estimate (3.26), gives

$$\|P^k f - f\|_{H^s(K)} \leq C_\epsilon \lambda_{\max}^{(r-s-\epsilon)k} \sum_{v \in K} \|f_v\|_{H^r_\Lambda(N_1)} = C_\epsilon \lambda_{\max}^{(r-s-\epsilon)k} \|f\|_{H^r(K)},$$

concluding the proof. $\square$

Constructing a family of approximation operators satisfying (3.26) and applying Theorem 43 for Loop's subdivision scheme yields the main result of this thesis. Recall Theorem 25 states that Loop's subdivision scheme is a $C^{2,1}_{\mathrm{loc}}$-scheme. We show in Chapter 4 that the required estimate (3.26) follows from the existence of a quasi-interpolant on K_n with certain polynomial reproducing properties. In Chapter 5 we construct a quasi-interpolant for each K_n for Loop's subdivision scheme. In particular, applying the result of Chapter 4, this shows that the estimate (3.26) holds for any $s < r \leq 3$. Therefore once we have these results we will have proved the main result of this thesis which we restate here.

Theorem 2. *Let $\mathcal{S}(K^k)$ be the space of Loop's subdivision functions on the k-times subdivided complex K^k. For integers $0 \leq s < r \leq 3$ and any $\epsilon > 0$ we have the following bound on the minimal $H^s(K)$-approximation error of a function $f \in H^r(K)$:*

$$\mathrm{dist}(f, \mathcal{S}(K^k))_{H^s(K)} \leq C_\epsilon \lambda_{\max}^{(r-s-\epsilon)k} \|f\|_{H^r(K)},$$

where the constant $C_\epsilon = C(\epsilon, K)$ is independent of k and f.

Chapter 4

APPROXIMATION THEORY ON THE n–REGULAR COMPLEX

In the previous chapter we proved a global approximation theorem for subdivision functions, Theorem 43. For each valence n and level $k > 0$, suppose we have a local, linear operator Q^k that approximates a function $f \in H^r(K_n)$ on the n-regular neighborhood K_n by a subdivision function $Q^k f \in \mathcal{S}(K_n^k)$. Furthermore, suppose the H^s-approximation error satisfies the estimate (3.26), which shows $\|Q^k f - f\|_{H^s} = O(\lambda^{(r-s-\epsilon)k})$, where $\lambda = \lambda_{\max}(K_n)$ and the H^s-norm is computed on some fixed bounded domain in characteristic coordinates. In Section 3.3 we used the operators Q^k to construct a global approximation operator P^k that approximates a function $f \in H^r(K)$ defined on a simplicial surface K by a subdivision function $P^k f \in \mathcal{S}(K^k)$. Theorem 43 shows that $\|P^k f - f\|_{H^s} = O(\lambda^{(r-s-\epsilon)k})$, where $\lambda = \lambda_{\max}(K)$. In this chapter, we define the properties of a quasi-interpolant on K_n and show that the approximation error of a quasi-interpolant satisfies the bound (3.26).

A quasi-interpolant on K_n is a generalization of a standard technique for proving approximation results on $\mathbb{R}^m$. We begin with an introductory example of quartic triangular splines in $\mathbb{R}^2$, demonstrating that the existence of a quasi-interpolant of order r implies an approximation power of at least order r. The quasi-interpolant technique is well known, see for instance [20, 3, 6].

Let $\mathcal{S}(K_6^k)$ be the space of quartic triangular splines on the subdivided regular grid K_6^k. As shown in Kowalski [12], the approximation power of quartic splines is demonstrated by constructing a family of local linear operators $Q^k : H^1(\mathbb{R}^2) \to \mathcal{S}(K_6^k)$, called a quasi-interpolant. We will show

$$\|Q^k f - f\|_{H^s} \leq C(\tfrac{1}{2})^{(r-s)k}\|f\|_{H^r} \qquad \text{for } 0 \leq s < r \leq 4. \tag{4.1}$$

The key properties of a quasi-interpolant are locality, boundedness and polynomial reproduction. Suppose Q^k is local with support width d as defined in (3.23). By boundedness,

we mean that there exists a constant $\|Q\|$ such that $\|Q^k f\|_{H^s(T)} \leq \|Q\| \|f\|_{H^s(N_d T)}$ for any level k and any face $T \in K_6^k$. To get fourth-order approximation power as in (4.1), we require that all polynomials of degree less than 4 are reproduced by Q^k; i.e., $Q^k g = g$ for any $g \in \mathbb{P}_3$ and $k \geq 0$.

To prove (4.1) we estimate the approximation error on a face $T \in K_6^k$. Let $g \in \mathbb{P}_{r-1}$ be a polynomial approximating f on $N_d T$. Then by the triangle inequality we have

$$\|Q^k f - f\|_{H^s(T)} \leq \|Q^k f - Q^k g\|_{H^s(T)} + \|Q^k g - g\|_{H^s(T)} + \|f - g\|_{H^s(T)}.$$

Applying locality, boundedness and polynomial reproduction we get

$$\|Q^k f - f\|_{H^s(T)} \leq (1 + \|Q\|) \|f - g\|_{H^s(N_d T)}.$$

To complete the proof of (4.1) we need an estimate of the error made in locally approximating f by a polynomial. If f were in class C^r, then we could take the approximation g as a degree $r-1$ Taylor polynomial of f, and by a simple estimate from the remainder in Taylor's Theorem we could conclude

$$\|f - g\|_{H^s(N_d T)} \leq C (\operatorname{diam} N_d T)^{r-s} \sup_{N_d T} |D^r f|.$$

We show, in Section 4.2, that an averaged version of a Taylor polynomial is well defined for a weakly differentiable function. The Bramble-Hilbert Lemma gives the bound

$$\|f - g\|_{H^s(N_d T)} \leq C (\operatorname{diam} N_d T)^{r-s} |f|_{H^r(N_d T)}.$$

The neighborhoods $N_d T$ are all similar and satisfy $\operatorname{diam}(N_d T) = C(\frac{1}{2})^k$ for some constant C. By summing the local errors we then get

$$\begin{aligned}
\|Q^k f - f\|_{H^s}^2 &= \sum_{T \in K_6^k} \|Q^k f - f\|_{H^s(T)}^2 \\
&\leq C(\tfrac{1}{2})^{2(4-s)k} \sum_{T \in K_6^k} |f|_{H^4(N_d T)}^2 .
\end{aligned} \tag{4.2}$$

Any face $T_0 \in K_6^k$ is summed over many times in the sum above, and the number of times is independent of T_0. Therefore, the sum on the right of (4.2) is bounded by $C|f|_{H^4}^2$, proving (4.1).

We want to apply this technique to the problem of approximation on K_n by subdivision functions. Specifically, we want to identify the properties of a quasi-interpolant so that we can prove the approximation error bound (3.26). The main challenge we face is reproducing polynomials. The norms in (3.26) are computed in characteristic coordinates. Let $\mathbb{P}_r(K_n)$ be the space of polynomials of degree at most r in characteristic coordinates. Clearly, if $g \in \mathbb{P}_r(K_n)$ is to be reproduced by a quasi-interpolant, then we must have $g \in \mathcal{S}(K_n)$. This is a severe limitation. Constant functions are in $\mathcal{S}(K_n)$, and linear polynomials are in $\mathcal{S}(K_n)$, since they are components of a characteristic map. However, $\mathbb{P}_2(K_n)$ is not contained in the space of Loop's subdivision functions, unless the central vertex is regular (see Prautzsch and Reif [15]).

Constructing a quasi-interpolant that only reproduces $\mathbb{P}_1(K_n)$ would only prove second-order approximation power, but Theorem 2 states that the approximation power of Loop's subdivision functions is order 3. To show that subdivision functions have this higher-order approximation power, we require that the quasi-interpolant also locally reproduce affine quadratic polynomials away from the central vertex.

Throughout this chapter we assume that $\mathcal{S}$ is a $C_{\mathrm{loc}}^{R,1}$-subdivision scheme and that $\mathcal{S}(K_n)$ is contained in $H_{\Lambda,\mathrm{loc}}^s(K_n)$, with $s \leq R$. In particular, to apply the theory developed here to prove Theorem 2 we have a $C_{\mathrm{loc}}^{2,1}$-subdivision scheme, and Loop's subdivision functions are in $H_{\Lambda,\mathrm{loc}}^2(K_n)$, by Theorem 42. Let λ be the subdominant eigenvalue of the subdivision map for valence n, and let $\lambda_{\max} = \max\{\lambda, \frac{1}{2}\}$. Throughout this chapter, C will denote a generic constant whose value can change from line to line.

In Section 4.1, we define the properties of a quasi-interpolant and state the local approximation theorem. We review polynomial approximation in Sobolev spaces in Section 4.2. In Section 4.3, we prove our local approximation theorem. In Section 4.4, we develop an equivalent characterization of a quasi-interpolant, one that is easier to verify.

4.1   Approximation by Quasi-Interpolants

We begin with the formal definition of a *quasi-interpolant* on the $n-$regular complex K_n. Consider a family of linear operators

$$\mathcal{Q} = \{Q^k : L^1_{\Lambda,\mathrm{loc}}(K_n) \to \mathcal{S}(K_n^k) \qquad \text{for } k \geq 0\}.$$

We say that Q^k is *rotation equivariant* if the following diagram commutes

$$
\begin{array}{ccc}
L^1_{\Lambda,\mathrm{loc}}(K_n) & \xrightarrow{\ Q^k\ } & \mathcal{S}(K_n^k) \\
{\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*} \\
L^1_{\Lambda,\mathrm{loc}}(K_n) & \xrightarrow[\ Q^k\ ]{} & \mathcal{S}(K_n^k) \ ,
\end{array}
\qquad (4.3)
$$

where g is an element of the automorphism group of K_n as defined in (2.11).

We say that $\mathcal{Q}$ is *contraction equivariant* if the following diagram commutes for all $k \geq 0$

$$
\begin{array}{ccc}
L^1_{\Lambda,\mathrm{loc}}(K_n) & \xrightarrow{\ Q^k\ } & \mathcal{S}(K_n^k) \\
{\scriptstyle c^*}\downarrow & & \downarrow{\scriptstyle c^*} \\
L^1_{\Lambda,\mathrm{loc}}(K_n) & \xrightarrow[\ Q^{k-1}\ ]{} & \mathcal{S}(K_n^{k-1}) \ ,
\end{array}
\qquad (4.4)
$$

where c is the contraction map defined in Section 2.3. Observe that if $\mathcal{Q}$ is contraction equivariant then

$$Q^k = c^{-k*}Q^0 c^{k*} \qquad \text{for all } k \geq 0,$$

i.e., $\mathcal{Q}$ is completely determined by Q^0. Also, if $\mathcal{Q}$ is contraction equivariant and Q^k is rotation equivariant for some k, then Q^k is rotation equivariant for all $k \geq 0$.

Suppose $\mathcal{Q}$ is local with support width d, as defined in (3.23). Observe that if $K \subset K_n^k$ is a simplicial surface and $N \subset |K_n|$ is a neighborhood containing $|N_d K|$, then $\mathcal{Q}$ induces an operator

$$Q^k : L^1_{\Lambda,\mathrm{loc}}(N) \to \mathcal{S}(K_n^k)_{|K}, \qquad (4.5)$$

where $\mathcal{S}(K_n^k)_{|K}$ denotes restriction to $|K|$. Recall, from Section 2.7, that $N_d^c T \subset |K_n|$ is the closed circumscribing ball in equivariant characteristic coordinates for each face $T \in \mathsf{Face}(K_n^k)$, and $N_d^c K = \cup N_d^c T$ for any sub-simplicial surface $K \subset K_n^k$, where the union is taken over faces $T \in K$.

We say that $\mathcal{Q}$ is *uniformly H_Λ^s-bounded* if there exists a constant $\|Q\|$ such that the inequality

$$\|Q^k f\|_{H_\Lambda^s(K)} \leq \|Q\| \, \|f\|_{H_\Lambda^s(N_d^c K)} \tag{4.6}$$

holds for every simplicial surface $K \subset K_n^k$.

Given a local family $\mathcal{Q}$ with support width d, we say that Q^k is *order r* if the following two conditions are satisfied:

(i) $Q^k f = f$ for f a polynomial of degree less than $r - 1$ in characteristic coordinates.

(ii) $Q^k f = f$ on $|K|$ for any (a) subcomplex $K \subset K_n^k$ such that $|N_d K|^\circ$ is an affine coordinate neighborhood, and (b) polynomial f of degree less than r in affine coordinates.

We say $\mathcal{Q}$ is *order r* if Q^k is order r for all $k \geq 0$.

Definition 44. *A local, rotation and contraction equivariant, uniformly H_Λ^s-bounded family of order r is called a* **quasi-interpolant** *on K_n.*

The key theorem of this chapter, which follows, shows that a quasi-interpolant on K_n satisfies the approximation error bound (3.26).

Theorem 45. *Suppose that $\mathcal{Q}$ is a quasi-interpolant on K_n with $s < r \leq R+1$. Then given a simplicial surface $K \subset K_n^{k_0}$ with $|K| \subset |N_1(v_0, K_n)|$ and any $\epsilon > 0$, there exists a constant $C_\epsilon = C(\epsilon, n, K)$ such that for any $k \geq k_0$, the approximation error satisfies the estimate*

$$\|Q^k f - f\|_{H_\Lambda^s(K)} \leq C_\epsilon \lambda_{\max}^{(r-s-\epsilon)k} \|f\|_{H_\Lambda^r(N_d^c K)}. \tag{4.7}$$

If $\lambda < \frac{1}{2}$, we can take $\epsilon = 0$.

4.2   Background on Averaged Taylor Polynomials

To prove Theorem 45, we approximate $f \in H_\Lambda^r(N_d^c K)$ locally by a polynomial in characteristic coordinates. The naive choice is the Taylor polynomial, but it is not well defined since the derivatives of f are weak derivatives which may not be defined pointwise.

Suppose that Ω is a domain in $\mathbb{R}^m$ and that $f \in L^1_{\text{loc}}(\Omega)$ has weak derivatives $D^\alpha f$ for all $|\alpha| < r$. We can define a weak Taylor polynomial of order r (i.e., of degree less than r) by

$$\widetilde{T}^r_y f(x) = \sum_{|\alpha|<r} \frac{1}{\alpha!} D^\alpha f(y)(x-y)^\alpha, \qquad y \in \Omega \text{ and } x \in \mathbb{R}^m. \tag{4.8}$$

Notice that (4.8) is only defined in the sense of distributions; to obtain a bona fide polynomial in x, we average (4.8) in the y variable with respect to a weighting function.

Definition 46. *Suppose $f \in L^1_{\Lambda,\text{loc}}(\Omega)$ has weak derivatives $D^\alpha f$ for all $|\alpha| < r$. Let $B = B_\rho(y_0) \subset \Omega$ be a closed ball of radius ρ centered at y_0. The order r **Taylor polynomial of f averaged over B** is defined as*

$$T^r_B f(x) = \int_B \left(\widetilde{T}^r_y f(x) \right) \phi_\rho(y - y_0) \, dy \,,$$

where

$$\phi_\rho(y) = \begin{cases} c_m \, \rho^{-m} \, e^{-\left(1-\left(\frac{|y|}{\rho}\right)^2\right)^{-1}} & \text{if } |y| < \rho \\ 0 & \text{otherwise.} \end{cases}$$

The constant c_m is chosen so that $\|\phi_1\|_{L^1} = 1$.

Remark 47. *Notice $\phi_\rho \in C^\infty(\mathbb{R}^m)$, and ϕ_ρ is supported on $B_\rho(0)$. Also $\|\phi_\rho\|_{L^1} = 1$ for all $\rho > 0$, and $\|\phi_\rho\|_{L^\infty} = c_m \, e^{-1} \rho^{-m}$.*

The approximation properties of averaged Taylor polynomials are given by the Bramble-Hilbert Lemma. Recall, from Definition 31, the concepts of star-shaped with respect to a ball and the chunkiness parameter $\gamma(\Omega)$ of a domain Ω.

Lemma 48 (Bramble-Hilbert [3]). *Fix non-negative integers n and r, and a real number $\gamma_0 \geq 2$. Let Ω be a domain in $\mathbb{R}^n$ such that $\gamma(\Omega) \leq \gamma_0$, and let B be a ball of radius $\rho > \frac{1}{2}\rho_{\max}(\Omega)$ such that Ω is star-shaped with respect to B. Then there exists a constant C_{n,r,γ_0} such that*

$$|T^r_B f - f|_{H^{m,p}(\Omega)} \leq C_{n,r,\gamma_0} \ diam(\Omega)^{r-m}|f|_{H^{r,p}(\Omega)}$$

for any $m = 0, \ldots, r$ and $f \in H^{r,p}(\Omega)$. The constant C_{n,r,γ_0} can be chosen to depend only on n, r and γ_0.

By imposing a bound on the diameter of Ω we can derive a bound on the full norm.

Corollary 49. *Let r, n, γ_0, Ω and B be as in the Bramble-Hilbert Lemma, with the additional constraint* $\mathrm{diam}(\Omega) < d_0$, *then for any $s \leq r$*

$$\|T_B^r f - f\|_{H^{s,p}(\Omega)} \leq C_{n,r,\gamma_0} \ \max\{1, d_0^s\}(s+1)^{1/p} \ diam(\Omega)^{r-s} |f|_{H^{r,p}(\Omega)}$$

for any $f \in H^{r,p}(\Omega)$.

Proof. By Lemma 48, for any $0 \leq m \leq s$ we have

$$\begin{aligned}
|T_B^r f - f|_{H^{m,p}(\Omega)} &\leq C_{n,r,\gamma_0} \, \mathrm{diam}(\Omega)^{r-m} |f|_{H^{r,p}(\Omega)} \\
&\leq C_{n,r,\gamma_0} d_0^{s-m} \, \mathrm{diam}(\Omega)^{r-s} |f|_{H^{r,p}(\Omega)} \\
&\leq C_{n,r,\gamma_0} \ \max\{1, d_0^s\} \ \mathrm{diam}(\Omega)^{r-s} |f|_{H^{r,p}(\Omega)} \, .
\end{aligned}$$

Summing $|T_B^r f - f|_{H^{m,p}(\Omega)}^p$ for $m = 0$ to s completes the proof. $\square$

4.3 Proof of the Approximation Theorem

Fix $k_0 \geq 0$ and $K \subset K_n^{k_0}$ as in Theorem 45. For any $k \geq k_0$ and $f \in H_\Lambda^r(N_d^c K)$ we estimate the approximation error on $|K|$ by summing the errors over each face. For each face $T \in K^{k-k_0} \subset K_n^k$, let $L_T f$ be the order-r Taylor polynomial of f averaged over $N_d^c T$, i.e.,

$$L_T f = T_{N_d^c T}^r f.$$

By the triangle inequality, we have

$$\begin{aligned}
\|Q^k f - f\|_{H_\Lambda^s(K)}^2 &= \sum_{T \in K^{k-k_0}} \|Q^k f - f\|_{H_\Lambda^s(T)}^2 \\
&\leq \sum_{T \in K^{k-k_0}} \Big(\|f - L_T f\|_{H_\Lambda^s(T)} + \|Q^k(f - L_T f)\|_{H_\Lambda^s(T)} \\
&\qquad\qquad\qquad\qquad + \|Q^k L_T f - L_T f\|_{H_\Lambda^s(T)} \Big)^2 \, .
\end{aligned}$$

Then by the H_Λ^s-boundedness of the quasi-interpolant, we get

$$\|Q^k f - f\|_{H_\Lambda^s(K)}^2 \leq \sum_{T \in K^{k-k_0}} \Big((1 + \|Q\|) \|f - L_T f\|_{H_\Lambda^s(N_d^c T)} + \|Q^k L_T f - L_T f\|_{H_\Lambda^s(T)} \Big)^2 \, .$$

Another application of the triangle inequality yields the estimate

$$\|Q^k f - f\|_{H_\Lambda^s(K)} \le (1 + \|Q\|) \left(\sum_{T \in K^{k-k_0}} \|f - L_T f\|_{H_\Lambda^s(N_d^c T)}^2 \right)^{1/2} +$$

$$\left(\sum_{T \in K^{k-k_0}} \|Q^k L_T f - L_T f\|_{H_\Lambda^s(T)}^2 \right)^{1/2} . \quad (4.9)$$

In the next two sections, we conclude the proof of Theorem 45 by showing that each of the terms on the right side of (4.9) is bounded by the right side of inequality (4.7).

4.3.1 Estimation of the First Term

Since $|K|$ is contained in a bounded region $|N_1(v_0, K_n)|$, the diameter $\mathrm{diam}_\Lambda(N_d^c T)$ for $T \in \mathsf{Face}(K^{k-k_0})$ is bounded independent of T, K, k_0 and k, say $\mathrm{diam}_\Lambda(N_d^c T) < d_0$. Observe that because $N_d^c T$ is a ball, its chunkiness parameter is 2. Applying Corollary 49 to the Bramble-Hilbert Lemma yields

$$\|f - L_T f\|_{H_\Lambda^s(N_d^c T)} \le C \ \mathrm{diam}_\Lambda(N_d^c T)^{r-s} |f|_{H_\Lambda^r(N_d^c T)} , \quad (4.10)$$

where the constant $C = C_{2,r,2} \ \max\{1, d_0^s\} \sqrt{s+1}$ is independent of K, k and T.

Corollary 28 shows that a constant C_1 exists such that for any $T \in \mathsf{Face}(N_1(v_0, K_n)^k)$ we have a bound on the diameter $\mathrm{diam}_\Lambda(N_d^c T) \le C_1 \lambda_{\max}^k$. Substituting this bound into (4.10) we get

$$\|f - L_T f\|_{H_\Lambda^s(N_d^c T)} \le C \lambda_{\max}^{(r-s)k} |f|_{H_\Lambda^r(N_d^c T)} . \quad (4.11)$$

Summing the square of (4.11) over all $T \in \mathsf{Face}(K^{k-k_0})$ we get

$$\sum_{T \in K^{k-k_0}} \|f - L_T f\|_{H_\Lambda^s(N_d^c T)}^2 \le C \lambda_{\max}^{2(r-s)k} \sum_{T \in K^{k-k_0}} |f|_{H_\Lambda^r(N_d^c T)}^2 . \quad (4.12)$$

The following proposition shows that the sum on the right side of (4.12) is bounded by $C|f|_{H_\Lambda^r(N_d^c K)}^2$ and hence by the full norm $C\|f\|_{H_\Lambda^r(N_d^c K)}^2$. Therefore, the first term in (4.9) is bounded by the right side of (4.7), as desired.

Proposition 50. *There exists a constant C such that for any simplicial surface $K \subset K_n^{k_0}$, function $f \in H_\Lambda^{r,q}(N_d^c K)$, and integer $k \geq k_0$, the following inequality is satisfied*

$$\left(\sum_{T \in K^{k-k_0}} |f|_{H_\Lambda^{r,q}(N_d^c T)}^q \right)^{\frac{1}{q}} \leq C |f|_{H_\Lambda^{r,q}(N_d^c K)} .$$

Proof. We write the semi-norm as an integral, using $\chi_{N_d^c T}$ to denote the characteristic function of the set $N_d^c T$,

$$\sum_{T \in K^{k-k_0}} |f|_{H_\Lambda^{r,q}(N_d^c T)}^q = \sum_{|\alpha|=r} \sum_T \int \chi_{N_d^c T} |D^\alpha f(y)|^q \, dy .$$

We exchange the order of the inner summation and integration. In the summation, the domains of integration overlap. The function $\eta^k(y)$, defined in (2.46), counts the number of overlapping balls. Using this function we replace repeated integration by a single, appropriately weighted integral, specifically,

$$\sum_{T \in K^{k-k_0}} |f|_{H_\Lambda^{r,q}(N_d^c T)}^q \leq \sum_{|\alpha|=r} \int_{N_d^c K^{k-k_0}} \eta^k(y) |D^\alpha f(y)|^q \, dy .$$

Proposition 30 shows that there is a bound $\eta^k(y) < C$. Thus

$$\sum_{T \in K^{k-k_0}} |f|_{H_\Lambda^{r,q}(N_d^c T)}^q \leq C |f|_{H_\Lambda^{r,q}(N_d^c K^{k-k_0})}^q .$$

Replacing the domain $N_d^c K^{k-k_0}$ with the larger domain $N_d^c K$ completes the proof. $\qquad\square$

4.3.2   Estimation of the Second Term

Each of the summands in the second term on the right side of (4.9) is a local error made in approximating a polynomial of degree less than r. In the first part of this section, we derive a general estimate for the approximation error of such a polynomial on a single face. Then in the second part, we sum these errors and derive an estimate for the second term.

Polynomial Approximation

Observe that since Q^k reproduces polynomials of degree at most $r - 2$ in characteristic coordinates, only the degree $r - 1$ part of $L_T f$ contributes to the approximation error. So,

given $L \in \mathbb{P}_{r-1}(K_n)$ we have the identity

$$Q^k L - L = Q^k L_h - L_h \,, \tag{4.13}$$

where $L_h \in \mathbb{P}_{r-1}^h(K_n)$ is the homogeneous part of L, and $\mathbb{P}_{r-1}^h(K_n)$ denotes the space of homogeneous polynomials of degree $r - 1$.

Recall, from Section 2.7, that a face $T \in K_n^k$ has an annulus index j_T and that $c^{-j_T}(T)$ is either a face in the fundamental annulus $|\Omega_0|$ or a face in $N_1(v_0, K_n)$. The next lemma shows that we can bound the approximation error on T in terms of the approximation error on $c^{-j_T}(T)$ for a fixed homogeneous polynomial.

Lemma 51. *For any face $T \in N_1(v_0, K_n)^k$ and any polynomial $L \in \mathbb{P}_{r-1}^h(K_n)$, we have*

$$\|Q^k L - L\|_{H_\Lambda^s(T)} \leq \lambda^{(r-s)j_T} \|Q^{k-j_T} L - L\|_{H_\Lambda^s(c^{-j_T}(T))} \,.$$

Proof. Since $c(y) = \lambda y$ in characteristic coordinates, its Jacobian determinant is λ^2. Hence for any $m = 0, \ldots, s$ we have

$$|Q^k L - L|_{H_\Lambda^m(T)}^2 = \sum_{|\alpha|=m} \int_T (D^\alpha Q^k L - D^\alpha L)^2 \, dy$$

$$= \lambda^2 \sum_{|\alpha|=m} \int_{c^{-1}T} (c^* D^\alpha Q^k L - c^* D^\alpha L)^2 \, dy \,.$$

Differentiating both the homogeneity identity $\lambda^{r-1} L = c^* L$ and the contraction equivariance identity (4.4) applied to a polynomial $c^* Q^k L = \lambda^{r-1} Q^{k-1} L$ yields

$$\lambda^{r-1} D^\alpha L = \lambda^{|\alpha|} c^* D^\alpha L$$

and

$$\lambda^{r-1} D^\alpha Q^{k-1} L = \lambda^{|\alpha|} c^* D^\alpha Q^k L$$

for any multi-index α. These two identities enable us to compute as follows:

$$|Q^k L - L|_{H_\Lambda^m(T)}^2 = \lambda^2 \sum_{|\alpha|=m} \int_{c^{-1}T} (\lambda^{r-1-m}(D^\alpha Q^{k-1} L - D^\alpha L))^2 \, dy$$

$$= \lambda^{2(r-m)} |Q^{k-1} L - L|_{H_\Lambda^m(c^{-1}T)}^2 \,. \tag{4.14}$$

Since $|T| \subset |N_1(v_0, K_n)|$ we have $j_T \geq 0$ and so applying (4.14) j_T times we get

$$\|Q^k L - L\|_{H^s_\Lambda(T)} = \left(\sum_{m=0}^{s} |Q^k L - L|^2_{H^m_\Lambda(T)} \right)^{1/2}$$

$$\leq \left(\lambda^{2(r-s)j_T} \sum_{m=0}^{s} |Q^{k-j_T} L - L|^2_{H^m_\Lambda(c^{-j_T}(T))} \right)^{1/2}$$

$$\leq \lambda^{(r-s)j_T} \|Q^{k-j_T} L - L\|_{H^s_\Lambda(c^{-j_T}(T))} . \qquad (4.15)$$

$\square$

To bound the approximation error of polynomials, we introduce a family of "polynomial norms" on $\mathbb{P}^h_{r-1}(K_n)$, denoted by $|L|_q$ for $1 \leq q \leq \infty$. Any $L \in \mathbb{P}^h_{r-1}(K_n)$ is given by $L(y) = \sum_{|\alpha|=r-1} \frac{1}{\alpha!} c_\alpha y^\alpha$, and we define

$$|L|_q = \|\{c_\alpha : |\alpha| = r - 1\}\|_{\ell^q} . \qquad (4.16)$$

We extend these to semi-norms on all of $\mathbb{P}_{r-1}(K_n)$ by $|L|_q = |L_h|_q$, where $L_h \in \mathbb{P}^h_{r-1}$ is the homogeneous part of L. Take the 2-norm as the default case $|L| \doteq |L|_2$.

Lemma 52. *There exists a constant C such that for any face $T \in N_1(v_0, K_n)^k$ and polynomial $L \in \mathbb{P}_{r-1}(K_n)$ we have the inequality*

$$\|Q^k L - L\|_{H^s_\Lambda(T)} \leq C \lambda^{j_T(r-s)} (\tfrac{1}{2})^{(k-j_T)(r-s+1)} |L| . \qquad (4.17)$$

Proof. By Lemma 51, it suffices to prove that (4.17) is satisfied for faces $T \in N_1(v_0, K_n)^k$ such that $j_T = 0$. Such a face is either a face in $N_1(v_0, K_n)$ or a face in the fundamental annulus Ω_0^k. Next, observe that because the space $\mathbb{P}^h_{r-1}$ is finite-dimensional, the operator $L \mapsto Q^k L - L$ is $H^s_\Lambda(T)$-bounded for any particular face T. Therefore, there is a constant C such that (4.17) holds for any finite collection of faces. Thus, we need only show that (4.17) holds for any face $T \in \Omega_0^k$ and any k sufficiently large, say $k \geq k'$, since there are only a finite number of faces $T \in N_1(v_0, K_n)^k$ with $k < k'$ and $j_T = 0$.

By the rotation equivariance of $\mathcal{Q}$, it suffices to find a constant C such that (4.17) holds for all T contained in a given wedge W of K_n. Let $k' > 0$ be the smallest integer such that $\widetilde{\Omega} = N^c_d(\Omega_0 \cap W)^{k'}$ is an affine coordinate neighborhood, and let x denote the

affine coordinate variable. For any $T \in \mathsf{Face}(\Omega_0 \cap W)^k$, with $k \geq k'$, and any polynomial $g(x) \in \mathbb{P}_{r-1}$ we have

$$\|Q^k L - L\|_{H_\Lambda^s(T)} = \|(Q^k - I)(L - g)\|_{H_\Lambda^s(T)} \leq (\|Q\| + 1)\|L - g\|_{H_\Lambda^s(N_d^c T)} \qquad (4.18)$$

by the polynomial reproducing and boundedness properties of $\mathcal{Q}$. The transition function $y = \tau(x)$ from affine to characteristic coordinates is a $C_{\mathrm{loc}}^{R,1}$-diffeomorphism. So both L and g are in $H_A^r(\widetilde{\Omega}) = H_\Lambda^r(\widetilde{\Omega})$, where the subscript A denotes computations in the affine coordinate system. By Proposition 40 the two norms are equivalent, so

$$\|L - g\|_{H_\Lambda^s(N_d^c T)} \leq C \|L - g\|_{H_A^s(N_d^c T)}, \qquad (4.19)$$

where the constant C, from the change of coordinates, is independent of k and T. Proposition 32 shows that there are constants k'' and γ_0 such that the affine coordinate regions $N_d^c T$ for $T \in \mathsf{Face}(\Omega_0 \cap W)^k$ and $k > k''$ are star-shaped with a chunkiness parameter satisfying $\gamma_A(N_d^c T) < \gamma_0$. Furthermore, for all such T, there is an upper bound on the diameter $\mathrm{diam}_A(N_d^c T) < d_0$. Thus, by Corollary 49 to the Bramble-Hilbert Lemma, there is a polynomial $g \in \mathbb{P}_{r-1}$ such that

$$\|L - g\|_{H_A^s(N_d^c T)} \leq C \, \mathrm{diam}_A(N_d^c T)^{r-s} |L|_{H_A^r(N_d^c T)}. \qquad (4.20)$$

Lemma 29 gives a bound on the diameters

$$\mathrm{diam}_A(N_d^c T) < C(\tfrac{1}{2})^k. \qquad (4.21)$$

Then, by Hölder's inequality and Lemma 29, we have

$$|L|_{H_A^r(N_d^c T)} \leq \|L\|_{H_A^{r,\infty}(\widetilde{\Omega})} \, \mathrm{area}_A(N_d^c T)^{1/2} \leq C \|L\|_{H_A^{r,\infty}(\widetilde{\Omega})} (\tfrac{1}{2})^k \leq C |L| (\tfrac{1}{2})^k. \qquad (4.22)$$

The final inequality in (4.22) follows since $|L|$ and $\|L\|_{H_A^{r,\infty}(\widetilde{\Omega})}$ are both norms on a finite-dimensional space $\mathbb{P}_{r-1}^h(K_n)$, and are hence equivalent. Combining inequalities (4.18) to (4.22) proves (4.17) for $T \in \mathsf{Face}((\Omega_0 \cap W)^k)$ and $k > \max\{k', k''\}$. $\qquad \square$

Summing the Approximation Errors

We proceed by summing the approximation errors to get a bound on the second term in (4.9). For any $k \geq k_0$ let

$$\text{Sum} = \sum_{T \in K^{k-k_0}} \|Q^k L_T f - L_T f\|^2_{H^s_\Lambda(T)} \, .$$

Applying Lemma 52 gives

$$\text{Sum} \leq C \sum_{T \in K^{k-k_0}} \lambda^{2j_T(r-s)} (\tfrac{1}{2})^{2(k-j_T)(r-s+1)} |L_T f|^2 . \tag{4.23}$$

We now express the polynomial norm $|L_T f|$ in terms of a Sobolev norm. Since $L_T f \in \mathbb{P}_{r-1}(K_n)$, the derivatives of order $r-1$ are constant, so for any $1 \leq q \leq \infty$ we have

$$|L_T f|^q_{H^{r-1,q}_\Lambda(N^c_d T)} = \sum_{|\alpha|=r-1} \int_{N^c_d T} |D^\alpha L_T f|^q \, dy = |L_T f|^q_q \ \text{area}_\Lambda(N^c_d T)$$

by the definition of the polynomial norm (4.16). Solving for $|L_T f|_q$ and noting that the polynomial 2-norm is equivalent to the polynomial q-norm gives

$$|L_T f| \leq C_q \, |L_T f|_q = C_q \, \text{area}_\Lambda(N^c_d T)^{-1/q} \, |L_T f|_{H^{r-1,q}_\Lambda(N^c_d T)} \, ,$$

where C_q is the constant relating the norms. Corollary 27 gives a lower bound on the diameter of the circular neighborhoods $N^c_d T$ and hence a lower bound on the area. Substituting this into the previous inequality yields the inequality

$$|L_T f| \leq C_q \left(\lambda^{2j_T} (\tfrac{1}{2})^{2(k-j_T)} \right)^{-1/q} |L_T f|_{H^{r-1,q}_\Lambda(N^c_d T)}. \tag{4.24}$$

The next lemma bounds $|L_T f|_{H^{r-1,q}_\Lambda(N^c_d T)}$ in terms of a Sobolev norm of f.

Lemma 53. *There exists a constant C such that*

$$|L_T f|_{H^{r-1,q}_\Lambda(N^c_d T)} \leq C |f|_{H^{r-1,q}_\Lambda(N^c_d T)} \tag{4.25}$$

for any function $f \in H^r_\Lambda(N^c_d T)$, face $T \in K^k_n$, and exponent $1 \leq q < \infty$.

Proof. From the Sobolev Embedding Theorem [1] there is an embedding $H_\Lambda^r(N_d^c T) \hookrightarrow H_\Lambda^{r-1,q}(N_d^c T)$ for any $q < \infty$. From the definition of an averaged Taylor polynomial (Definition 46), one easily shows that $D^\alpha T_B^r f = T_B^{r-|\alpha|} D^\alpha f$ for $|\alpha| \leq r - 1$. Thus for any multi-index $|\alpha| = r - 1$, the constant function $D^\alpha L_T f$ is given by

$$D^\alpha L_T f = D^\alpha T_{N_d^c T}^r f = T_{N_d^c T}^1 D^\alpha f = \int \phi_\rho(y - y_0) \, D^\alpha f \, dy,$$

where $N_d^c T = B_\rho(y_0)$. Applying Hölder's inequality with p as the conjugate exponent to q yields

$$\begin{aligned}
|D^\alpha L_T f| &\leq \|\phi_\rho\|_{L^p} \|D^\alpha f\|_{L_\Lambda^q(N_d^c T)} \\
&\leq (\pi \rho^2)^{1/p} \|\phi_\rho\|_{L^\infty} \|D^\alpha f\|_{L_\Lambda^q(N_d^c T)} && \text{(since } B_\rho = \text{supp}(\phi_\rho)\text{)} \\
&\leq C \rho^{2/p} \rho^{-2} \|D^\alpha f\|_{L_\Lambda^q(N_d^c T)} && \text{(by Remark 47)} \\
&\leq C \rho^{-2/q} \|D^\alpha f\|_{L_\Lambda^q(N_d^c T)} && \text{(by } \tfrac{1}{q} + \tfrac{1}{p} = 1\text{).} && (4.26)
\end{aligned}$$

We compute the semi-norm on the left side of (4.25) by

$$\begin{aligned}
|L_T f|_{H_\Lambda^{r-1,q}(N_d^c T)}^q &= \sum_{|\alpha|=r-1} \|D^\alpha L_T f\|_{L_\Lambda^q(N_d^c T)}^q \\
&= \sum_{|\alpha|=r-1} \pi \rho^2 \, |D^\alpha L_T f|^q.
\end{aligned}$$

Substituting inequality (4.26) proves the Lemma. $\qquad\square$

We now substitute (4.24) with exponent $2q$ into (4.23), and then apply Lemma 53 to obtain the estimate

$$\text{Sum} \leq C_q \sum_{T \in K^{k-k_0}} \lambda^{2j_T(r-s-\frac{1}{q})} \left(\tfrac{1}{2}\right)^{2(k-j_T)(r-s+1-\frac{1}{q})} |f|_{H_\Lambda^{r-1,2q}(N_d^c T)}^2. \qquad (4.27)$$

Let p be the conjugate exponent to q. Rewriting the exponents in (4.27) in terms of p we get

$$\text{Sum} \leq C_p \sum_{T \in K^{k-k_0}} \lambda^{2j_T(p(r-s-1)+1)\frac{1}{p}} \left(\tfrac{1}{2}\right)^{2(k-j_T)(p(r-s)+1)\frac{1}{p}} |f|_{H_\Lambda^{r-1,2q}(N_d^c T)}^2. \qquad (4.28)$$

Now, we apply Hölder's inequality to the sum on the right side of (4.28) to get

$$\text{Sum} \leq C_p \, A \left(\sum_{T \in K^{k-k_0}} |f|_{H_\Lambda^{r-1,2q}(N_d^c T)}^{2q} \right)^{\frac{1}{q}}, \qquad (4.29)$$

where

$$A = \left(\sum_{T \in K^{k-k_0}} \lambda^{2j_T(p(r-s-1)+1)} \left(\tfrac{1}{2}\right)^{2(k-j_T)(p(r-s)+1)} \right)^{\frac{1}{p}}. \tag{4.30}$$

By Proposition 50 we see that the last term in (4.29) is bounded by $C|f|^2_{H^{r-1,2q}_\Lambda(N^c_d K)}$. Then by the Sobolev Embedding Theorem, there is an embedding $H^r_\Lambda(N^c_d K) \hookrightarrow H^{r-1,2q}_\Lambda(N^c_d K)$ for any K and q; i.e., there is a constant C such that $\|f\|_{H^{r-1,2q}_\Lambda(N^c_d K)} \leq C\|f\|_{H^r_\Lambda(N^c_d K)}$. Consequently,

$$\mathrm{Sum} \leq C_p A \|f\|^2_{H^r_\Lambda(N^c_d K)}. \tag{4.31}$$

We still need to estimate A. The summands in (4.30) only depend on the annulus index j_T, which runs from 0 to k. Furthermore, the number of faces $T \in K^{k-k_0}$ with a given annulus index j is bounded by $C2^{2(k-j)}$. Thus

$$A \leq C \left(\sum_{j=0}^{k} \lambda^{2j(p(r-s-1)+1)} \left(\tfrac{1}{2}\right)^{2(k-j)p(r-s)} \right)^{\frac{1}{p}}$$

$$= C\left(\tfrac{1}{2}\right)^{2k(r-s)} \left(\sum_{j=0}^{k} \left(\lambda^{2p(r-s-1)+2} \left(\tfrac{1}{2}\right)^{-2p(r-s)} \right)^j \right)^{\frac{1}{p}}. \tag{4.32}$$

Let $R = \lambda^{2p(r-s-1)+2}\left(\tfrac{1}{2}\right)^{-2p(r-s)}$ be the ratio between consecutive terms in the geometric series of (4.32). We consider two cases. First, suppose $\lambda < \tfrac{1}{2}$, which implies $\lambda_{\max} = \tfrac{1}{2}$. In the limit, as $p \to 1$, we have $R \to \left(\tfrac{\lambda}{1/2}\right)^{2(r-s)} < 1$. So there exists a $p > 1$ such that $R < 1$, and for any $k > 0$, the sum in (4.32) is a partial sum of a convergent geometric series. In particular, these partial sums are bounded by some constant C, and therefore

$$A < C\, \lambda_{\max}^{2k(r-s)}.$$

In the other case, we have $\tfrac{1}{2} \leq \lambda$, which implies $\lambda_{\max} = \lambda$. In this case, $R > 1$ for any $p > 1$. So we use the estimate $\sum_0^k R^j < C_p R^k$ in inequality (4.32) yielding the inequality

$$A < C_p\left(\tfrac{1}{2}\right)^{2k(r-s)} R^{k/p} = C_p \lambda^{2k(r-s-1+\frac{1}{p})} = C_p \lambda^{2k(r-s-\frac{1}{q})}.$$

For a given $0 < \epsilon < 1$, let $q = 1/\epsilon$, then we have

$$A < C_\epsilon \lambda_{\max}^{2k(r-s-\epsilon)}. \tag{4.33}$$

Combining the estimates of A with (4.31) shows that the second term on the right side of (4.9) is bounded by the right side of (4.7), completing the proof of Theorem 45.

4.4 An Alternative Characterization of a Quasi-Interpolant

We present an alternative characterization of a quasi-interpolant. We replace the uniform boundedness property (4.6) with semi-norm estimates in affine coordinates. These estimates only involve the level zero quasi-interpolant $Q : L^1_{\Lambda,\mathrm{loc}}(K_n) \to \mathcal{S}(K_n)$ and estimates on faces $T \in K_n$. In Chapter 5, to verify that the map we construct is a quasi-interpolant, we will show that it satisfies the alternative boundedness criteria.

Given a local linear map

$$Q : L^1_{\Lambda,\mathrm{loc}}(K_n) \to \mathcal{S}(K_n) \tag{4.34}$$

with support width d, we can define a contraction equivariant family of linear operators by

$$\mathcal{Q} = \{Q^k = c^{-k*}Qc^{k*} : k \geq 0\}. \tag{4.35}$$

We define a set of conditions on Q that ensure $\mathcal{Q}$ is a quasi-interpolant.

Let W be a wedge in K_n. We define a simplicial neighborhood $W_{\mathrm{cen}}(d) \subset W$ of the central vertex in terms of d so that $N^c_d(W \setminus W_{\mathrm{cen}}(d))$ is an affine coordinate neighborhood. Let $W_{\mathrm{cen}}(d) \subset W$ be the subcomplex given by

$$W_{\mathrm{cen}}(d) = \bigcup \overline{T} \qquad \text{such that } T \in \mathsf{Face}(W) \text{ and } v_0 \in N^c_d T. \tag{4.36}$$

Recall that $\overline{T}$ is the subcomplex containing T and all of its subsets. The contraction map $c : |W| \to |W|$ generates a group action on $|W|$. Using $W_{\mathrm{cen}}(d)$, we define a fundamental region with respect to this group action. Specifically, we define $W_0 \subset W$ by

$$W_0 = \bigcup \overline{T} \qquad \text{such that } T \in \mathsf{Face}(W) \text{ and } |cT| \subset |W_{\mathrm{cen}}(d)|.$$

Let $(N^c_d(W \setminus W_{\mathrm{cen}}(d)), x)$ be an affine coordinate system, and denote Sobolev norms computed in this affine coordinate system by H_A .

Definition 54. *We say a rotation equivariant map* $Q : L^1_{\Lambda,\mathrm{loc}} \to \mathcal{S}(K_n)$ *satisfies the* **H^s-alternative boundedness criteria** *if there exists a constant* $\|Q\|$ *such that for any*

function $f \in H_A^m(N_d^c T)$, face $T \in W \setminus W_{cen}(d)$, and integer $0 \leq m \leq s$, the following semi-norm estimates hold:

$$|Qf|_{H_A^m(T)} \leq \|Q\| \, |f|_{H_A^m(N_d^c T)} \, . \tag{4.37}$$

Lemma 55. *Suppose the rotation equivariant linear map* $Q : L_{\Lambda,\mathrm{loc}}^1(K_n) \to S(K_n)$ *(i) is local with support width* d, *as in* (3.23) *with* $k = 0$, *(ii) is order* r, *and (iii) satisfies the* H^s*-alternative boundedness criteria. Then* (4.35) *is a quasi-interpolant on* K_n *which is uniformly* H_Λ^s*-bounded, has support width* d, *and is order* r.

We first prove a technical Proposition.

Proposition 56. *Let* Ω *and* Ω' *be domains in* $\mathbb{R}^2$ *and let* $c(x) = \lambda x$ *be a dilation map on* $\mathbb{R}^2$ *with pull back* $c^* : f \mapsto f \circ c$. *Suppose* $L : H^s(\Omega) \to H^s(\Omega')$ *is a linear map that satisfies the following semi-norm estimates*

$$|Lf|_{H^m(\Omega')} \leq \|L\| \, |f|_{H^m(\Omega)} \tag{4.38}$$

for some constant $\|L\|$ *and each* $0 \leq m \leq s$. *Then for each* $0 \leq m \leq s$ *we have*

$$\|c^{-1*} L c^* f\|_{H^m(c\Omega')} \leq \|L\| \, \|f\|_{H^m(c\Omega)} \qquad \text{for } f \in H^m(c\Omega).$$

Proof. Fix $0 \leq m \leq s$ and $f \in H^m(c\Omega)$ and compute as follows:

$$
\begin{aligned}
\|c^{-1*} L c^* f\|_{H^m(c\Omega')}^2 &= \sum_{i=0}^m |c^{-1*} L c^* f|_{H^i(c\Omega')}^2 \\
&= \sum_{i=0}^m \lambda^{-2(i-1)} |L c^* f|_{H^i(\Omega')}^2 && \text{by Proposition 41} \\
&\leq \sum_{i=0}^m \lambda^{-2(i-1)} \|L\|^2 \, |c^* f|_{H^i(\Omega)}^2 && \text{by (4.38)} \\
&= \sum_{i=0}^m \|L\|^2 \, |f|_{H^i(c\Omega)}^2 && \text{by Proposition 41} \\
&= \|L\|^2 \, \|f\|_{H^m(c\Omega)}^2 \, .
\end{aligned}
$$

$\square$

We break the proof of Lemma 55 into five steps.

Step 1. We first show that $\mathcal{Q}$ satisfies the locality and polynomial reproducing properties in Section 4.1. To prove locality we consider $k \geq 0$, a simplicial surface $K \subset K_n^k$ and a function $f \in L_\Lambda^{1,\mathrm{loc}}(K_n)$ such that $f \equiv 0$ on $|N_d K|$. We need to show that $Q^k f = c^{-k*} Q c^{k*} f \equiv 0$ on $|K|$. We have $c^{k*} f \equiv 0$ on $|c^{-k} N_d K| = |N_d c^{-k} K|$, and since $c^{-k} K \subset K_n$ we have $Q c^{k*} f \equiv 0$ on $|c^{-k} K|$ by the hypothesis for Q. Therefore $Q^k f = c^{-k*} Q c^{k*} f \equiv 0$ on $|K|$. Polynomial reproduction follows since c is linear in either affine or characteristic coordinates.

In the remaining steps we show that (4.6) holds on each face $T \in W^k$. From this, it follows that (4.6) holds for all simplicial surfaces $K \subset K_N^k$ by applying rotational equivariance, and summing over all the faces in K. Therefore, we have shown that $\mathcal{Q}$ is uniformly H_Λ^s-bounded.

Step 2. In this step we show

$$\|Q^k f\|_{H_A^m(T)} \leq \|Q\| \, \|f\|_{H_A^m(N_d^c T)} \tag{4.39}$$

for any function $f \in H_A^m(N_d^c T)$, face $T \in W_0^k$, and integer $0 \leq m \leq s$. This follows from Proposition 56 with domains $\Omega' = c^{-k} T$ and $\Omega = N_d^c(c^{-k} T)$, dilation map c^k, and operator Q.

Step 3. For either $k \geq 0$ and $T \in W_0^k$, or $k = 0$ and $T \in W_{\mathrm{cen}}(d)$ we prove

$$\|Q^k f\|_{H_\Lambda^m(T)} \leq C \, \|Q\| \, \|f\|_{H_\Lambda^m(N_d^c T)} \qquad \text{for } 0 \leq m \leq s. \tag{4.40}$$

Since Q is linear, (4.40) holds for $k = 0$ and any single face $T \in W$. Since there are only finitely many faces $T \in W_{\mathrm{cen}}(d)$, there is a constant C such that (4.40) holds for all $T \in W_{\mathrm{cen}}(d)$. So we need only consider faces $T \in W_0^k$. The transition function to characteristic coordinates $y = \tau(x)$ is class C^R on the closed and bounded set $N_d^c(W_0)$. Therefore by Proposition 40 and (4.39) we get

$$\frac{1}{C} \|Q^k f\|_{H_\Lambda^m(T)} \leq \|Q^k f\|_{H_A^m(T)} \leq \|Q\| \|f\|_{H_A^m(N_d^c T)} \leq C \|Q\| \|f\|_{H_\Lambda^m(N_d^c T)}. \tag{4.41}$$

Step 4. For either $k \geq 0$ and $T \in W_0^k$, or $k = 0$ and $T \in W_{\mathrm{cen}}(d)$ we prove

$$|Q^k f|_{H_\Lambda^m(T)} \leq C |f|_{H_\Lambda^m(N_d^c T)} \qquad \text{for } 0 \leq m \leq s. \tag{4.42}$$

Suppose $g \in \mathbb{P}_{m-1}(K_n)$ is a characteristic coordinate polynomial. Notice $\mathbb{P}_{m-1}(K_n) \subset \mathbb{P}_{r-2}(K_n)$, and therefore

$$|Q^k f|_{H^m_\Lambda(T)} = |Q^k(f-g)|_{H^m_\Lambda(T)}, \tag{4.43}$$

since Q^k reproduces $\mathbb{P}_{r-2}(K_n)$. Then applying (4.40) we get

$$|Q^k f|_{H^m_\Lambda(T)} \leq \|Q^k(f-g)\|_{H^m_\Lambda(T)} \leq C\|(f-g)\|_{H^m_\Lambda(N^c_d T)}. \tag{4.44}$$

In particular, we can take $g = T^m_{N^c_d T} f$ as the averaged Taylor polynomial approximation. The chunkiness parameter for any disk is 2, so by Corollary 49 we get

$$\|f - T^m_{N^c_d T} f\|_{H^m_\Lambda(N^c_d T)} \leq C|f|_{H^m_\Lambda(N^c_d T)}, \tag{4.45}$$

where the constant C is independent of T and k. Substituting (4.45) into (4.44) gives (4.42).

Step 5. We show that the uniform boundedness property (4.6) holds for any face $T \in W^k$ and any $k \geq 0$. Now $c^{-k}T \in \mathsf{Face}(W)$, and we consider two cases: (i) $c^{-k}T \notin \mathsf{Face}(W_{\mathrm{cen}}(d))$ and (ii) $c^{-k}T \in \mathsf{Face}(W_{\mathrm{cen}}(d))$. In case (i) we have $c^{-k}T$ in a region $W_j = c^{-j}W_0 \subset W$ for some $j \geq 0$. Let $\widetilde{T} = c^j c^{-k}T = c^{j-k}T$, which is a face in W^j_0. In case (ii) we set $j = 0$ and $\widetilde{T} = c^{-k}T \in \mathsf{Face}(W_{\mathrm{cen}}(d))$.

Now (4.6) follows from Step 4 and Proposition 56, where Proposition 56 is applied with linear map $L = Q^j$, dilation c^{k-j}, and domains $\Omega = N^c_d \widetilde{T}$ and $\Omega' = \widetilde{T}$. Indeed, with these substitutions, the hypothesis (4.38) becomes

$$|Q^j f|_{H^m_\Lambda(\widetilde{T})} \leq C \, |f|_{H^m_\Lambda(N^c_d \widetilde{T})},$$

which follows from Step 4. Therefore, the conclusion of Proposition 56 implies (4.6) since

$$c^{-(k-j)*}Q^j c^{(k-j)*} = c^{-k*}Q c^{k*} = Q^k.$$

Chapter 5

CONSTRUCTING QUASI-INTERPOLANTS

In this chapter we complete the proof of the main result, Theorem 2, by constructing a quasi-interpolant on K_n for Loop's subdivision functions. The main result follows from the global approximation theorem, Theorem 43 in Chapter 3, and a bound on the local approximation error of the quasi-interpolant Q^k given by

$$\|Q^k f - f\|_{H^s_\Lambda(\widehat{K_n})} \leq C_\epsilon \lambda_{\max}^{(r-s-\epsilon)k} \|f\|_{H^r_\Lambda(N^c_{d_n} \widehat{K_n})}, \tag{5.1}$$

where d_n is the support width of Q^k on K_n and $\widehat{K_n} = N_3(v_0, K_n^2)$. Theorem 45 in Chapter 4 shows that the local approximation bound (5.1) is satisfied by a quasi-interpolant of order 3. To be precise, Theorem 2 is implied by the following theorem.

Theorem 57. *For Loop's subdivision scheme there exists a quasi-interpolant on K_n of order 3, which is uniformly H^2_Λ-bounded for each valence n.*

A quasi-interpolant on K_n is a family of linear maps $Q^k : L^1_{\Lambda,\mathrm{loc}}(K_n) \to S(K_n^k)$ satisfying the conditions of Definition 44. Recall, by Lemma 55, that to define Q^k we need only construct the map

$$Q : L^1_{\Lambda,\mathrm{loc}}(K_n) \to S(K_n),$$

and then define Q^k in general by $Q^k = c^{-k*} Q c^{k*}$. The linear map Q must be rotation equivariant, local, bounded, and reproduce a class of polynomials.

To show that Q is local and of order 3 we need to demonstrate the following:

(i) There exists a constant d such that for any simplicial sub-surface $K \subset K_n$, we have the implication

$$\text{if } f \equiv 0 \text{ on } |N_d K| \quad \text{then} \quad Q f \equiv 0 \text{ on } |K|. \tag{5.2}$$

(ii) Linear polynomials in characteristic coordinates are reproduced by Q.

(iii) Local quadratic polynomials in affine coordinates, sufficiently far from the central vertex, are reproduced by Q. Specifically, there is some d such that for any face $T \in K_n$ with $|N_dT|$ contained in an affine coordinate neighborhood, then $Qf = f$ on $|T|$ for f a polynomial of degree at most 2 in affine coordinates on $|N_dT|$.

Notice that the constants d appearing in (i) and (iii) can be chosen to be the same by taking the largest of the two. The minimum value d satisfying both (i) and (iii) is called the support width of Q. Also recall from Theorem 43 that the support width d_n for the quasi-interpolant on K_n is allowed to depend on the valence n.

To show that Q satisfies the H^2-alternative boundedness criteria we must verify the semi-norm estimates

$$|Qf|_{H_A^m(T)} \leq \|Q\| \, |f|_{H_A^m(N_d^cT)} \tag{5.3}$$

for $m = 0, 1$ or 2 and any face $T \in W \setminus W_{\text{cen}}(d)$. Notice in particular that these estimates are computed in affine coordinates away from the central vertex.

We construct Q as a composition of three maps

$$Q = \mathcal{S}^\infty \circ \mathcal{A} \circ \mathcal{R}.$$

The operator $\mathcal{R} : L^1_{\Lambda,\text{loc}}(K_n) \to \mathsf{CN}(K_n)$ is called the restriction operator. The averaging operator $\mathcal{A} : \mathsf{CN}(K_n) \to \mathsf{CN}(K_n)$ averages the neighbors at each vertex, and $\mathcal{S}^\infty : \mathsf{CN}(K_n) \to \mathcal{S}(K_n)$ is the subdivision limit operator for Loop's subdivision scheme.

We define restriction and averaging so that they are rotation equivariant. Then since the subdivision operator $\mathcal{S}^\infty$ is also rotation equivariant, the composition Q is rotation equivariant, as can be seen from the following commutative diagram

$$
\begin{array}{ccccccc}
L^1_{\Lambda,\text{loc}}(K_n) & \xrightarrow{\mathcal{R}} & \mathsf{CN}(K_n) & \xrightarrow{\mathcal{A}} & \mathsf{CN}(K_n) & \xrightarrow{\mathcal{S}^\infty} & \mathcal{S}(K) \\
{\sigma^*}\downarrow & & \downarrow{\sigma^*} & & \downarrow{\sigma^*} & & \downarrow{\sigma^*} \\
L^1_{\Lambda,\text{loc}}(K_n) & \xrightarrow{\mathcal{R}} & \mathsf{CN}(K_n) & \xrightarrow{\mathcal{A}} & \mathsf{CN}(K_n) & \xrightarrow{\mathcal{S}^\infty} & \mathcal{S}(K) \ ,
\end{array}
\tag{5.4}
$$

where σ is a rotation on K_n.

The technique of decomposing the quasi-interpolant into components which are analyzed separately is motivated by Kowalski [12]. In this paper, Kowalski studied approximation

by box splines in Sobolev spaces on $\mathbb{R}^n$. Since our semi-norm estimates are local, in the neighborhood of a face T away from v_0, and are computed in affine coordinates, we can easily adapt the theory from this paper. Our development is slightly different though since we need results on domain of $\mathbb{R}^2$ instead of all of $\mathbb{R}^2$.

In Section 5.1 we review the properties of box splines. In Section 5.2 we define the restriction operator $\mathcal{R}$ and show that it and the subdivision limit operator $\mathcal{S}^\infty$ satisfy the necessary semi-norm estimates. In Section 5.3 we derive conditions that the averaging operator must satisfy. These conditions are represented by a linear system of equations at each vertex of K_n. In the remaining sections, we prove that the linear systems have satisfactory solutions.

5.1   Box Splines

Loop's subdivision functions are locally given by box splines away from the extraordinary vertices. Box splines are a multi-dimensional generalization of uniform B-splines. The spline functions generated by Loop's subdivision on a regular grid are bivariate splines defined on a 3-direction grid. We present only a quick catalogue of the needed results. A thorough presentation of box splines may be found in deBoor, Höllig and Riemenschneider [6].

We are interested in bivariate box splines on a 3-direction equilateral grid in $\mathbb{R}^2$, i.e. a regular embedding of K_6 into $\mathbb{R}^2$. The grid is defined by a pair of linearly independent direction vectors ξ_1 and $\xi_2 \in \mathbb{R}^2$, which in turn define a third direction vector $\xi_3 = \xi_1 + \xi_2$. We use the 3-direction equilateral grid defined by

$$X = [\xi_1, \xi_2, \xi_3] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix}. \tag{5.5}$$

For any multi-index $\beta \in \mathbb{Z}_+^3$, we define the *direction set* X^β as a $2 \times |\beta|$ matrix given by

$$X^\beta = [\overbrace{\xi_1, \ldots, \xi_1}^{\beta_1}, \overbrace{\xi_2, \ldots, \xi_2}^{\beta_2}, \overbrace{\xi_3, \ldots, \xi_3}^{\beta_3}],$$

which we also interpret as a linear map $X^\beta : \mathbb{R}^{|\beta|} \to \mathbb{R}^2$.

Definition 58. *Given a multi-index $\beta \in \mathbb{Z}_+^3$, we define the associated* **box spline basis function** *as*

$$M_{X^\beta}(x) = vol\left((X^\beta)^{-1}(x) \cap [0,1]^{|\beta|}\right) / \sqrt{\det(X^\beta(X^\beta)^t)}$$

for $x \in \mathbb{R}^2$. Here $(X^\beta)^{-1}(x)$ is a $(|\beta| - 2)$−dimensional subspace of $\mathbb{R}^{|\beta|}$ and vol is the $(|\beta| - 2)$-dimensional volume.

We write M_β for M_{X^β} when the direction set X is clear from context. We state a few general properties of box splines.

Proposition 59. *For any multi-index $\beta \in \mathbb{Z}_+^3$, with at most one component that is zero, the box spline basis function M_β satisfies the following:*

(i) $M_\beta \geq 0$ *and* $\mathrm{supp}(M_\beta) = X^\beta([0,1]^{|\beta|})$.

(ii) $\int M_\beta = 1$.

(iii) $\sum_{v \in K_6} M_\beta(x - v) \equiv 1$.

(iv) M_β *is a polynomial of degree at most $|\beta| - 2$ on each face of K_6.*

(v) $M_\beta \in C^{m-1}(\mathbb{R}^2)$, *where $m = |\beta| - \beta_i - 1$ and β_i is the largest index of β. Moreover, $M_\beta \in C^{m-1,1}$, that is, the order $m - 1$ derivatives of M_β are Lipschitz continuous, since M_β consists of a finite number of pieces where each is smooth to its boundary.*

A *spline function* is a linear combination of translates of a box spline basis function of the form

$$f(x) = \sum_{v \in K_6} u_v M_\beta(x - v), \tag{5.6}$$

where the sum is taken over vertices in K_6. For any point $x \in \mathbb{R}^2$ and region $\Omega \subset \mathbb{R}^2$ we define the *index sets* for the direction set X^β by

$$J(x, \beta) = \{x - X^\beta(0,1)^2\} \cap \mathsf{Vertex}(K_6) \quad \text{and} \quad J(\Omega, \beta) = \bigcup_{x \in \Omega} J(x, \beta). \tag{5.7}$$

The index set $J(x, \beta)$ or $J(\Omega, \beta)$ contains all indices $v \in \mathsf{Vertex}(K_6)$, such that $M_\beta(\cdot - v)$ is non-zero on x or Ω respectively. To define a box spline on a region, we only need coefficients on the index set. For any region $\Omega \subset \mathbb{R}^2$ with f given by (5.6) we have

$$f(x) = \sum_{v \in J(\Omega, \beta)} u_v M_\beta(x - v) \qquad \text{for } x \in \Omega. \tag{5.8}$$

To compute the Sobolev norm of a spline function we must take its derivatives. The derivative of a box spline is another box spline of lower order. We define a backward difference operator $\nabla_{\xi_i} = \nabla_i$ for $i = 1, 2$ or 3 acting on a continuous function $f \in C(\mathbb{R}^2)$ by $\nabla_i f(x) = f(x) - f(x - \xi_i)$, and we let $\nabla_{X^\alpha} = \nabla_\alpha = \nabla_{\xi_1}^{\alpha_1} \nabla_{\xi_2}^{\alpha_2} \nabla_{\xi_3}^{\alpha_3}$, for $\alpha \in \mathbb{Z}_+^3$, denote higher-order differences. We also apply these operators to control nets. For $u \in \mathsf{CN}(K_6)$ we define $\nabla_i u$ by $(\nabla_i u)_v = u_v - u_{v - \xi_i}$. Also we define a centered difference operator $(\nabla_{X^\beta}^c f)(x) = (\nabla_{X^\beta} f)(x + c)$, where $c = \frac{1}{2}(\beta_1 \xi_1 + \beta_2 \xi_2 + \beta_3 \xi_3)$. Note the centered difference operator is not affected by a sign change or permutation in the direction set. Specifically, if $Y = X \cdot \mathrm{diag}(\pm 1, \pm 1, \pm 1) \cdot P$ is another direction set, differing only by the orientation of the individual directions and a permutation matrix P, then $\nabla_{X^\beta}^c = \nabla_{Y^\beta}^c$. Also we define directional derivative operators $D_i f = \xi_{i,1} \frac{\partial f}{\partial x_1} + \xi_{i,2} \frac{\partial f}{\partial x_2}$ and higher-order differential operators $D_\alpha = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}$.

The key differentiation identity for a box spline basis function M_β is that for any $\alpha \leq \beta$ (meaning $\alpha_i \leq \beta_i$ for all i) we have

$$D_\alpha M_\beta = \nabla_\alpha M_{\beta - \alpha}. \tag{5.9}$$

Applying this identity to the box spline function (5.8) on a subcomplex $L \subset K_6$ we get

$$D_\alpha f(x) = \sum (\nabla_\alpha u)_v M_{\beta - \alpha}(x - v) \qquad \text{for } x \in |L|, \tag{5.10}$$

where the sum is taken over all vertices $v \in J(L, \beta)$ such that $(\nabla_\alpha u)_v$ is defined.

For each multi-index $\beta \in \mathbb{Z}_+^3$, the truncated power function T_β is a simpler piecewise polynomial function related to M_β by the identity

$$M_\beta = \nabla_\beta T_\beta. \tag{5.11}$$

The *truncated power function* T_β is defined by

$$T_\beta(x) = \mathrm{vol}\left((X^\beta)^{-1}(x) \cap \mathbb{R}_+^{|\beta|} \right) / \sqrt{\det(X^\beta (X^\beta)^t)},$$

where $\mathbb{R}_+^n = \{(x_1, \ldots, x_n) : x_1 \geq 0\}$. It is a polynomial on each face of K_6, is homogeneous of degree $|\beta| - 2$, and agrees with the box spline basis function M_β in the neighborhood of the origin. A box spline function with bounded support can be represented as a truncated power series. Suppose u_v is zero for all but finitely many $v \in K_6$, then $f(x)$ given by (5.6) may be written as

$$f(x) = \sum_{v \in K_6} u_v (\nabla_\beta T_\beta)(x - v) = \sum_{v \in K_6} (\nabla_\beta u)_v T_\beta (x - v). \tag{5.12}$$

Example: Loop's subdivision on a regular equilateral grid generates subdivision limit functions which are box spline functions. Using equilateral affine coordinates, the box spline basis function is M_{222}, shown in Figure 5.1(a), with support shown in Figure 5.1(b). The basis function, and hence Loop's subdivision functions, are piecewise quartic polynomials and C^2. The truncated power function is composed of two non-zero polynomial pieces as shown in Figure 5.1(c). These polynomial pieces can be computed explicitly as

$$T_+(x) = 2x_1^4 - 4x_1^2 x_2^2 + \tfrac{16}{3\sqrt{3}} x_1 x_2^3 - \tfrac{2}{3} x_2^4 \tag{5.13}$$

and

$$T_-(x) = 2x_1^4 - 4x_1^2 x_2^2 - \tfrac{16}{3\sqrt{3}} x_1 x_2^3 - \tfrac{2}{3} x_2^4. \tag{5.14}$$

5.2 Boundedness of Loop's Subdivision and Restriction

In this section we show that the subdivision limit operator for Loop's scheme S^∞ is bounded. To do this we need Sobolev norms and semi-norms on $\mathsf{CN}(L)$ for an arbitrary simplicial sub-surface $L \subset K_6$. In this section, we also define the restriction operator $\mathcal{R}$ and show that it satisfies similar semi-norm estimates. The semi-norm estimates that we prove will later be used to show that the composition $Q = S^\infty \circ \mathcal{A} \circ \mathcal{R}$ satisfies the semi-norm estimates in (5.3).

On the regular grid K_6, in equilateral affine coordinates, Loop's subdivision functions are quartic box splines. Given $u \in \mathsf{CN}(K_6)$, the subdivision limit function is given by the formula

$$S^\infty u(x) = \sum_{v \in K_6} u_v M_{222}(x - v + 2\xi_3),$$
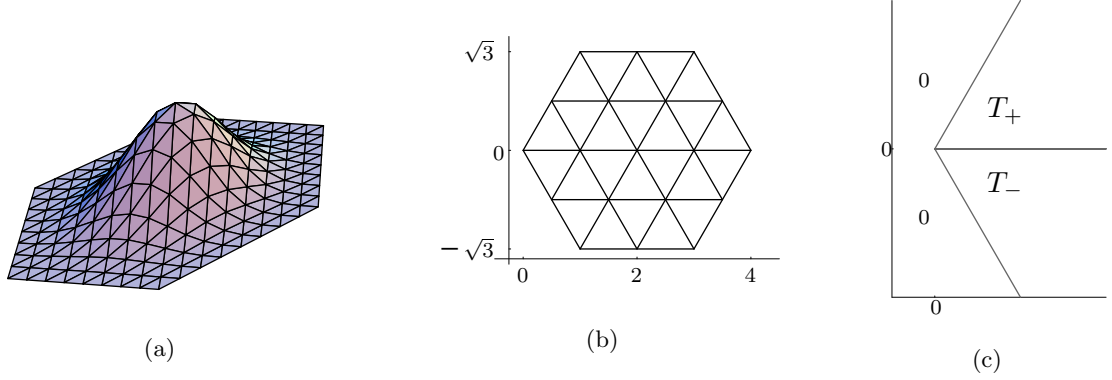
Figure 5.1: Quartic box spline basis function. (a) The M_{222} basis function. (b) The support of M_{222}. (c) The support of T_{222}.

where the sum is over $v \in \mathsf{Vertex}(K_6)$. Recall the mask width of Loop's subdivision scheme is 2, so by Proposition 8, if $L \subset K$ is a subcomplex of a simplicial surface K without boundary, and $N_1(L) \hookrightarrow K_6$ is an embedding into the equilateral grid, then the embedding defines an affine coordinate system $(|N_1(L)|^\circ, x)$. With respect to these coordinates we have

$$\mathcal{S}^\infty u(x) = \sum_{v \in N_1(L)} u_v M_{222}(x - v + 2\xi_3) \qquad \text{for } x \in |L|.$$

We now define norms and semi-norms for the control net spaces $\mathsf{CN}(L)$, where $L \subset K_6$ is a simplicial sub-surface of K_6 with the equilateral realization. The $L^2_{\mathsf{CN}}(L)$-norm of $u \in \mathsf{CN}(L)$ is defined by

$$\|u\|^2_{L^2_{\mathsf{CN}}(L)} = \sum_{v \in L} u_v^2.$$

Then the $H^m_{\mathsf{CN}}(L)$-semi-norms and norms are defined using the backward difference operators ∇_α for $\alpha \in \mathbb{Z}^3_+$ by

$$|u|^2_{H^m_{\mathsf{CN}}(L)} = \sum_{|\alpha|=m} \|\nabla_\alpha u\|^2_{L^2_{\mathsf{CN}}} \quad \text{and} \quad \|u\|^2_{H^m_{\mathsf{CN}}(L)} = \sum_{s=0}^{m} |u|^2_{H^s_{\mathsf{CN}}(L)},$$

where each L^2_{CN}-norm $\|\nabla_\alpha u\|_{L^2_{\mathsf{CN}}}$ is computed over the set of vertices where the differenced control net $\nabla_\alpha u$ is defined.

Let

$$Z^m(L) = \{u \in \mathsf{CN}(L) : |u|_{H^m_{\mathsf{CN}}(L)} = 0\} \tag{5.15}$$

be the H_{CN}^m-null space. Given a function $f \in C(|L|)$, let $f_{|L} \in \mathsf{CN}(L)$ be the control net obtained by restricting the function to the vertices of the control net.

Proposition 60. *Let $L \subset K_6$ be a non-empty, connected simplicial sub-surface. Then for any $m = 0, 1$ or 2 we have*

$$Z^m(L) = \{p_{|L} : p \in \mathbb{P}_{m-1}\},$$

where we interpret $\mathbb{P}_{-1}$ to be $\{0\}$.

Proof. It is clear that $\{p_{|L} : p \in \mathbb{P}_{m-1}\} \subset Z^m(L)$ for $m = 0, 1$ or 2, since a second-order difference of a linear function is 0. Conversely, we show that $Z^m(L) \subset \{p_{|L} : p \in \mathbb{P}_{m-1}\}$. This is clear for $m = 0$ or 1, so we need only consider $m = 2$. Let $u \in Z^2(L)$. For any face $T \in L$, there is a unique $p \in \mathbb{P}_1$ that interpolates u on the vertices of T, i.e., $u_v = p(v)$ for all $v \in \mathsf{Vertex}(T)$. The control net $p_{|L} \in \mathsf{CN}(L)$ is an extension $\widetilde{u}$ of $u_{|T} \in \mathsf{CN}(T)$ such that $\widetilde{u} \in Z^2(L)$. We show that, given $u_{|T} \in \mathsf{CN}(T)$, there is a unique extension $\widetilde{u} \in Z^2(L)$, and therefore $u = p_{|L}$, completing the proof.

To show that the extension $\widetilde{u} \in Z^2(L)$ is unique, we prove the following claim:

> Suppose $K \subset K_6$ is a non-empty, connected simplicial surface, and T is a face of K_6 such that $K \cup \overline{T} \subset K_6$ is also a connected simplicial surface. Then for any $u \in Z^2(K)$, there is a unique extension $\widetilde{u} \in Z^2(K \cup \overline{T})$.

The proposition follows from the claim, since starting with only $\overline{T}$ we can add one face at a time until we get $L = \overline{T} + \overline{T_1} + \cdots + \overline{T_n}$. As each face is added we extend $\widetilde{u}$ in a unique way.

To prove the claim, notice there is at most one vertex w in $K \cup \overline{T}$ which is not in K. If there is no such vertex, then u is already a control net on $K \cup \overline{T}$. If there is such a vertex w, let T' be a face in K such that $T \cap T'$ is an edge of K. There is a vertex $v_0 \in \mathsf{Vertex}(\overline{T} \cup \overline{T'})$ and a multi-index $\alpha \in Z_+^3$ with $|\alpha| = 2$ and with indices i_1 and i_2 such that $\alpha_{i_1} = \alpha_{i_2} = 1$, and such that

$$\left\{v_0, \quad v_0 - \xi_{i_1}, \quad v_0 - \xi_{i_2}, \quad v_0 - \xi_{i_1} - \xi_{i_2}\right\} = \mathsf{Vertex}(\overline{T} \cup \overline{T'}).$$

Now, any $\widetilde{u} \in Z^2(K \cup \overline{T})$ must satisfy

$$(\nabla_\alpha \widetilde{u})_{v_0} = 0.$$

This is a linear equation with one unknown value $\widetilde{u}_w$, so there is a unique solution. $\qquad\square$

We redefine the H_A^m-Sobolev norms on $|L|$ so that they have a parallel construction with the control net norms. They are redefined as

$$|f|_{H_A^m(L)}^2 = \sum_{|\alpha|=m} \|D_\alpha f\|_{L_A^2(L)}^2 \qquad \text{and} \qquad \|f\|_{H_A^m(L)}^2 = \sum_{s=0}^{m} |f|_{H_A^s(L)}^2 .$$

These norms are equivalent to the standard Sobolev norms on $|L|$ given by

$$\|f\|_{H^m(L)}^2 = \sum_{|\gamma|\leq m} \int_{|L|} (D^\gamma f(x))^2 \, dx ,$$

where $\gamma \in \mathbb{Z}_+^2$ and $D^\gamma = (\frac{\partial}{\partial x_1})^{\gamma_1}(\frac{\partial}{\partial x_2})^{\gamma_2}$. Note that we use the upper indices to denote partial derivatives and lower indices to denote directional derivatives in the grid directions. To prove that the norms are equivalent, we consider the linear map $x = Bz = \begin{pmatrix} 1/2 & 1/2 \\ -\sqrt{3}/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Then by Proposition 40 we have an equivalence of norms

$$c\|f\|_{H^m(L)} \leq \|f \circ B\|_{H^m(B^{-1}L)} \leq C\|f\|_{H^m(L)} . \tag{5.16}$$

The linear map B transforms the standard grid into the equilateral grid, and therefore for a multi-index $\alpha \in \mathbb{Z}_+^3$

$$(D_\alpha f)(Bz) = D^{(\alpha_1+\alpha_3,\alpha_2)}(f \circ B)(z) + D^{(\alpha_1,\alpha_2+\alpha_3)}(f \circ B)(z) .$$

So we have

$$\|f\|_{H_A^m(L)}^2 = \sum_{|\alpha|\leq m} \int_{|L|} (D_\alpha f(x))^2 dx = \sum_{|\alpha|\leq m} \int_{B^{-1}|L|} (D_\alpha f(Bz))^2 |\det B| \, dz$$

$$= \sum_{|\alpha|\leq m} \int_{B^{-1}|L|} \left(D^{\alpha_1+\alpha_3,\alpha_2}(f \circ B)(z) + D^{\alpha_1,\alpha_2+\alpha_3}(f \circ B)(z) \right)^2 |\det B| \, dz .$$

Therefore, there are constants such that

$$c\|f \circ B\|_{H^m(B^{-1}L)}^2 \leq \|f\|_{H_A^m(L)}^2 \leq C\|f \circ B\|_{H^m(B^{-1}L)}^2 . \tag{5.17}$$

Combining (5.16) and (5.17) proves that these are equivalent norms.

Proposition 61. *Let T be a face in K_n with an embedding $N_1(T) \hookrightarrow K_6$, then for any control net $u \in \mathsf{CN}(N_1 T)$ we have*

$$|\mathcal{S}^\infty u|_{H_A^m(T)} \leq |u|_{H_{\mathsf{CN}}^m(N_1 T)}$$

for any $m = 0, 1$ or 2.

In computing the bounds for the subdivision and restriction operators we will use a general form of Young's inequality, which as Folland [10] says "should be more widely known".

Proposition 62 (The Generalized Young's Inequality). *Let (X, μ) and (Y, ν) be σ-finite measure spaces, and let $1 \leq p \leq \infty$ and $C > 0$. Suppose K is a measurable function on $X \times Y$ such that*

$$\sup_{x \in X} \int_Y |K(x, y)| \, d\nu(y) \leq C \qquad and \qquad \sup_{y \in Y} \int_X |K(x, y)| \, d\mu(x) \leq C. \qquad (5.18)$$

Given $f \in L^p(Y)$, the function

$$Tf(x) = \int_Y K(x, y) f(y) d\mu(y) \qquad (5.19)$$

is well defined almost everywhere, and Tf is in $L^p(X)$ with $\|Tf\|_{L^p(X)} \leq C\|f\|_{L^p(Y)}$.

The proof can be found in Folland [10]. It relies on some clever applications of Hölder's inequality.

Proof of Proposition 61. For any $u \in \mathsf{CN}(N_1 T)$ and any multi-index $|\alpha| = m$ we have, by (5.10),

$$D_\alpha(\mathcal{S}^\infty u)(x) = \sum_v (\nabla_\alpha u)_v M_{222-\alpha}(x - v + 2\xi_3) \qquad \text{for } x \in |T|, \qquad (5.20)$$

where the sum is taken over the index set J on which $\nabla_\alpha u$ is defined. The operator $\nabla^\alpha u \mapsto D_\alpha \mathcal{S}^\infty u$ in (5.20) may be viewed as an "integral" operator of the form (5.19) in the Generalized Young's Inequality. Specifically, take $X = |T|$ with Lebesgue measure, $Y = J$ with counting measure, and the kernel K to be

$$K(x, v) = M_{222-\alpha}(x - v + 2\xi_3).$$

We take $C = 1$ for the constant in Young's Inequality, since by Proposition 59 we have

$$\sum_{j \in J} M_{222-\alpha}(x - j) = 1 \text{ for } x \in |T| \quad \text{and} \quad \int_T M_{222-\alpha}(x - j)\, dx \leq 1 \text{ for } j \in J.$$

Therefore by Young's inequality we get

$$\|D_\alpha(\mathcal{S}^\infty u)\|_{L^2(T)} \leq \|\nabla_\alpha u\|_{L^2_{\mathsf{CN}}(J)}.$$

Summing over all multi-indices $|\alpha| = m$ yields

$$|\mathcal{S}^\infty u|^2_{H_A^m(T)} = \sum_{|\alpha|=m} \|D_\alpha(\mathcal{S}^\infty u)\|^2_{L^2(T)} \leq \sum_{|\alpha|=m} \|\nabla_\alpha u\|^2_{L^2_{\mathsf{CN}}} = |u|^2_{H_{\mathsf{CN}}^m(N_1 T)}.$$

$\square$

To define rotation equivariant operators, we first define the operator on a wedge W of K_n, then extend in a rotation equivariant way to all of K_n. Let $(|N_3(W \setminus N_3(v_0))|^\circ, x)$ be an affine equilateral coordinate system as shown in Figure 5.2.

$$W \setminus N_3(v_0)$$

PSfrag replacements

$$v_0$$
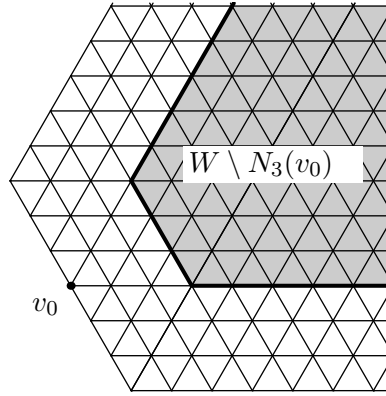
Figure 5.2: The equilateral affine coordinate system of $(|N_3(W \setminus N_3(v_0))|^\circ, x)$.

We define the restriction operator $\mathcal{R} : L^1_{\Lambda,\mathrm{loc}}(K_n) \to \mathsf{CN}(K_n)$ by integrating the function over balls. We fix a constant $0 < \rho < \sqrt{3}/2$ which will define the radius of these balls. For each $v \in \mathsf{Vertex}(W) \setminus v_0$ and $f \in L^1_{\Lambda,\mathrm{loc}}(K_n)$ we define

$$(\mathcal{R}f)_v = \frac{1}{\pi\rho^2} \int_{B_\rho} f(v + x)\, dx, \tag{5.21}$$

where $B_\rho = \{x \in \mathbb{R}^2 : |x| < \rho\}$. We extend restriction to a vertex $v \in K_n \setminus v_0$ using a rotation σ of K_n such that $\sigma^{-1}v \in W$, then define $(\mathcal{R}f)_v = (\mathcal{R}(f \circ \sigma))_{\sigma^{-1}v}$ so that $\mathcal{R}$ is rotation equivariant.

For the central vertex $v_0 \in K_n$, let $y : |K_n| \to \mathbb{R}^2$ be an equivariant characteristic coordinate chart such that $|y(v)| = 1$ for $v \in N_1(v_0) \setminus v_0$. We define restriction at the central vertex by

$$(\mathcal{R}f)_{v_0} = \frac{1}{\pi\rho^2} \int_{B_\rho} f(v_0 + y)\,dy\,, \tag{5.22}$$

where $B_\rho = \{y \in \mathbb{R}^2 : |y| < \rho\}$. The bound on ρ gives the following locality property of $\mathcal{R}$

$$\text{if } f \equiv 0 \text{ on } |N_1(v)| \text{ then } (\mathcal{R}f)_v = 0\,. \tag{5.23}$$

Proposition 63. *For any $L \subset K_n$ such that there is an embedding $N_1(L) \hookrightarrow K_6$, we have the semi-norm estimate*

$$|\mathcal{R}f|_{H^m_{CN}(L)} \leq C|f|_{H^m_A(N_1 L)} \qquad \text{for } m = 0, 1 \text{ or } 2,$$

computed in the affine coordinates on $N_1(L)$.

To prove the lemma, we introduce a more general restriction operator for a subcomplex $L \subset K_6$. The restriction operator $\mathcal{R}^A : L^1_{\text{loc}}(L - \Omega) \to \mathsf{CN}(L)$ is defined in terms of a function $A \in L^\infty(\mathbb{R}^2)$ that is supported on a bounded set Ω. The set difference notation means $A \pm B = \{x \pm y : x \in A \text{ and } y \in B\}$. The general restriction operator is defined by

$$(\mathcal{R}^A f)_v = \int A(v - x)f(x)dx \qquad \text{for } f \in L^1_{\text{loc}}(L - \Omega). \tag{5.24}$$

Lemma 64. *Suppose $A \in L^\infty(\mathbb{R}^2)$ is supported in a bounded set Ω. For any subcomplex $L \subset K_6$, the restriction operator is bounded in the L^2-norm as follows*

$$\|\mathcal{R}^A f\|_{L^2_{CN}(L)} \leq |\mathcal{R}^A|\,\|f\|_{L^2(L-\Omega)}\,, \tag{5.25}$$

where

$$|\mathcal{R}^A| = \max\left\{\|A\|_{L^1},\ \left\|\sum_{v \in K_6} |A(\cdot - v)|\right\|_{L^\infty}\right\} \tag{5.26}$$

is a constant independent of L.

Proof. The operator $\mathcal{R}^A$ is an integral operator of the form (5.19) in the Generalized Young's Inequality. Specifically, let $X = \mathsf{Vertex}(L)$ with counting measure, $Y = |L| - \Omega$ with Lebesgue measure, and $K(v, y) = A(v - y)$ be the kernel. Both the bounds in (5.18) are satisfied by (5.26), proving the Lemma. $\qquad\square$

We introduce an integral operator, which we use in the proof of Proposition 63,

$$I_i A(x) = \int_{-1}^{0} A(x - t\xi_i)\, dt \qquad \text{for } i = 1, 2 \text{ or } 3 \tag{5.27}$$

and the iterated version $I^\alpha = I_1^{\alpha_1} I_2^{\alpha_2} I_3^{\alpha_3}$ for a multi-index $\alpha \in \mathbb{Z}_+^3$.

Lemma 65. *Suppose $A \in L^\infty(\mathbb{R}^2)$ is supported in a bounded domain Ω. Then we have the following:*

(i) *For any subcomplex $L \subset K_6$, multi-index $\alpha \in \mathbb{Z}_+^3$, and function $f \in L^1_{\mathrm{loc}}(L - \Omega)$, we have*

$$\nabla_\alpha \mathcal{R}^A f = \mathcal{R}^{I^\alpha A} D_\alpha f$$

wherever the differenced control net is defined.

(ii) *The constant in (5.26) satisfies the inequality*

$$|\mathcal{R}^{I^\alpha A}| \leq |\mathcal{R}^A|.$$

Proof. (i) For any function A as in the hypothesis and $i = 1, 2$ or 3 we have

$$
\begin{aligned}
(\nabla_i \mathcal{R}^A f)_w &= (\mathcal{R}^A f)_w - (\mathcal{R}^A f)_{w - \xi_i} \\
&= \int A(w - x)\big(f(x) - f(x - \xi_i)\big)\, dx \\
&= \int A(w - x) \int_{-1}^{0} D_i f(x + t\xi_i)\, dt\, dx \\
&= \int D_i f(x) \int_{-1}^{0} A(w - x - t\xi_i)\, dt\, dx \\
&= \int D_i f(x) I_i A(w - x)\, dx \\
&= (\mathcal{R}^{I_i A} D_i f)_w.
\end{aligned}
$$

(ii) Computing for $i = 1, 2$ or 3 we get the inequalities

$$\|I_i A\|_{L^1} = \int |I_i A(x)| \, dx \leq \int_{-1}^0 \int |A(x - t\xi_i)| \, dx \, dt = \|A\|_{L^1}$$

and

$$\sum_{v \in K_6} I_i A(x - v) = \int_{-1}^0 \sum_{v \in K_6} A(x - v - t\xi_i) \, dt \leq \left\| \sum_{v \in K_6} A(\cdot - v) \right\|_{L^\infty}.$$

Therefore $|\mathcal{R}^{I_i A}| = \max\{\|I_i A\|_{L^1}, \|\sum_{v \in K_6} I_i A(x - v)\|_{L^\infty}\} \leq |\mathcal{R}^A|.$

$\square$

Proof of Proposition 63. The embedding $N_1 L \hookrightarrow K_6$ defines an equilateral affine coordinate system $(|N_1 L|^\circ, x)$. In this coordinate system we have

$$\mathcal{R}f = \mathcal{R}^A f \quad \text{on } L, \text{ where } A = \frac{1}{\pi \rho^2} \chi(B_\rho) \, ;$$

i.e., A is a multiple of the characteristic function of the ball of radius ρ centered at the origin. Let $\alpha \in \mathbb{Z}_+^3$ be a multi-index with $|\alpha| = m$, and let J_α be the index set on which $\nabla_\alpha u$ is defined for $u \in \mathsf{CN}(L)$. By Lemmas 64 and 65 we have

$$
\begin{aligned}
\|\nabla_\alpha \mathcal{R}^A f\|_{L^2_{\mathsf{CN}}(J_\alpha)} &= \|\mathcal{R}^{I^\alpha A} D^\alpha f\|_{L^2_{\mathsf{CN}}(J_\alpha)} && \text{by Lemma 65 part (i)} \\
&\leq |\mathcal{R}^{I^\alpha A}| \, \|D^\alpha f\|_{L^2(L - B_\rho)} && \text{by Lemma 64} \\
&\leq |\mathcal{R}^A| \, \|D^\alpha f\|_{L^2(N_1 L)} && \text{by Lemma 65 part (ii).}
\end{aligned}
$$

Summing over α we get

$$|\mathcal{R}f|^2_{H^m_{\mathsf{CN}}(L)} = |\mathcal{R}^A f|^2_{H^m_{\mathsf{CN}}(L)} = \sum_{|\alpha| = m} \|\nabla_\alpha \mathcal{R}^A f\|^2_{L^2_{\mathsf{CN}}} \leq C \sum_{|\alpha| = m} \|D^\alpha f\|^2_{L^2(N_1 L)} = C \, |f|^2_{H^m(N_1 L)}.$$

$\square$

5.3 A Quasi-Interpolant on $\mathbf{K_n}$

We construct the averaging operator $\mathcal{A} : \mathsf{CN}(K_n) \to \mathsf{CN}(K_n)$ in this section, completing the definition of the quasi-interpolant $Q = \mathcal{S}^\infty \circ \mathcal{A} \circ \mathcal{R}$ on K_n for Loop's subdivision scheme. We then prove Theorem 57, showing that the quasi-interpolant Q has the desired properties.

This theorem depends on the solutions to a family of linear systems. The existence of these solutions is the subject of the remaining sections of the chapter.

The quasi-interpolant Q must reproduce both components of the characteristic map and a constant function. Let $\tau : |K_n| \to \mathbb{R}^2$ be an equivariant characteristic map, generated by a control net $\tau \in \mathsf{CN}(K_n)$ of the same name. The quasi-interpolant Q will reproduce these functions if for all vertices $v \in K_n$

$$(\mathcal{A}\,\mathcal{R}\,\tau)_v = \tau_v \qquad (5.28)$$

and

$$(\mathcal{A}\,1)_v = 1. \qquad (5.29)$$

Indeed if these equations are satisfied for every $v \in K_n$, then

$$Q\tau = \mathsf{S}^\infty \mathcal{A}\,\mathcal{R}(\tau) = \sum (\mathcal{A}\,\mathcal{R}\,\tau)_v\,\phi_v = \sum \tau_v \phi_v = \tau,$$

and similarly $Q1 = 1$. Notice, if (5.28) is satisfied for a single characteristic map τ then the same equation is satisfied for any characteristic map, since any characteristic map can be represented as a composition of τ with a linear transformation.

In some neighborhood of the origin we need only reproduce the linear polynomials in characteristic coordinates. We denote the restriction of $\mathcal{A}$ to a subcomplex $K \subset K_n$ by $\mathcal{A}_{|K}$: $\mathsf{CN}(K_n) \to \mathsf{CN}(K)$. We say the operator $\mathcal{A}_{|K}$ has *support width* d' if for any subcomplex $L \subset K \subset K_n$ we have the implication

$$\text{if } u \equiv 0 \text{ on } N_{d'}(L, K_n) \quad \text{then} \quad \mathcal{A}_{|K} u \equiv 0 \text{ on } \mathsf{Vertex}(L). \qquad (5.30)$$

Lemma 66. *For any valence n and integer $d' \geq 2$, we can define $\mathcal{A}_{|N_{d'}(v_0)}$ so that it has support width d' and satisfies (5.28) and (5.29) for every vertex $v \in N_{d'}(v_0)$.*

Proof. For each $v \in N_{d'}(v_0)$, let $v_1(v)$ and $v_2(v)$ be vertices in $N_1(v_0, K_n) \setminus v_0$ closest to v. Suppose $\mathcal{A}_{|N_{d'}(v_0)}$ is of the form

$$(\mathcal{A}u)_v = w_v^0 u_{v_0} + w_v^1 u_{v_1(v)} + w_v^2 u_{v_2(v)}, \qquad (5.31)$$

with undetermined coefficients w_v^0, w_v^1 and w_v^2. Since $\{v_0, v_1, v_2\} \subset N_{d'}(v)$, the operator $\mathcal{A}_{|N_{d'}(v_0)}$ has support width d'. We write (5.28) and (5.29) as a linear system

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & (\mathcal{R}\tau)_{v_1} & (\mathcal{R}\tau)_{v_2} \\ 0 & & \end{pmatrix} \begin{pmatrix} w_v^0 \\ w_v^1 \\ w_v^2 \end{pmatrix} = \begin{pmatrix} 1 \\ \tau_v \\ \end{pmatrix}. \tag{5.32}$$

We show that (5.32) has a solution. Note $(\mathcal{R}\tau)_{v_1}$ is non-zero. Let σ be the rotation of K_n such that $\sigma v_1 = v_2$. Since τ is equivariant, there is a 2×2 rotation matrix $\widetilde{\sigma}$ such that $\tau \circ \sigma = \widetilde{\sigma} \cdot \tau$. Then by the equivariance of $\mathcal{R}$ and τ we have

$$(\mathcal{R}\tau)_{v_2} = (\mathcal{R}\tau)_{\sigma v_1} = \big(\mathcal{R}(\tau \circ \sigma)\big)_{v_1} = \mathcal{R}(\widetilde{\sigma} \cdot \tau)_{v_1} = \widetilde{\sigma} \cdot (\mathcal{R}\tau)_{v_1}.$$

Therefore, since $\widetilde{\sigma}$ is a rotation other than the identity, the lower right 2x2 submatrix in (5.32) is non-degenerate, and so the system has a unique solution. $\qquad\square$
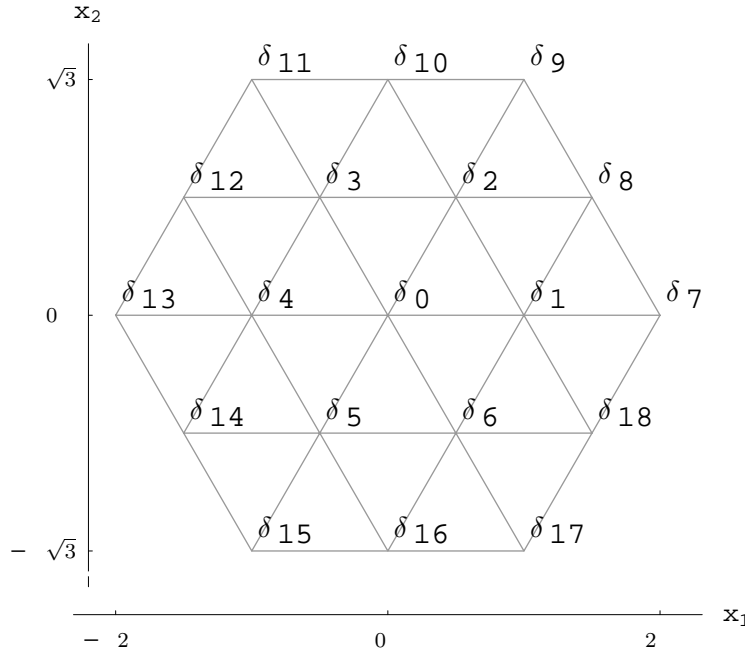

Figure 5.3: The stencil of offsets for the 2 neighborhood.

Away from the central vertex we must reproduce affine quadratic polynomials as well as characteristic linear polynomials. We work in the affine coordinate system $(|N_3(W \setminus$

$N_3(v_0))|^\circ, x)$ shown in Figure 5.2. If we define $\mathcal{A}$ on $W \setminus N_{d'}(v_0)$, then we can extend $\mathcal{A}$ to $K_n \setminus N_{d'}(v_0)$, so that it is rotation equivariant. Specifically, for any $v \in K_n \setminus N_{d'}(v_0)$, let σ be a rotation of K_n such that $\sigma^{-1}v \in W$, and define

$$(\mathcal{A}u)_v = (\mathcal{A}(u \circ \sigma))_{\sigma^{-1}v}. \tag{5.33}$$

For a fixed vertex $v \in W \setminus N_3(v_0)$ with coordinate representation $x(v) = (v_1, v_2)$, we consider the following basis for $\mathbb{P}_2$:

$$
\begin{aligned}
f_0(x) &= 1 & f_3(x) &= (x_1 - v_1)(x_2 - v_2) \\
f_1(x) &= x_1 - v_1 & f_4(x) &= (x_1 - v_1)^2 - (x_2 - v_2)^2 \\
f_2(x) &= x_2 - v_2 & f_5(x) &= (x_1 - v_1)^2 + (x_2 - v_2)^2 - \tfrac{1}{2}\rho^2
\end{aligned}
\tag{5.34}
$$

We construct convolution operators $L_j : \mathsf{CN}(N_2(W \setminus N_3(v_0))) \to \mathsf{CN}(W \setminus N_3(v_0))$, which have a certain duality property with the basis $\{f_k\}$. Let $\{\delta_i\}_{i=0}^{18}$ be the sequence of x-coordinate offset vectors, shown in Figure 5.3. The convolution operator L_j for $j = 0, \ldots, 7$ is defined by the formula

$$(L_j u)_w = \sum_{i=0}^{18} m_j^i \, u_{w+\delta_i} \qquad \text{for } w \in W \setminus N_3(v_0), \tag{5.35}$$

where the masks $m_j = \{m_j^i\}_{i=0}^{18}$ are shown in Figure 5.4.

Proposition 67. *For any vertex $v \in W \setminus N_3(v_0)$, let $\{f_k\}$ be the basis of $\mathbb{P}_2$ given by (5.34). Then $(L_j \, \mathcal{R} \, f_k)_v = \delta_{jk}$ for any $0 \le j \le 7$ and $0 \le k \le 5$.*

Proof. Since the operators L_j are translation equivariant and the basis functions are also translation equivariant, it suffices to prove the proposition for a single vertex v. Thus the proof reduces to 48 cases that need to be checked. Let $\widetilde{f}_k(x) = f_k(x)$ for $k = 0, \ldots, 4$, and let $\widetilde{f}_5(x) = (x_1 - v_1)^2 + (x_2 - v_2)^2$. We first claim that $(\mathcal{R}f_k)_w = \widetilde{f}_k(w)$ for any $w \in N_2(v)$ and $k = 0, \ldots, 5$. This follows from the mean value property, except for case $k = 5$, which must be computed explicitly. Most of the 48 cases can be checked easily by finding a suitable reflection symmetry of the mask and the control net $\widetilde{f}_{k|N_2(v)}$, such as a reflection about an axis. For instance, the mask m_1 has an odd symmetry with respect to reflection across the line $x_1 = v_1$, and $\widetilde{f}_0, \widetilde{f}_2, \widetilde{f}_4$ and $\widetilde{f}_5$ are even with respect to this reflection, therefore $L_1 f_0 = L_1 f_2 = L_1 f_4 = L_1 f_5 = 0$. Other cases must be checked individually. $\qquad\square$
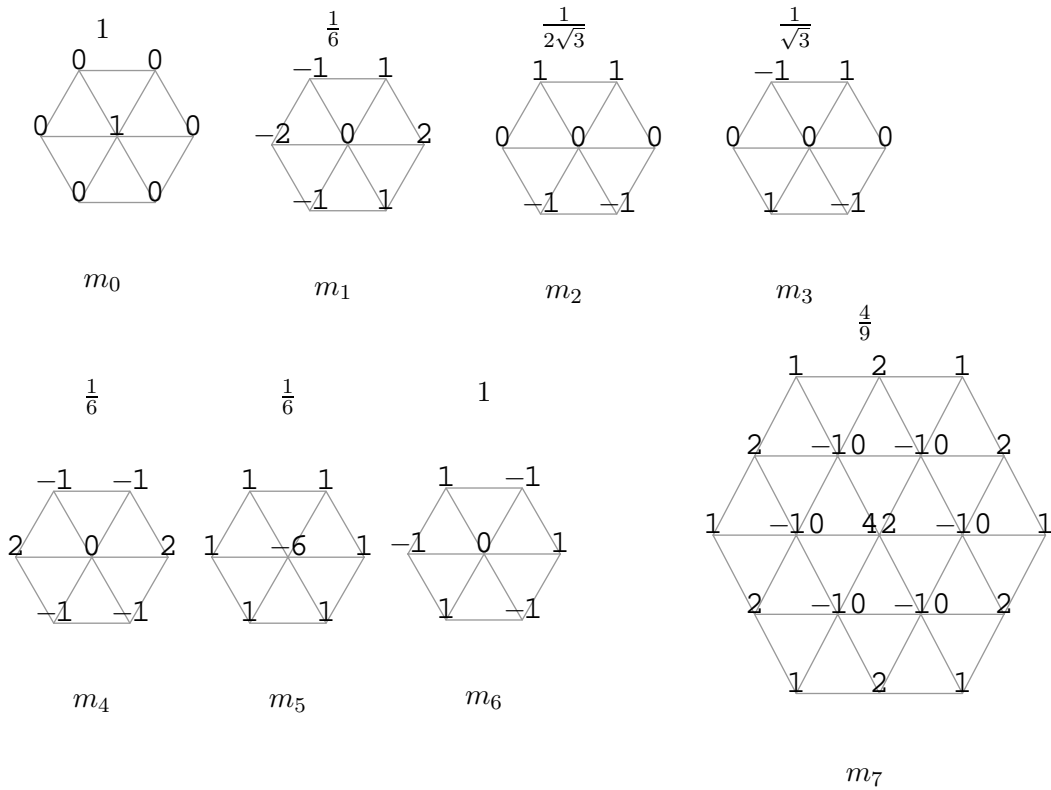
Figure 5.4: The masks $m_0, \ldots, m_7$. The number above the stencil is a common factor, which is multiplied into all elements of the mask. Elements of the mask which are not shown have a value of 0.

We define $\mathcal{A}_{W \setminus N_{d'}(v_0)}$ by

$$(\mathcal{A}u)_v = \sum_{j=0}^{7} w_v^j (L_j u)_v \qquad \text{for } v \in W \setminus N_{d'}(v_0). \tag{5.36}$$

The parameters d' and w_v^j are, for now, undetermined.

We use the quasi-interpolant theory on the regular grid K_6 to help determine the coefficients w_v^j. From Chapter III of deBoor, et. al. [6], we know that $\widehat{Q} : C(\mathbb{R}^2) \to \mathcal{S}(K_6)$ given by

$$\widehat{Q}f(x) = \sum_{v \in K_6} (\eta f)_v \, M_{222}(x - v + 2\xi_3),$$

with

$$(\eta f)_v = \frac{3}{2} f(v) - \frac{1}{12} \sum_{w \in N_1(v) \setminus v} f(w) \tag{5.37}$$

is a quasi-interpolant which reproduces all polynomials of degree at most 3. Since our quasi-interpolant Q is supposed to locally reproduce $\mathbb{P}_2$, we expect $\mathcal{A} \circ \mathcal{R}$ to behave exactly like η on $\mathbb{P}_2$. That is, for each $v \in W \setminus N_{d'}(v_0)$ we require that

$$(\mathcal{A} \mathcal{R} f_k)_v = (\eta f_k)_v \qquad \text{for } k = 0, \ldots, 5. \tag{5.38}$$

Notice that if (5.38) is satisfied at a vertex v, then by linearity $(\mathcal{A} \mathcal{R} f)_v = (\eta f)_v$ for any $f \in \mathbb{P}_2$. By the definition of $\mathcal{A}$ (5.36) and Proposition 67 we get

$$(\mathcal{A} \mathcal{R} f_k)_v = \sum_{j=0}^{7} w_v^j (L_j \mathcal{R} f_k)_v = w_v^k. \tag{5.39}$$

A simple computation shows

$$(\eta f_k)_v = \frac{3}{2} f_k(0) - \frac{1}{12} \sum_{i=1}^{6} f_k(\delta_i) = \begin{cases} 1 & k = 0, \\ 0 & k = 1, 2, 3, 4 \\ -\frac{1}{2}(1 + \rho^2) & k = 5. \end{cases} \tag{5.40}$$

So from (5.38), (5.39) and (5.40) we conclude $w_v^0 = 1, w_v^1 = \cdots = w_v^4 = 0$, and $w_v^5 = -\frac{1}{2}(1 + \rho^2)$.

Substituting (5.36) and the known values of $w_v^0, \ldots, w_v^5$ into the left side of (5.28) yields a 2×2 system that w_v^6 and w_v^7 must satisfy

$$\begin{pmatrix} L_6 \mathcal{R}\tau & L_7 \mathcal{R}\tau \end{pmatrix}_v \begin{pmatrix} w^6 \\ w^7 \end{pmatrix}_v = \left(\tau - L_0 \mathcal{R}\tau + \tfrac{1}{2}(1 + \rho^2) L_5 \mathcal{R}\tau \right)_v \tag{5.41}$$

at each $v \in W \setminus N_3(v_0)$. We call (5.41) the polynomial reproducing system.

Lemma 68. *Suppose for some valence n and integer $d' \geq 3$ we define $\mathcal{A}_{|W \setminus N_{d'}(v_0)}$ by (5.36), where the coefficients w_v^j satisfy*

$$w_v^0 = 1, \qquad w_v^1 = \cdots = w_v^4 = 0, \qquad w_v^5 = -\tfrac{1}{2}(1 + \rho^2), \tag{5.42}$$

and equation (5.41). We extend $\mathcal{A}$ to $K_n \setminus N_{d'}(v_0)$ using (5.33) so that it is rotation equivariant, and extend it to all of K_n by defining $\mathcal{A}_{|N_{d'-1}(v_0)}$ as in Lemma 66. Then $Q = \mathcal{S}^\infty \mathcal{A} \mathcal{R}$ is order 3 with support width $d = d' + 1$.

Proof. Since $\mathcal{A}_{|K_n \setminus N_{d'}(v_0)}$ has support width 2 and $\mathcal{A}_{|N_d(v_0)}$ has support width $d' - 1 \geq 2$, the support width of $\mathcal{A}$ is $d' - 1$. Then the support width of Q is $d = d' + 1$ since the mask width of Loop's subdivision scheme is 2 and (5.23) gives the support width of $\mathcal{R}$. To show that Q is order 3 we first show that Q reproduces characteristic functions. We do this by showing that (5.28) is satisfied for every vertex $v \in K_n$. Equation (5.28) is satisfied for $v \in N_{d'-1}(v_0)$ by Lemma 66 and for $v \in W \setminus N_{d'}(v_0)$ by the derivation of (5.41). For a vertex $v \in K_n \setminus N_{d'}(v_0)$, let σ be a rotation of K_n such that $\sigma^{-1} v \in W \setminus N_{d'}(v_0)$. Then by equivariance

$$(\mathcal{A} \mathcal{R} \tau)_v = (\mathcal{A} \mathcal{R} (\tau \circ \sigma))_{\sigma^{-1} v},$$

and since $\tau \circ \sigma$ is also a characteristic map, by (5.28) we have

$$(\mathcal{A} \mathcal{R} \tau)_v = (\tau \circ \sigma)_{\sigma^{-1} v} = \tau_v.$$

Similarly Q reproduces constant functions on all of K_n.

Secondly, we need to show that for any $K \subset K_n$ with an affine coordinate system $(|N_d K|, \widetilde{x})$, and $f(\widetilde{x})$ a polynomial of degree at most 2, we have $Qf = f$ on $|K|$. By the quasi-interpolant $\widehat{Q}$ on the regular grid we have

$$f(\widetilde{x}) = \sum_{v \in N_1 K} (\eta f)_v M_{222}(\widetilde{x} - v + 2\xi_3) \qquad \text{for } \widetilde{x} \in |K|.$$

Now, $v \in N_1 K$ implies $v \in K_n \setminus N_{d'}(v_0)$. Let σ be a rotation of K_n such that $\sigma^{-1} v \in W \setminus N_{d'}(v_0)$. The coordinate representation of σ, namely $\widetilde{x} = \sigma(x)$, is an affine linear map, so $f \circ \sigma$ is polynomial of degree at most 2 in x-coordinates on $N_3(\sigma^{-1} v)$, and by rotation equivariance

$$(\eta f)_v = \left(\eta(f \circ \sigma) \right)_{\sigma^{-1} v} = \left(\mathcal{A} \, \mathcal{R} \, (f \circ \sigma) \right)_{\sigma^{-1} v}$$
$$= (\mathcal{A} \, \mathcal{R} f)_v \,.$$

Therefore,

$$f(\widetilde{x}) = \sum (\mathcal{A} \, \mathcal{R} \, f)_v \phi_v(\widetilde{x}) = (\mathcal{S}^\infty \, \mathcal{A} \, \mathcal{R} \, f)(\widetilde{x}) = Q f(\widetilde{x}) \qquad \text{for } \widetilde{x} \in |K| \,.$$

$\square$

The following lemma, which we prove in the remaining sections of the chapter, states that the required solutions to (5.41) exist. Assuming this lemma holds we can prove Theorem 57.

Lemma 69. *For each valence n there exists an integer $d'_n \geq 3$ and a constant C, such that for each $v \in W \setminus N_{d'_n}(v_0)$, equation (5.41) has a solution w_v^j, for $j = 6$ or 7 satisfying $|w_v^j| < C$.*

Proof of Theorem 57. By Lemmas 68 and 69 there is a map $Q : L^1_{\Lambda,\text{loc}}(K_n) \to \mathcal{S}(K_n)$ and a constant $d_n = d'_n + 1$ for each valence n, such that Q is local with support width d_n and of order 3. We need only show for each n that Q satisfies the alternative boundedness criteria, i.e., there exists a constant $\|Q_n\|$ such that

$$|Qf|_{H_A^m(T)} \leq \|Q_n\| |f|_{H_A^m(N_{d_n}^c T)}$$

for $m = 0, 1$ or 2 and any face $T \in W \setminus W_{\text{cen}}(d_n)$. Then, by Lemma 55, $Q^k = c^{-k*} Q c^{k*}$ defines a quasi-interpolant on K_n.

For any face $T \in W \setminus W_{\text{cen}}(d_n)$, the quasi-interpolant on T is decomposed as follows

$$Q_{|T} : H_A^m(N_4 T) \xrightarrow{\mathcal{R}} \mathsf{CN}(N_3 T) \xrightarrow{\mathcal{A}_{|N_1 T}} \mathsf{CN}(N_1 T) \xrightarrow{\mathcal{S}^\infty} H_A^m(T) \,. \tag{5.43}$$

By Propositions 61 and 63 we have semi-norm estimates for $\mathcal{S}^\infty$ and $\mathcal{R}$, specifically

$$|\mathcal{S}^\infty u|_{H_A^m(T)} \leq |u|_{H_{\mathsf{CN}}^m(N_1 T)} \tag{5.44}$$

and

$$|\mathcal{R}f|_{H^m_{\mathsf{CN}}(N_3T)} \le |\mathcal{R}|\,|f|_{H^m_A(N_4T)} \tag{5.45}$$

for $m = 0, 1$ or 2. Notice $|f|_{H^m_A(N_4T)} \le |f|_{H^m_A(N^c_{d_n}T)}$ since $d_n = d'_n + 1 \ge 4$ and $N_dT \subset N^c_dT$. So we only need to prove a semi-norm estimate for the averaging operator

$$|\mathcal{A}u|_{H^m_{\mathsf{CN}}(N_1T)} \le |\mathcal{A}|\,|u|_{H^m_{\mathsf{CN}}(N_3T)} \qquad \text{for } m = 0, 1 \text{ or } 2. \tag{5.46}$$

To analyze the averaging operators $\mathcal{A}_{|N_1T}$, first notice that any two neighborhoods $\mathsf{CN}(N_3T)$ and $\mathsf{CN}(N_3\widetilde{T})$ are isometric. To be precise, let $\widetilde{T}$ be a face of $W \setminus W_{\mathrm{cen}}(d_n)$. Then for any other face $T \in W \setminus W_{\mathrm{cen}}(d_n)$, there is a simplicial map $\iota : N_3\widetilde{T} \to N_3T$ so that ι^* preserves the H^m_{CN}-semi-norms. Using this identification, we consider $\mathcal{A}_{|N_1T} : \mathsf{CN}(N_3\widetilde{T}) \to \mathsf{CN}(N_1\widetilde{T})$, for every face $T \in W \setminus W_{\mathrm{cen}}(d_n)$.

For each $m = 0, 1$ or 2 we decompose $\mathsf{CN}(N_3\widetilde{T})$ into $Y^m \oplus Z^m$, where $Z^m = Z^m(N_3\widetilde{T})$ is the null space of the $H^m_{\mathsf{CN}}(N_3\widetilde{T})$-semi-norm, as defined in (5.15).

Lemma 69 implies a uniform bound on the operators $\mathcal{A}_{|N_1T}$. The map $\mathcal{A}_{|N_1T}$ can be represented as a matrix. Lemma 69 implies that the entries of these matrices are bounded, and hence the operators are uniformly bounded in any norm. The $H^m_{\mathsf{CN}}(N_3\widetilde{T})$-semi-norm is a norm on the subspace Y^m. So there is a constant $|\mathcal{A}|$ such that for any $T \in W \setminus W_{\mathrm{cen}}(d_n)$ we have

$$|\mathcal{A}u|_{H^m_{\mathsf{CN}}(N_1T)} \le |\mathcal{A}|\,|u|_{H^m_{\mathsf{CN}}(N_3T)} \qquad \text{for } u \in Y^m. \tag{5.47}$$

Now we consider $\mathcal{A}_{|N_1T}$ applied to $z \in Z^m$. By Proposition 60, $z = p_{|N_3\widetilde{T}}$ for some polynomial $p \in \mathbb{P}_{m-1}$. Notice that $\mathcal{R}p = p_{|N_3\widetilde{T}}$ and $\eta p = p_{|N_3\widetilde{T}}$. So by (5.38) we see that

$$\mathcal{A}_{|N_1T}\, p_{|N_3T} = p_{|N_1T}$$

for any $T \in W \setminus W_{\mathrm{cen}}(d_n)$ and $p \in \mathbb{P}_1$. Then by the definition of $Z^m(N_1\widetilde{T})$ we have

$$|\mathcal{A}_{|N_1T}\, z|_{H^m_{\mathsf{CN}}(N_1T)} = 0 \tag{5.48}$$

for any face T, control net $z \in Z^m$, and integer $m = 0, 1$ or 2.

Decomposing $u \in CN(N_3T)$ into $u = y + z$, with $y \in Y^m$ and $z \in Z^m$, we get the desired semi-norm estimate (5.46) by (5.47) and (5.48), completing the proof. $\qquad\square$

5.4 Solving the Polynomial Reproducing System

The goal for the remainder of the chapter is to prove Lemma 69, showing that the polynomial reproducing system (5.41) has solutions for all $v \in W$ sufficiently far from the central vertex, and moreover, that the solutions are uniformly bounded. We apply the contraction identity for the characteristic map τ to rewrite (5.41) as an equation in terms of τ in a bounded neighborhood independent of $v \in W$.

We define scale dependent convolution operators $L_j^h : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ based on the discrete convolutions $L_0, \ldots, L_7$ and restriction $\mathcal{R}$. For any $h > 0$, let

$$L_j^h f(y) = h^{-s_j} \sum_{i=0}^{18} m_j^i \frac{1}{\pi(h\rho)^2} \int_{B(h\rho)} f(y + h\delta_i + x) \, dx \,. \tag{5.49}$$

Notice $L_j^1 = L_j \circ \mathcal{R}$. The constants s_j are defined by

j	0	1	2	3	4	5	6	7
s_j	0	1	1	2	2	2	3	4

.

We will see in Section 5.5 that the constant s_j is chosen, as in a finite difference scheme, so that L_j^h approximates a differential operator as $h \to 0$.

Recall that we have a coordinate system $(|N_3(W \setminus N_3(v_0))|^\circ, x)$ shown in Figure 5.2, and in these coordinates $c(x) = x/2$, therefore the contraction identity (2.15) for τ is $\tau(x/2) = \lambda\tau(x)$. For any $v \in W^k$, let $h = 2^{-k}$, then we have $v/h = c^{-k}(v) \in \mathsf{Vertex}(W)$. Applying a change of variables and using the contraction identity for τ, we derive a scaling relation for $L_j^h\tau$

$$L_j^h\tau(v) = h^{-s_j} \sum_{i=0}^{18} m_j^i \frac{1}{\pi\rho^2} \int_{B(\rho)} \tau\left(h\left(\frac{v}{h} + \delta_i + x\right)\right) dx$$

$$= \lambda^k h^{-s_j} \sum_{i=0}^{18} m_j^i \frac{1}{\pi\rho^2} \int_{B(\rho)} \tau\left(\frac{v}{h} + \delta_i + x\right) dx = \lambda^k h^{-s_j} (L_j^1\tau)_{v/h} \,. \tag{5.50}$$

There is also a contraction identity for the control net $\tau^k \in \mathsf{CN}(K_n^k)$ which generates the characteristic map. Since $(\mathcal{S}\tau)_{cv} = \lambda\tau_v$, we have $\tau_v^k = (\mathcal{S}^k\tau)_v = \lambda^k\tau_{c^{-k}v} = \lambda^k\tau_{v/h}$. Substituting this identity and (5.50) into (5.41), and dividing by $\lambda^{-k}h^3$, we get

$$\left(\begin{array}{cc} L_6^h\tau(v) & hL_7^h\tau(v) \end{array}\right)\left(\begin{array}{c} w^6(v, h) \\ w^7(v, h) \end{array}\right) = \frac{1}{h^3}\left(\tau_v^k - L_0^h\tau(v)\right) + \frac{1+\rho^2}{2h}L_5^h\tau(v)\,, \tag{5.51}$$

where we have written $w^j(v, h)$ for $w^j_{v/h}$.

Next, we simplify the right side of (5.51) by expressing the control net in terms of the spline function it generates.

Proposition 70. *Suppose $f(x) = \sum_{v \in hK_6} f_v M_{222}(\frac{x-v}{h} + 2\xi_3)$ is a quartic box spline on a scaled equilateral 3-direction grid. Then for any $v \in \mathsf{Vertex}(hK_6)$*

$$f_v = f(v) - \tfrac{1}{8}h^2 \Delta f(v). \tag{5.52}$$

Proof. We can express the Laplacian in terms of derivatives along the grid directions by $\Delta = \frac{2}{3}(D_1^2 + D_2^2 + D_3^2)$. Then from the differentiation formula for a box spline (5.10) we have

$$D_i^2 \left(\sum_{v \in hK_6} f_v M_{222}(\tfrac{x-v}{h} + 2\xi_3) \right) = \tfrac{1}{h^2} \sum_{v \in hK_6} \left(\nabla_i^2 f \right)_v M_{222-2e_i}(\tfrac{x-v}{h} + 2\xi_3) .$$

The box spline M_{222-2e_i} is continuous and is supported on the convex hull of the remaining directions. For example, the support of M_{022} is the convex hull of $\{0, 2\xi_2, 2\xi_3, 2(\xi_2 + \xi_3)\}$. From this we see that $M_{222-2e_i}(v) = 0$ for all $v \in K_6$, except $v = \xi_1 + \xi_2 + \xi_3 - \xi_i$. Since $\sum_{v \in K_6} M_{222-2e_i} \equiv 1$ by Proposition 59 part (iii) we conclude that $M_{222-2e_i}(\xi_1+\xi_2+\xi_3-\xi_i) = 1$. Thus for any $v \in hK_6$,

$$D_i^2 \left(\sum_{j \in hK_6} f_j M_{222}(\tfrac{x-j}{h} + 2\xi_3) \right)_{|x=v} = \tfrac{1}{h^2} \left(\nabla_i^2 f \right)_{v+h\xi_i} = \tfrac{1}{h^2}(f_{v+h\xi_i} - 2f_v + f_{v-h\xi_i}) ,$$

and hence

$$\Delta f(v) = \frac{2}{3}(D_1^2 + D_2^2 + D_3^2)_{|v} f$$
$$= \frac{1}{h^2}\left(-4f_v + \frac{2}{3} \sum_{N_1(v,hK_6)\backslash v} f_u \right). \tag{5.53}$$

By an explicit computation we get $M_{222}(1,0) = \frac{1}{12}$ and $M_{222}(2,0) = \frac{1}{2}$, and therefore we have

$$f(v) = \frac{1}{2}f_v + \frac{1}{12} \sum_{N_1(v,hK_6)\backslash v} f_u . \tag{5.54}$$

Combining (5.53) and (5.54) we derive (5.52). $\qquad \square$

Using this proposition to rewrite the right side of (5.51) we get

$$\left(\begin{array}{cc} L_6^h \tau(v) & h L_7^h \tau(v) \end{array} \right) \left(\begin{array}{c} w^6(v,h) \\ w^7(v,h) \end{array} \right) = R^h \tau(v), \tag{5.55}$$

where

$$R^h = \frac{1}{h^3} I - \frac{1}{8h} \Delta - \frac{1}{h^3} L_0^h + \frac{1+\rho^2}{2h} L_5^h. \tag{5.56}$$

Using (5.55) we can rewrite the polynomial reproducing system at a vertex in W in terms of scaled operators acting near a fundamental region. Define the subcomplex $W_1 \subset W$ by $W_1 = N_2(v_0, W) \setminus N_1(v_0, W)$. The region $|W_1|$ is a fundamental domain of $|K_n| \setminus v_0$ with respect to the group generated by a rotation of $|K_n|$ and the contraction c. Let $E = \mathsf{Edge}(W_1) \setminus \{\xi_2, \xi_3\}$ be the edges of W_1 except the edge nearest the central vertex, and let $A = 2\xi_3$ and $B = \xi_2 + \xi_3$ be vertices of W_1, as shown in Figure 5.5.
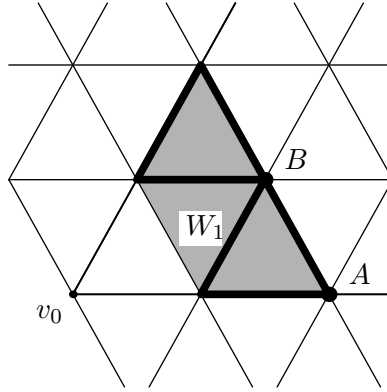
Figure 5.5: The fundamental region $|W_1|$. The edges $E \subset \mathsf{Edge}(W_1)$ are highlighted.

We represent a vertex in $N_3(W \setminus N_3(v_0, K_n))^k$ for some $k \geq 0$ by a pair (v, h), where v is the x-coordinate of the vertex and $h = 2^{-k}$ is the scale. We consider the following three disjoint collections of vertex and scale pairs:

(i) Let V_I be the collection of pairs (v, h) such that for some face $F \in W_1$, we have $|N_2(v, K_n^k)| \subset |F|$.

(ii) Let V_{II} be the collection of pairs $(v, h) \notin V_I$ such that $|N_3(v, K_n^k)| \subset |F_-| \cup |F_+|$, where $F_\pm \in \mathsf{Face}(K_n)$ and $F_- \cap F_+ \in E$.

(iii) Let V_{III} be the collection of pairs $(v, h) \notin V_I \cup V_{II}$ such that $|N_3(v, K_n^k)| \subset |N_1(u_0, K_n)|$, where $u_0 = A$ or B.

Figure 5.6 shows an assignment of all but finitely many of the vertices of W to one of three classes. A vertex $v \in W$ is considered to be in class I, for instance, if by applying repeated contraction c^k and possibly a rotation σ, we get $(\sigma c^k(v), 2^{-k}) \in V_I$. Notice not every vertex in W falls into one of the three classes above, but every vertex $v \in K_n \setminus N_{10}(v_0)$ does. We consider each of these classes in the next three sections. In each case we show that the solutions to (5.41) exist and are bounded. Therefore we have solutions to (5.41) for all $v \in W$ sufficiently far from v_0.
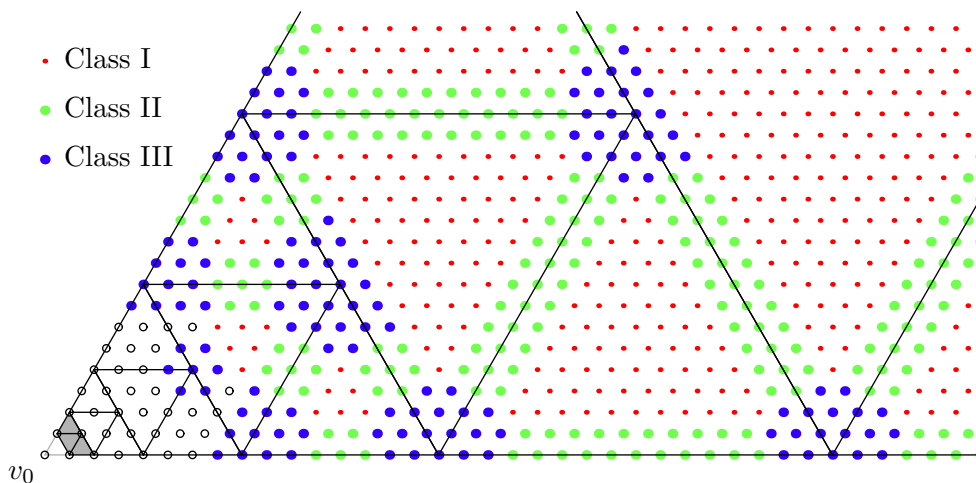
Figure 5.6: Three classes of vertices in W. The shaded region is W_1. The lines are the images of $\mathsf{Edge}(W_1)$ under the maps c^{-k}.

5.5   Class I: Away From Edges

Suppose $(v, h) \in V_I$ and $k = -\log_2 h$. We show that $w^6(v, h) = w^7(v, h) = 0$ is a trivial solution to (5.55). We see that $L_0^h \tau(v)$ and $L_5^h \tau(v)$ depend on τ on $|N_2(v, K_n^k)|$. This domain is contained in a single face of K_n and so τ is a polynomial on this domain.

We first develop the relationship between the convolution operators L_j^h and certain differential operators. The differential operators we consider are the Laplacian $\Delta = \frac{\partial^2}{\partial x_1^2} +$

$\frac{\partial^2}{\partial x_2^2}$, the bi-Laplacian Δ^2, and $D_{111} = \frac{1}{4}\frac{\partial^3}{\partial x_1^3} - \frac{3}{4}\frac{\partial^3}{\partial x_1 \partial x_2^2}$, which is differentiation in the three grid directions of the equilateral grid.

The box spline functions spanned by translates of M_{222} are piecewise quartic polynomials as we see from Proposition 59 part (iv). However, not all quartic polynomials are M_{222} box spline functions. By differentiating the two polynomial pieces of T_{222}, given by (5.13) and (5.14), we see that $\Delta^2 T_\pm = 0$. So, by (5.11) it follows that $\Delta^2 M_{222} = 0$ on each face of K_6. We denote the space of bi-harmonic, degree-4 polynomials by

$$\mathbb{P}_4^\Delta = \{p \in \mathbb{P}_4 : \Delta^2 p = 0\}.$$

The following proposition shows that the right side of (5.55) is zero for $(v, h) \in V_I$, and so $w^6(v, h) = w^7(v, h) = 0$ is a solution to (5.55).

Proposition 71. *For any polynomial $p \in \mathbb{P}_4^\Delta$ we have the following identities:*

$$L_0^h p = p + \tfrac{1}{8}(h\rho)^2 \Delta p \qquad\qquad L_6^h p = D_{111} p$$

$$L_5^h p = \tfrac{1}{4}\Delta p \qquad\qquad\qquad\quad L_7^h p = 0$$

$$R^h p = 0$$

Proof. We expand p as a finite Taylor series about v, i.e., $p(v + x) = \sum \frac{1}{\alpha!}(D^\alpha p)_{|v} x^\alpha$, where $D^\alpha p = 0$ for $|\alpha| > 4$. Scaled moments of the ball $B(h\rho)$, defined as $\frac{1}{\pi(h\rho)^2}\int_{B(h\rho)} x^\alpha\, dx$ for a multi-index $\alpha \in \mathbb{Z}_+^2$, are listed below for $|\alpha| \le 4$:

		α_2			
α_1	0	1	2	3	4
0	1	0	$\frac{h^2\rho^2}{4}$	0	$\frac{h^4\rho^4}{8}$
1		0	0	0	
2			$\frac{h^4\rho^4}{24}$		

The moments are symmetric in the indices. So we only show the cases where $\alpha_2 \ge \alpha_1$. Using these moments we compute

$$L_0^h p(v) = \frac{1}{\pi(h\rho)^2}\sum_\alpha \frac{1}{\alpha!}(D^\alpha p)_{|v}\int_{B(h\rho)} x^\alpha\, dx$$

$$= p(v) + \tfrac{1}{8}(h\rho)^2 \Delta p(v) + \tfrac{1}{192}(h\rho)^4 \Delta^2 p(v).$$

For the other convolution operators, we first analyze the finite difference operators. We substitute the Taylor series expansion of $p(v+x)$ into the finite difference to get the following:

$$h^{-2}\sum_{i=0}^{6}m_5^i\,p(v+h\delta_i) = \tfrac{1}{4}\Delta p(v) + \tfrac{1}{64}h^2\Delta^2 p(v)$$

$$h^{-3}\sum_{i=0}^{6}m_6^i\,p(v+h\delta_i) = -D_{111}p(v)$$

$$h^{-4}\sum_{i=0}^{18}m_7^i\,p(v+h\delta_i) = \Delta^2 p(v)$$

Integrating over $B(h\rho)$ we get the following:

$$L_5^h p(v) = \frac{1}{\pi(h\rho)^2}\int_{B(h\rho)} h^{-2}\sum_{i=0}^{7}m_5^i\,p(v+x+h\delta_i)dx$$

$$= \frac{1}{\pi(h\rho)^2}\int_{B(h\rho)} \tfrac{1}{4}\Delta p(v+x) + \tfrac{1}{64}h^2\Delta^2 p(v+x)\,dx$$

$$= \tfrac{1}{4}\Delta p(v) + \tfrac{(1+2\rho^2)}{64}h^2\Delta^2 p(v),$$

$$L_6^h p(v) = \frac{1}{\pi(h\rho)^2}\int_{B(h\rho)} h^{-3}\sum_{i=0}^{7}m_6^i\,p(v+x+h\delta_i)\,dx$$

$$= \frac{1}{\pi(h\rho)^2}\int_{B(h\rho)} D_{111}p(v+x)\,dx$$

$$= D_{111}p(v),$$

and

$$L_7^h p(v) = \frac{1}{\pi(h\rho)^2}\int_{B(h\rho)} h^{-4}\sum_{i=0}^{18}m_7^i\,p(v+x+h\delta_i)\,dx$$

$$= \frac{1}{\pi(h\rho)^2}\int_{B(h\rho)} \Delta^2 p(v+x)\,dx$$

$$= \Delta^2 p(v).$$

Combining the other parts of this proposition, we see also that $R^h p = 0$. $\qquad\square$

5.6 Class II: Near an edge

We show that there is a solution to (5.55) for any $(v,h) \in V_{II}$ and moreover that the solutions are uniformly bounded. For a fixed $(v,h) \in V_{II}$ let F_- and $F_+ \in \mathsf{Face}(K_n)$

be such that $|N_3(v, K_n^k)| \subset |F_-| \cup |F_+|$. Let $U = \overline{F}_+ \cup \overline{F}_-$ be a subcomplex of K_n. From the definition of $E \subset \mathsf{Edge}(W_1)$ (Figure 5.5) one sees that there is an embedding $N_1U \hookrightarrow K_6$. So we can define an equilateral affine coordinate chart x on $|N_1U|^\circ$ such that $x(F_- \cap F_+) = \{0, \xi_3\} \in \mathsf{Edge}(K_6)$ with $x(v) = (v_1, v_2)$ and $v_1 \leq 0$. The mask width of Loop's subdivision scheme is 2 so, by Proposition 8, we can write the characteristic map on $|U|$ as

$$\tau(x) = \sum_{u \in N_1U} \tau_u M_{222}(x - u + 2\xi_3) \qquad \text{for } x \in |U|. \tag{5.57}$$

We can extend the control net $\tau_{|N_1U}$ to all of K_6 by setting $\tau_u = 0$ for all vertices not in N_1U. This extension does not effect the value of τ on $|U|$.

We use the truncated power series form of a box spline (5.12) to write $\tau_{|U} = p + G$ with $p \in \mathbb{P}_4^\Delta$ and G a piecewise polynomial that is zero on $|F_-|$. Using (5.12) and (5.57) we can write τ as a truncated power series

$$\tau(x) = \sum_{u \in K_6} \left(\nabla_{222}^c \tau \right)_u T_{222}(x - u). \tag{5.58}$$

Recall that ∇_α^c is a centered difference operator defined in Section 5.1.

Most of the terms of (5.58) are in $\mathbb{P}_4^\Delta$ on $|U|$. Only when $u = -j\xi_3$ with $j \geq 0$ is $T_{222}(x - u)$ non-polynomial on $|U|$. We decompose these non-polynomial terms into a polynomial component and a non-polynomial component that is zero on F_-. Define

$$g_1(x) = \tfrac{32}{3\sqrt{3}} x_{2+}^3 \qquad \text{and} \qquad g_2(x) = \tfrac{32}{3\sqrt{3}} x_1 x_{2+}^3, \tag{5.59}$$

where x_+^n is the single variable truncated power, i.e., $x_+^n = x^n$ when $x \geq 0$ and $x_+^n = 0$ otherwise. Using the explicit form of T_{222} in (5.13) we get

$$T_{222}(x) = \left(T_{222}(x) - T_-(x) \right) + T_-(x) = g_2(x) + T_-(x) \qquad \text{for } x \in |U|.$$

Then the non-polynomial piece of τ on $|U|$ is given by

$$G(x) = \sum_{j=0}^\infty (\nabla_{222}^c \tau)_{-j\xi_3} g_2(x + j\xi_3)$$

$$= \left(\sum_{j=0}^\infty (\nabla_{222}^c \tau)_{-j\xi_3} \right) g_2(x) + \left(\sum_{j=0}^\infty j (\nabla_{222}^c \tau)_{-j\xi_3} \right) g_1(x).$$

We conclude that we can decompose τ on $|U|$ into

$$\tau(x) = p(x) + G(x), \quad \text{where} \quad G(x) = c_1 g_1(x) + c_2 g_2(x) \tag{5.60}$$

for some $c_i \in \mathbb{R}^2$ and $p \in \mathbb{P}_4^\Delta$.

Proposition 72. *At any point $y = (y_1, y_2) \in \mathbb{R}^2$ and for any $h \geq 0$ we have*

$$L_j^h \, g_2(y) = y_1 L_j^h \, g_1(y) \qquad \text{for } j = 0, 5 \text{ or } 7. \tag{5.61}$$

Proof. Let $F_2(x_1, x_2) = (-x_1, x_2)$ be reflection about the x_2 axis. We also let F_2 denote the corresponding permutation of the stencil indices, so that $\delta_{F_2(i)} = F_2(\delta_i)$. The mask m_j is symmetric with respect to F_2, i.e., $m_j^i = m_j^{F_2(i)}$. We compute $L_j^h \, g_2(y)$ as follows:

$$L_j^h \, g_2(y)$$

$$= \frac{1}{2} h^{-s_j} \sum_{i=0}^{18} m_j^i \frac{1}{\pi(h\rho)^2} \int_{B(h\rho)} g_2(y + h\delta_i + x) + g_2(y + h\delta_{F_2(i)} + x) dx$$

$$= \frac{16}{3\sqrt{3}} h^{-s_j} \sum_{i=0}^{18} m_j^i \frac{1}{\pi(h\rho)^2} \int_{B(h\rho)} \left((y_1 + h\delta_{i,1} + x_1) + (y_1 - h\delta_{i,1} + x_1) \right) (y_2 + h\delta_{i,2} + x_2)_+^3 dx$$

$$= \frac{32}{3\sqrt{3}} h^{-s_j} \sum_{i=0}^{18} m_j^i \frac{1}{\pi(h\rho)^2} \left(\int_{B(h\rho)} y_1 (y_2 + h\delta_{i,2} + x_2)_+^3 dx + \int_{B(h\rho)} x_1 (y_2 + h\delta_{i,2} + x_2)_+^3 dx \right).$$

The second integral in the previous line is zero since the integrand is odd with respect to F_2. Thus the right side is equal to $y_1 L_j^h \, g_1(y)$, proving (5.61). $\square$

The operator R^h on the right side of the polynomial reproducing equation (5.55) is linear. Applying R^h to the decomposition (5.60), we see, by Proposition 71, that $R^h p = 0$. Since $G = 0$ on $|F_-|$ we have $G(v) = \Delta G(v) = 0$ and therefore

$$R^h \tau(v) = -\frac{1}{h^3} L_0^h G(v) + \frac{1 + \rho^2}{2h} L_5^h G(v). \tag{5.62}$$

Now substituting $G = c_1 g_1 + c_2 g_2$ into (5.62) and using Proposition 72 we get

$$R^h \tau(v) = -\frac{1}{h^3} L_0^h (c_1 g_1 + c_2 g_2)(v) + \frac{1 + \rho^2}{2h} L_5^h (c_1 g_1 + c_2 g_2)(v)$$

$$= \left(-\frac{1}{h^3} L_0^h \, g_1(v) + \frac{1 + \rho^2}{2h} L_5^h \, g_1(v) \right) (c_1 + v_1 c_2). \tag{5.63}$$

In particular the right side of (5.55) is a scalar multiple of $c_1 + v_1c_2 \in \mathbb{R}^2$.

We now show that the second column of the matrix in (5.55) is also a scalar multiple of $c_1 + v_1c_2$. Computing we get

$$hL_7^h\,\tau(v) = hL_7^h\,(p + c_1g_1 + c_2g_2)(v) \qquad\qquad \text{by (5.60)}$$

$$= hc_1(L_7^h\,g_1)(v) + hc_2(L_7^h\,g_2)(v) \qquad\qquad \text{by Prop. 71}$$

$$= h(c_1 + v_1c_2)(L_7^h\,g_1)(v) \qquad\qquad \text{by Prop. 72.}$$

We have either $v_2 = 0$ or $-\frac{\sqrt{3}}{2}h$, and $g_1(x)$ is a homogeneous function of degree 3 in the variable x_2, i.e., $g_1(hx) = h^3g_1(x)$. Therefore $L_j^h\,g_1$ is homogeneous of degree $(3 - s_j)$, and so the scalar in (5.63) is a function of v_2/h. Similarly $hL_7\,g_1(v)$ is a function of v_2/h whose values can be computed as follows:

$$hL_7^h\,g_1(v_1, 0) = \tfrac{128}{9}(1 - \tfrac{1}{2}\rho^2 + \tfrac{16}{15\sqrt{3}\pi}\rho^3)$$

$$hL_7^h\,g_1(v_1, -\tfrac{\sqrt{3}}{2}h) = \tfrac{64}{9}(1 + \rho^2 - \tfrac{128}{45\sqrt{3}\pi}\rho^3)$$

Therefore for any $(v, h) \in V_{II}$ a solution to (5.55) is given by

$$w^6(v, h) = 0 \qquad \text{and} \qquad w^7(v, h) = \frac{\frac{1+\rho^2}{2h}L_5^h\,g_1(v) - \frac{1}{h^3}L_0^h\,g_1(v)}{hL_7^h\,g_1(v)}, \qquad\qquad (5.64)$$

with $|w^7(v, h)| < C$ for some constant C.

5.7 Class III: Near a Corner

We solve the polynomial reproducing system at $(v, h) \in V_{III}$ for sufficiently small h. The pairs $(v, h) \in V_{III}$ are divided into two cases $V_{III.A}$ and $V_{III.B}$ based on whether v is near the point A or $B \in K_n$. Consider case $V_{III.A}$ (case $V_{III.B}$ is similar) let x be an equilateral affine coordinate system centered at A. A vertex v in the pair $(v, h) \in V_{III.A}$ is represented in x-coordinates as $v = hu$ where u is one of the 32 points shown in Figure 5.7. Many of the computations in this section are done with the aid of a computer algebra program such as Mathematica.

We write the characteristic map $\tau(x)$ in the form

$$\tau(x) = p(x) + G_3(x) + G_4(x)\,, \qquad\qquad (5.65)$$
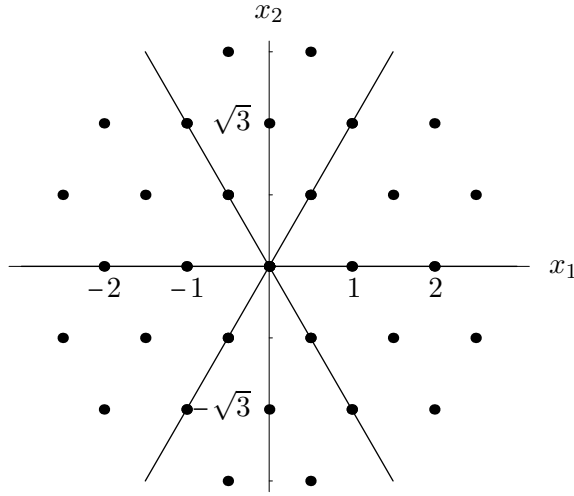
Figure 5.7: Offsets u to a vertex.

where $p \in \mathbb{P}_4^\Delta$ and $G_3(x)$ and $G_4(x)$ are homogeneous of degree 3 and 4 respectively. This decomposition will allow us to isolate the effect of the scale h on the systems of equations and will give us asymptotic descriptions of the resulting systems. When L_j^h and R^h are applied to the function G_k we get homogeneity identities

$$L_j^h G_k(hu) = h^{k-s_j} L_j^1 G_k(u) \qquad \text{and} \qquad R^h G_k(hu) = h^{k-3} R^1 G_k(u) \,.$$

From Proposition 71 we know $L_6^h p = D_{111}p$, which is a linear polynomial. We decompose this linear polynomial into

$$L_6^h \, p(x) = D_{111}p(x) = D_{111}p(u_0) + \Psi(x), \tag{5.66}$$

where $\Psi(x)$ is a homogeneous polynomial of degree 1.

Now using the decomposition (5.65) we can rewrite (5.55) at $v = hu$ and scale h as $M(hu, h) \left(\begin{smallmatrix} w^6(hu,h) \\ w^7(hu,h) \end{smallmatrix} \right) = R^h \tau(hu)$, where

$$M(hu, h) = \Big(D_{111}p(u_0) + L_6^1 G_3 \quad L_7^1 G_3 \Big)_{|u} + h \Big(\Psi + L_6^1 G_4 \quad L_7^1 G_4 \Big)_{|u} \tag{5.67}$$

and

$$R^h \tau(hu) = R^1 G_3(u) + h R^1 G_4(u) \,. \tag{5.68}$$

5.7.1 Two Symmetric Box Spline Decompositions

We exploit the symmetry of τ in organizing the computations that follow. We decompose the spline function τ on $|N_1(A, K_n)|$ or $|N_1(B, K_n)|$ into components that are even or odd with respect to certain reflections.

For each case $V_{III.A}$ or $V_{III.B}$ we choose an equilateral affine coordinate chart x and an equivariant characteristic chart τ. In case $V_{III.A}$ the x-coordinate system is centered at A and satisfies $x(v_0) = (-2, 0)$, and we choose τ so that $\tau(A)$ is on the x_1-axis. Notice, such a choice exists since an equivariant characteristic map composed with a rotation is still an equivariant characteristic map. Let F_i be the reflection across the x_i-axis for $i = 1$ or 2. Now since τ is equivariant we have $\tau \circ F_1 = F_1 \circ \tau$ on $N_1(A)$, showing that the components of $\tau = (\tau_1, \tau_2)$ are even and odd respectively with respect to the reflection F_1. Similarly for Case $V_{III.B}$ we choose an equilateral affine coordinate chart centered at B such that $x(v_0) = (0, \sqrt{3})$, and we choose τ so that $\tau(B)$ is on the x_2-axis. Then since $\tau \circ F_2 = F_2 \circ \tau$ on $N_1(B)$ the components τ_1 and τ_2 are odd and even respectively, with respect to the reflection F_2.

In this section we decompose a quartic box spline function in the 1-neighborhood of a vertex into terms which are symmetric with respect to one of the reflections F_1 or F_2. The first proposition deals with the F_1 reflection, while the second proposition describes a decomposition with respect to F_2. Let $\sigma(x_1, x_2) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be rotation by $\pi/3$ radians. Define 6 wedges $W_i = \{(r, \theta) : \frac{\pi}{3} i \leq \theta \leq \frac{\pi}{3}(i+1)\}$ for $i = 0, \ldots, 5$ in polar coordinates; i.e., $x_1 = r \cos(\theta)$ and $x_2 = r \sin(\theta)$. As shorthand notation we write $W_{i,j}$ for $W_i \cup W_j$. Let $T = T_{222}$ be the truncated power function for the quartic splines. A quartic spline on $N_1(0, K_6)$ is given by

$$f(x) = \sum_{u \in N_2(0, K_6)} f_u M_{222}(x - u + 2\xi_3), \qquad \text{for } x \in |N_1(0, K_6)|. \tag{5.69}$$

Proposition 73. *The quartic box spline on $N_1(0)$ as given in (5.69) has a representation*

$$f(x) = p(x) + \sum C_i E_i(x) \qquad \text{for } x \in |N_1(0, K_6)|, \tag{5.70}$$

where $p \in \mathbb{P}_4^\Delta$ and the sum is taken over $i = 0, 1, 2, 4, 5, 6$ and 8. The functions E_i are given

as follows:

$$E_0(x) = g_1(x) + g_1(F_1 x) \qquad\qquad E_4(x) = g_2(x) + g_2(F_1 x)$$

$$E_1(x) = g_1(\sigma x) - g_1(\sigma F_1 x) \qquad\qquad E_5(x) = g_2(\sigma x) - g_2(\sigma F_1 x)$$

$$E_2(x) = g_1(\sigma x) + g_1(\sigma F_1 x) \qquad\qquad E_6(x) = g_2(\sigma x) + g_2(\sigma F_1 x) \qquad (5.71)$$

$$E_8(x) = T(x)$$

The function E_j is even or odd with respect to F_1 when j is even or odd respectively, and E_0, E_1 and E_2 are homogeneous degree 3 while the other functions are homogeneous degree 4.

The coefficients C_i are computed in terms of the truncated power series coefficients $a_u = (\nabla_{222}^c f)_u$ for $u \in K_6$ where we have extended the control net f by $f_u = 0$ for $u \notin N_2(0, K_6)$. The coefficients are given as follows:

$$C_0 = \frac{1}{2} \sum_{j=1}^{4} j a_{-\xi_3 j} \qquad\qquad C_4 = \frac{1}{2} \sum_{j=1}^{4} a_{-\xi_3 j}$$

$$C_1 = \frac{1}{2} \sum_{j=1}^{4} j \left(a_{-\xi_1 j} - a_{-\xi_2 j} \right) \qquad\qquad C_5 = \frac{1}{2} \sum_{j=1}^{4} \left(a_{-\xi_1 j} - a_{-\xi_2 j} \right)$$

$$C_2 = \frac{1}{2} \sum_{j=1}^{4} j \left(a_{-\xi_1 j} + a_{-\xi_2 j} \right) \qquad\qquad C_6 = \frac{1}{2} \sum_{j=1}^{4} \left(a_{-\xi_1 j} + a_{-\xi_2 j} \right) \qquad (5.72)$$

$$C_8 = a_0$$

Proof. We first comment that the functions E_j clearly satisfy the symmetry and homogeneity conditions. We write f as a truncated power series in terms of T using (5.12), namely,

$$f(x) = \sum_{u \in K_6} (\nabla_{222}^c f)_u T(x - u) = \sum_{u \in K_6} a_u T(x - u) \qquad \text{for } x \in |N_1(0, K_6)|. \qquad (5.73)$$

For most $u \in K_6$ the shifted truncated power $T(x - u)$ is in $\mathbb{P}_4^\Delta$ on $|N_1(0, K_6)|$. So these terms all contribute to $p(x)$ in the decomposition (5.70). We list the other terms in (5.73) explicitly

$$f(x) = p(x) + a_0 T(x) + \sum_{j=1}^{4} a_{-\xi_3 j} T(x + \xi_3 j) + \sum_{j=1}^{4} a_{-\xi_1 j} T(x + \xi_1 j) + \sum_{j=1}^{4} a_{-\xi_2 j} T(x + \xi_2 j). \qquad (5.74)$$

Analyzing the first sum of (5.74) we see that for $x \in |N_1(0, K_6)|$ and $j \geq 1$ we have $x + \xi_3 j = (x_1 + j, x_2) \in W_{0,5}$. So, from (5.13) we have

$$T(x) = p(x) + \tfrac{16}{3\sqrt{3}} x_1 |x_2^3| \qquad \text{for } x \in W_{0,5},$$

where $p \in \mathbb{P}_4^\Delta$. Therefore,

$$\sum_{j=1}^4 a_{-\xi_3 j} T(x + \xi_3 j) = p(x) + \tfrac{16}{3\sqrt{3}} \sum_{j=1}^4 a_{-\xi_3 j}(x_1 + j)|x_2|^3$$

$$= p(x) + \frac{1}{2}\left(\sum_{j=1}^4 j a_{-\xi_3 j}\right)\left(g_1(x) + g_1(F_1 x)\right) \qquad (5.75)$$

$$+ \frac{1}{2}\left(\sum_{j=1}^4 a_{-\xi_3 j}\right)\left(g_2(x) + g_2(F_1 x)\right)$$

$$= p(x) + C_0 E_0(x) + C_4 E_4(x).$$

Next we analyze the second sum in (5.74). Let $\tilde{x} = \sigma x$ be a rotated coordinate system. Expressing T in this coordinate system we get

$$T(\tilde{x}) = \begin{cases} -2\,\tilde{x}_1^4 + 4\,\tilde{x}_1^2\,\tilde{x}_2^2 + \tfrac{16}{3\sqrt{3}}\,\tilde{x}_1\,\tilde{x}_2^3 + \tfrac{2}{3}\,\tilde{x}_2^4 & \text{for} \quad \tilde{x} \in W_0 \\ \tfrac{32}{3\sqrt{3}}\tilde{x}_1\tilde{x}_2^3 & \tilde{x} \in W_5 \\ 0 & \text{otherwise}. \end{cases}$$

In particular we have $T(\tilde{x}) = \tfrac{32}{3\sqrt{3}}\tilde{x}_1\tilde{x}_{2+}^3$ for $\tilde{x} \in W_{4,5}$. In $\tilde{x}$-coordinates we have $\tilde{x}(-\xi_1) = (-1, 0)$ and if $\tilde{x} \in |N_1(0, K_6)|$ and $j \geq 1$ then $\tilde{x} + \xi_1 j = (\tilde{x}_1 + j, \tilde{x}_2) \in W_{4,5}$. Therefore we have

$$\sum_{j=1}^4 a_{-\xi_1 j} T(\tilde{x} + \xi_1 j) = \tfrac{32}{3\sqrt{3}} \sum_{j=1}^4 a_{-\xi_1 j}(\tilde{x}_1 + j)\tilde{x}_{2+}^3,$$

and in x-coordinates

$$\sum_{j=1}^4 a_{-\xi_1 j} T(x + \xi_1 j) = \left(\sum_{j=1}^4 j a_{-\xi_1 j}\right) g_1(\sigma x) + \left(\sum_{j=1}^4 a_{-\xi_1 j}\right) g_2(\sigma x). \qquad (5.76)$$

Similarly, for the third summation term of (5.74), let $\hat{x} = \sigma F_1 x$ be a rotated and reflected coordinate system. The truncated power function is given by $T(\hat{x}) = \tfrac{32}{3\sqrt{3}}\hat{x}_1\hat{x}_{2+}^3$

for $\hat{x} \in W_{0,1}$. In $\hat{x}$-coordinates we have $\hat{x}(\xi_2) = (1,0)$ and if $\hat{x} \in |N_1(0, K_6)|$ and $j \geq 1$ then $\hat{x} + \xi_2 j = (\hat{x}_1 + j, \hat{x}_2) \in W_{0,1}$. Therefore we have

$$\sum_{j=1}^{4} a_{-\xi_2 j} T(\hat{x} + \xi_2 j) = \tfrac{32}{3\sqrt{3}} \sum_{j=1}^{4} a_{-\xi_2 j}(\hat{x}_1 + j)\hat{x}_{2+}^3$$

and in x-coordinates

$$\sum_{j=1}^{4} a_{-\xi_2 j} T(x + \xi_2 j) = \left(\sum_{j=1}^{4} j a_{-\xi_2 j} \right) g_1(\sigma F_1 x) + \left(\sum_{j=1}^{4} a_{-\xi_2 j} \right) g_2(\sigma F_1 x). \tag{5.77}$$

Substituting (5.75), (5.76), and (5.77) into (5.74) and rearranging terms gives the decomposition. $\qquad\square$

The next proposition is similar except we construct a decomposition which is symmetric with respect to the reflection F_2. Let $X = [\xi_1, \xi_2, \xi_3]$ be as in the definition of the equilateral grid (5.5). Let $X_1 = [\xi_3, \xi_2, -\xi_1]$ and $X_2 = [-\xi_3, \xi_2, -\xi_1]$, be alternative direction sets for the equilateral grid, and let $T_i = T_{X_i^{222}}$ for $i = 1$ or 2 be the corresponding truncated power function. The corresponding centered difference operators are equal since we have only changed the orientation of directions, i.e., $\nabla^c_{X_1^{222}} = \nabla^c_{X_2^{222}} = \nabla^c_{222}$. Expressing a quartic box spline function as a truncated power series (5.12) we get

$$f(x) = \sum_{v \in K_6} (\nabla^c_{222} f)_v T_1(x - v) = \sum_{v \in K_6} (\nabla^c_{222} f)_v T_2(x - v).$$

Therefore with $T = \tfrac{1}{2}(T_1 + T_2)$ we have

$$f(x) = \sum_{v \in K_6} a_v T(x - v), \tag{5.78}$$

where $a_v = (\nabla^c_{222} f)_v$. The piecewise polynomial representation of T is

$$T(x) = \begin{cases} T_+(x) &= \frac{16}{3\sqrt{3}} x_1 x_2^3 & \text{for} \quad x \in W_0, \\ T_0(x) &= -2x_1^4 + 4x_1^2 x_2^2 + \frac{2}{3}x_2^4 & \text{for} \quad x \in W_1, \\ T_-(x) &= -\frac{16}{3\sqrt{3}} x_1 x_2^3 & \text{for} \quad x \in W_2, \\ 0 & & \text{for} \quad x \in W_3, W_4, \text{ or } W_5. \end{cases} \tag{5.79}$$

Proposition 74. *The quartic box spline function on $N_1(0)$ given in (5.69) can be written*

$$f(x) = p(x) + \sum_{i=0}^{6} \widetilde{C}_i \widetilde{E}_i(x) \qquad \text{for } x \in |N_1(0, K_6)|, \tag{5.80}$$

where $p \in \mathbb{P}_4^\Delta$. The functions $\widetilde{E}_i$ are as follows:

$$\widetilde{E}_0(x) = g_1(x) \qquad\qquad\qquad \widetilde{E}_3(x) = g_2(x)$$

$$\widetilde{E}_1(x) = g_1(\sigma x) - g_1(\sigma F_2 x) \qquad \widetilde{E}_4(x) = g_2(\sigma x) + g_2(\sigma F_2 x) \tag{5.81}$$

$$\widetilde{E}_2(x) = g_1(\sigma x) + g_1(\sigma F_2 x) \qquad \widetilde{E}_5(x) = g_2(\sigma x) - g_2(\sigma F_2 x)$$

$$\widetilde{E}_6(x) = T(x)$$

The function $\widetilde{E}_j$ is even or odd with respect to F_2 when j is even or odd respectively, and E_0, E_1, and E_2 are homogeneous of degree 3 while the other functions $\widetilde{E}_j$ are homogeneous of degree 4.

The coefficients $\widetilde{C}_i$ are computed in terms of the truncated power series coefficients $a_u = (\nabla_{222}^c f)_u$, where we have extended the control net f by $f_u = 0$ for $u \notin N_2(0, K_6)$. The coefficients are given as follows:

$$\widetilde{C}_0 = \frac{1}{2} \sum_{j=1}^{4} j(a_{-j\xi_3} + a_{j\xi_3}) \qquad\qquad \widetilde{C}_1 = \frac{1}{2} \sum_{j=1}^{4} j\left(a_{+j\xi_1} - a_{-j\xi_2}\right)$$

$$\widetilde{C}_2 = \frac{1}{2} \sum_{j=1}^{4} j\left(a_{+j\xi_1} + a_{-j\xi_2}\right) \qquad\qquad \widetilde{C}_3 = \frac{1}{2} \sum_{j=1}^{4}(a_{-j\xi_3} - a_{j\xi_3}) \tag{5.82}$$

$$\widetilde{C}_4 = \frac{1}{2} \sum_{j=1}^{4} a_{-j\xi_2} + a_{j\xi_1} \qquad\qquad \widetilde{C}_5 = \frac{1}{2} \sum_{j=1}^{4} a_{-j\xi_2} - a_{j\xi_1}$$

$$\widetilde{C}_6 = a_0$$

Proof. We first comment that the basis functions $\widetilde{E}_j$ clearly satisfy the symmetry and homogeneity conditions. Combining all the terms in (5.78) that are polynomial on $|N_1(0, K_6)|$ into a single term $p(x)$ we have

$$f(x) = p(x) + a_0 T(x) + \sum_{1 \le |j| \le 4} a_{j\xi_3} T(x - j\xi_3) + \sum_{j=1}^{4} a_{-j\xi_2} T(x + j\xi_2) + \sum_{j=1}^{4} a_{j\xi_1} T(x - j\xi_1). \tag{5.83}$$

Thus for $x \in |N_1(0, K_6)| \cap W_{0,1,2}$ the first sum in (5.83) can be written as

$$
\begin{aligned}
\sum_{1 \leq |j| \leq 4} a_{j\xi_3} T(x - j\xi_3) &= \sum_{j=1}^{4} a_{-j\xi_3} T_+(x + j\xi_3) + a_{j\xi_3} T_-(x - j\xi_3) \\
&= \frac{16}{3\sqrt{3}} \sum_{j=1}^{4} a_{-j\xi_3}(x_1 + j)x_2^3 - a_{j\xi_3}(x_1 - j)x_2^3 \\
&= \frac{16}{3\sqrt{3}} \sum_{j=1}^{4} (a_{-j\xi_3} - a_{j\xi_3}) x_1 x_2^3 + \frac{16}{3\sqrt{3}} \sum_{j=1}^{4} j (a_{-j\xi_3} + a_{j\xi_3}) x_2^3 \\
&= \widetilde{C}_0 \widetilde{E}_0(x) + \widetilde{C}_3 \widetilde{E}_3(x) .
\end{aligned}
\tag{5.84}
$$

Let $\tilde{x} = \sigma x$ be a rotated coordinate system. We can represent T in $W_{1,2}$ as a sum of $p \in \mathbb{P}_4^\Delta$ and a truncated power function, namely,

$$
T(\tilde{x}) = (\tilde{x}_1^4 - 2\tilde{x}_1^2 \tilde{x}_2^2 + \tfrac{8}{3\sqrt{3}} \tilde{x}_1 \tilde{x}_2^3) - \tfrac{32}{3\sqrt{3}} \tilde{x}_1 \tilde{x}_{2+}^3 \qquad \text{for } x \in W_{1,2} .
$$

So if $\tilde{x} \in |N_1(0, K_6)|$ and $j \geq 1$ then $\tilde{x} - j\xi_1 = (\tilde{x}_1 - j, \tilde{x}_2) \in W_{1,2}$ and thus

$$
\sum_{j=1}^{4} a_{j\xi_1} T(\tilde{x} - j\xi_1) = p(\tilde{x}) - \frac{32}{3\sqrt{3}} \sum_{j=1}^{4} a_{j\xi_1}(\tilde{x}_1 - j)\tilde{x}_{2+}^3
$$

and in x-coordinates we have

$$
\sum_{j=1}^{4} a_{j\xi_1} T(x - j\xi_1) = p(x) + \left(\sum_{j=1}^{4} j a_{j\xi_1} \right) g_1(\sigma x) - \left(\sum_{j=1}^{4} a_{j\xi_1} \right) g_2(\sigma x) .
\tag{5.85}
$$

Similarly, let $\hat{x} = \sigma F_2(x)$ be another coordinate system. On $W_{0,1}$ we can again represent T by $T(\hat{x}) = p(\hat{x}) - \frac{32}{3\sqrt{3}} \hat{x}_1 \hat{x}_{2+}^3$, where $p \in \mathbb{P}_4^\Delta$. So if $\hat{x} \in |N_1(0, K_6)|$ and $j \geq 1$ then $\hat{x} + j\xi_2 = (\hat{x}_1 - j, \hat{x}_2) \in W_{0,1}$ and thus

$$
\sum_{j=1}^{4} a_{-j\xi_2} T(\hat{x} + j\xi_2) = p(\hat{x}) - \frac{32}{3\sqrt{3}} \sum_{j=1}^{4} a_{-j\xi_2}(\hat{x}_1 - j)\hat{x}_{2+}^3
$$

and in x-coordinates we have

$$
\sum_{j=1}^{4} a_{-j\xi_2} T(x + j\xi_2) = p(x) + \left(\sum_{j=1}^{4} j a_{-j\xi_2} \right) g_1(\sigma F_2 x) - \left(\sum_{j=1}^{4} a_{-j\xi_2} \right) g_2(\sigma F_2 x) .
\tag{5.86}
$$

The proposition follows from (5.85) and (5.86) by rearranging terms. $\qquad \square$

We end this section with a proposition about the basis functions.

Proposition 75. *For $j = 0, 1$ or 2 we have $L_6^h E_j = L_6^h \widetilde{E}_j = 0$.*

Proof. We begin by noting that $L_6^h g_1 = 0$. Indeed, since $g_1(x_1, x_2)$ is only a function of x_2, we see that the integral in

$$L_6^h g_1(v) = h^{-3} \sum_{i=0}^{6} m_6^i \frac{1}{\pi (h\rho)^2} \int_{B(h\rho)} g_1(v + h\delta_i + x) dx$$

is the same for i and i' if $\delta_{i,2} = \delta_{i',2}$. But in this case, $m_6^i = -m_6^{i'}$, so $L_6^h g_1 = 0$.

Next notice that for $j = 0, 1$ or 2, the basis function E_j is composed of a sum of functions of the form $g_1 \circ \sigma$, where σ is an orthogonal map which leaves the stencil invariant. In particular, we write $\sigma(\delta_i) = \delta_{\sigma i}$ and we have $m^{\sigma i} = \pm m^i$. This allows us to conclude that

$$L_6^h g_i \circ \sigma(v) = h^{-3} \sum_{i=0}^{6} m_6^i \frac{1}{\pi (h\rho)^2} \int_{B(h\rho)} g_1(\sigma v + h\delta_{\sigma i} + \sigma x) dx$$

$$= \pm h^{-3} \sum_{i=0}^{6} m_6^i \frac{1}{\pi (h\rho)^2} \int_{B(h\rho)} g_1(\sigma v + h\delta_i + x) dx$$

$$= \pm L_6^h g_1(\sigma v) = 0 \,.$$

$\square$

5.7.2 Corner Case A

Since $\tau_1(x)$ is even with respect to F_1, while $\tau_2(x)$ is odd, we can apply Proposition 73 as in (5.65) to get $\tau = p + G_3 + G_4$ on $|N_1 A|$, where

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad G_3 = \begin{pmatrix} b_1^0 E_0 + b_1^2 E_2 \\ b_2^1 E_1 \end{pmatrix} \quad \text{and} \quad G_4 = \begin{pmatrix} b_1^4 E_4 + b_1^6 E_6 + b_1^8 E_8 \\ b_2^5 E_5 \end{pmatrix}. \tag{5.87}$$

Starting with the 2-neighborhood control net for an equivariant characteristic map as in Figure 2.6, we can apply Loop's subdivision rules twice and rescale to compute the 5-neighborhood control net. Then we extract the control net on $N_2(A)$, which determines τ on $|N_1 A|$, compute the truncated power series coefficients $a = \nabla^c_{222} \tau$, and apply (5.72) to

get the following:

$$b_1^0 = \frac{-3\left(1 + 2\cos\left(\frac{4\pi}{n}\right)\right)}{2\left(3 + 2\cos\left(\frac{2\pi}{n}\right)\right)\left(5 + 4\cos\left(\frac{2\pi}{n}\right)\right)} \qquad b_2^1 = -\frac{\sin\left(\frac{6\pi}{n}\right)}{\left(3 + 2\cos\left(\frac{2\pi}{n}\right)\right)^2}$$

$$b_1^2 = \frac{2\cos\left(\frac{\pi}{n}\right)^2\left(1 + 2\cos\left(\frac{4\pi}{n}\right)\right)}{\left(3 + 2\cos\left(\frac{2\pi}{n}\right)\right)^2}$$

$$b_1^4 = \frac{6\cos\left(\frac{\pi}{n}\right)^2\left(1 + 2\cos\left(\frac{4\pi}{n}\right)\right)}{\left(3 + 2\cos\left(\frac{2\pi}{n}\right)\right)\left(5 + 4\cos\left(\frac{2\pi}{n}\right)\right)} \qquad b_2^5 = \frac{\sin\left(\frac{6\pi}{n}\right)}{2\left(3 + 2\cos\left(\frac{2\pi}{n}\right)\right)^2} \qquad (5.88)$$

$$b_1^6 = \frac{5\cos\left(\frac{\pi}{n}\right)^2\left(1 + 2\cos\left(\frac{4\pi}{n}\right)\right)}{\left(3 + 2\cos\left(\frac{2\pi}{n}\right)\right)^2}$$

$$b_1^8 = \frac{-6\cos\left(\frac{\pi}{n}\right)^2\left(1 + 2\cos\left(\frac{4\pi}{n}\right)\right)}{\left(3 + 2\cos\left(\frac{2\pi}{n}\right)\right)^2}$$

To compute the $D_{111}p(A)$ term in (5.67), notice that $D_{111}E_i(0) = 0$ for all i, therefore $D_{111}\tau(A) = D_{111}p(A)$. The left side of this equality can be computed from the control net. By (5.10) we have

$$D_{111}\tau(x) = \sum (\nabla_{111}\tau)_v \, M_{111}(x - v + 2\xi_3).$$

The box spine M_{111} is the piecewise linear hat function defined by $M_{111}(v) = \delta(v, \xi_3)$, so

$$D_{111}\tau(A) = (\nabla_{111}\tau)_{A+\xi_3} = (\nabla_{111}^c\tau)_A.$$

So applying the difference operator to the characteristic control net we get

$$D_{111}\tau(A) = \left(\frac{-2 - 4\cos\left(\frac{4\pi}{n}\right)}{\left(3 + 2\cos\left(\frac{2\pi}{n}\right)\right)\left(5 + 4\cos\left(\frac{2\pi}{n}\right)\right)}, 0\right), \qquad (5.89)$$

which is zero only if $n = 3$ or 6.

Now we write the matrix $M(hu, h)$ in (5.67), using Proposition 75 to get

$$M(hu, h) = \begin{pmatrix} D_{111}\tau_1(A) & b_1^0 L_7^1 E_0(u) + b_1^2 L_7^1 E_2(u) \\ 0 & b_2^1 L_7^1 E_1(u) \end{pmatrix} + O(h).$$

Lemma 76. *There are constants h_0 and $C > 0$ such that for any $(v, h) \in V_{III.A}$ with $h < h_0$ we have solutions $w^6(v, h)$ and $w^7(v, h)$ to the polynomial reproducing system (5.55) such that $|w^6(v, h)|$ and $|w^7(v, h)| < C$.*

u	$L_7^1 E_1(u)$
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\dfrac{64\left(-64\sqrt{3}\,\rho^3+81\,\pi\left(-1+\rho^2\right)\right)}{243\,\pi}$
$\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$	$\dfrac{-64\left(-32\sqrt{3}\,\rho^3+27\,\pi\left(1+\rho^2\right)\right)}{243\,\pi}$
$\left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$	$\dfrac{-2048\,\rho^3}{405\sqrt{3}\,\pi}$
$\left(1, \sqrt{3}\right)$	$\dfrac{128\left(-16\sqrt{3}\,\rho^3+27\,\pi\left(-2+\rho^2\right)\right)}{243\,\pi}$
$\left(2, \sqrt{3}\right)$	$\dfrac{-64\left(-128\sqrt{3}\,\rho^3+135\,\pi\left(1+\rho^2\right)\right)}{1215\,\pi}$

Figure 5.8: Values of $L_7^1 E_1(u)$.

Proof.

Case 1: n = 3 or 6. From (5.88) we see $b_1^i = b_2^i = 0$ for $i = 0, \ldots, 8$, so the components of the Loop characteristic map are polynomials on $|N_1 A|$. Therefore by Proposition 71, the right side of (5.55) is 0. So $w^6(v, h) = w^7(v, h) = 0$ is a trivial solution.

Case 2: n ≠ 3 or 6, and (hu, h) ∈ V$_{\text{III.A}}$ with u$_1$ ≠ 0 and u$_2$ ≠ 0. Since $b_2^1 \neq 0$ and $L_7^1 E_1(u) \neq 0$ (as shown in Figure 5.8), we see that in the limit, as h goes to 0, the matrix $M(hu, h)$ is invertible. Hence solutions exist and are bounded for sufficiently small h.

Case 3: n ≠ 3 or 6, and (hu, h) ∈ V$_{\text{III.A}}$ with u$_2$ = 0. Since τ_2 is odd with respect to F_1, we have $L_6^1 \tau_2(u) = L_7^1 \tau_2(u) = R^1 \tau_2(u) = 0$, i.e., the bottom rows of (5.67) and (5.68) are 0. Then since $D_{111}\tau_1(A)$ is non-zero, we have solutions $w^6(hu, h)$ and $w^7(hu, h) = 0$ for sufficiently small h satisfying a bound $|w^6(hu, h)| < C$.

Case 4: n ≠ 3 or 6, and (hu, h) ∈ V$_{\text{III.A}}$ with u$_1$ = 0 and u$_2$ = ±√3. In this case the analysis is more complicated. From an explicit computation one can show $L_7^1 E_1(u) =$

$R^1 E_1(u) = 0$. So (5.67) reduces to

$$M(hu, h) = \begin{pmatrix} D_{111}\tau_1(A) & L_7^1 G_{3,1} \\ 0 & 0 \end{pmatrix} + h \begin{pmatrix} \Psi_1 + L_6^1 G_{4,1} & L_7^1 G_{4,1} \\ \Psi_2 + L_6^1 G_{4,2} & L_7^1 G_{4,2} \end{pmatrix} \qquad (5.90)$$

and

$$R^h(u) = \begin{pmatrix} R^1 G_{3,1}(u) \\ 0 \end{pmatrix} + h R^1 G_4(u). \qquad (5.91)$$

Computing the determinate of $M(hu, h)$ we get

$$\det(M(hu, h)) = h \left(D_{111}\tau_1(A) \cdot L_7^1 G_{4,2}(u) - L_7^1 G_{3,1}(u) \cdot (\Psi_2(u) - L_6^1 G_{4,2}(u)) \right) + O(h^2).$$

By an explicit computation we show that the coefficient of h is non-zero. Let $M_j(hu, h)|R^h\tau(u)$ be the matrix formed by replacing the j-th column of $M(hu, h)$ with $R^h\tau(u)$. Then clearly $\det(M_j(hu, h)|R^h\tau(hu)) = O(h)$. So by Cramer's rule

$$w^{5+j}(hu, h) = \frac{\det(M_j(hu, h)|R^h\tau(u))}{\det(M(hu, h))}$$

for $j = 1$ or 2. Thus for a sufficiently small h, solutions exist and are bounded.

$\square$

5.7.3   Corner Case B

Here $\tau_1(x)$ is odd and $\tau_2(x)$ is even with respect to F_2. Starting with the 2-neighborhood control net for an equivariant characteristic map, as in Figure 2.6, we can apply Loop's subdivision rules twice and rescale to compute the 5-neighborhood control net. Then we extract the control net on $N_2(B)$ which determines τ on $|N_1 B|$, compute the truncated power series coefficients $a = \nabla_X \tau$, and apply (5.82) to get the following:

$$b_1^1 = 0 \qquad\qquad b_2^0 = \frac{-3 \cos(\frac{\pi}{n}) \left(1 + 2 \cos(\frac{4\pi}{n})\right)}{\left(3 + 2 \cos(\frac{2\pi}{n})\right)^2}$$

$$b_1^3 = \frac{4 \cos(\frac{\pi}{n})^2 \left(\sin(\frac{\pi}{n}) - \sin(\frac{3\pi}{n}) + \sin(\frac{5\pi}{n})\right)}{\left(3 + 2 \cos(\frac{2\pi}{n})\right)^2} \qquad b_2^2 = 0 \qquad\qquad (5.92)$$

$$b_1^5 = 0 \qquad\qquad b_2^4 = \frac{-4 \left(\cos(\frac{\pi}{n}) + \cos(\frac{3\pi}{n}) + \cos(\frac{5\pi}{n})\right)}{3 + 2 \cos(\frac{2\pi}{n})}$$

$$b_2^6 = \frac{2 \left(\cos(\frac{\pi}{n}) + \cos(\frac{3\pi}{n}) + \cos(\frac{5\pi}{n})\right)}{3 + 2 \cos(\frac{2\pi}{n})}.$$

Applying Proposition 74, as in (5.65), we get $\tau_{||N_1B|} = p + G_3 + G_4$, where

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 0 \\ b_2^0 \widetilde{E}_0 \end{pmatrix} \quad G_4 = \begin{pmatrix} b_1^3 \widetilde{E}_3 \\ b_1^4 \widetilde{E}_4 + b_1^6 \widetilde{E}_6 \end{pmatrix}.$$

One can compute $D_{111}\widetilde{E}_i(0) = 0$ for all i, therefore $D_{111}\tau(B) = D_{111}p(B)$ which can be computed from the control net using (5.7.2). We compute

$$D_{111}\tau(B) = \left(\frac{2\left(\sin(\frac{\pi}{n}) - \sin(\frac{3\pi}{n}) + \sin(\frac{5\pi}{n})\right)}{(3 + 2\cos(\frac{2\pi}{n}))(5 + 4\cos(\frac{2\pi}{n}))}, 0 \right), \tag{5.93}$$

which is zero only if $n = 3$ or 6. Now we write the matrix on the left side of (5.67) in terms of u and h. We use Proposition 75 to get

$$M(hu, h) = \begin{pmatrix} D_{111}\tau_1(B) & 0 \\ 0 & b_2^0 L_7^1 E_0(u) \end{pmatrix} + O(h).$$

Lemma 77. *There are constants h_0 and $C > 0$ such that for any $(v, h) \in V_{III.B}$ with $h < h_0$ we have solutions $w^6(v, h)$ and $w^7(v, h)$ to the polynomial reproducing system (5.55) with $|w^6(v, h)|$ and $|w^7(v, h)| < C$.*

Proof.

Case 1: n = 3 or 6. From (5.88) we see $b_1^i = b_2^i = 0$ for $i = 0, \ldots, 8$, so the components of the Loop characteristic map are polynomials on $|N_1B|$. Therefore, by Proposition 71, the right side of (5.55) is 0. So $w^6(v, h) = w^7(v, h) = 0$ is a trivial solution.

Case 2: n $\neq$ 3 or 6, and (hu, h) $\in$ V$_{III.A}$ with $|u_2| \leq \sqrt{3}$. We claim that $M(hu, h)$ is invertible in the limit. Since $D_{111}\tau_1(B)$ and b_2^0 are non-zero, we need only show that $L_7^1\widetilde{E}_0(u)$ is also non-zero.

For any $u = (u_1, u_2) \in W$ we have

$$L_7^1\widetilde{E}_0(u) = \sum_{i=0}^{18} m_7^i \frac{1}{\pi\rho^2} \int_{B(\rho)} \widetilde{E}_0(v + x + \delta^i) dx$$

$$= \sum_{i=0}^{18} m_7^i \frac{32}{3\sqrt{3}\pi\rho^2} \int_{B(\rho)} (v_2 + x_2 + \delta_2^i)_+^3 dx_1 dx_2.$$

So this quantity depends only on $\tilde{u}_2$ and ρ. Figure 5.9 tabulates these values showing that they are not zero.

$\tilde{u}_2$	$L_7^1 \widetilde{E}_0(u)$
$\pm\sqrt{3}$	$\dfrac{2048\,\rho^3}{405\,\sqrt{3}\,\pi}$
$\pm\sqrt{3}/2$	$\dfrac{64\left(-128\,\sqrt{3}\,\rho^3+135\,\pi\left(1+\rho^2\right)\right)}{1215\,\pi}$
0	$\dfrac{-128\left(-32\,\sqrt{3}\,\rho^3+45\,\pi\left(-2+\rho^2\right)\right)}{405\,\pi}$

Figure 5.9: Values of $L_7^1 \widetilde{E}_0(u)$.

Case 3: $n \neq 3$ or 6 and $(hu, h) \in V_{III.B}$ with $|u_2| > \sqrt{3}$. Here, the analysis of case 2 does not apply because $L_7^1 \widetilde{E}_0(u) = 0$. Since $\widetilde{E}_0$ and $\widetilde{E}_3$ are in $\mathbb{P}_4^\Delta$ on $N_3(u)$, we have $L_7^1 \widetilde{E}_j = R^1 \widetilde{E}_j = 0$ for $j = 0$ or 3.

Thus the linear system reduces to

$$M(hu, h) = \begin{pmatrix} D_{111}\tau_1(B) + O(h) & 0 \\ O(h) & h\left(b_2^4 L_7^1 \widetilde{E}_4 + b_2^6 L_7^1 \widetilde{E}_6\right) \end{pmatrix} \tag{5.94}$$

and

$$R^h \tau(u) = h \begin{pmatrix} 0 \\ b_2^4 R^1 \widetilde{E}_4 + b_2^6 R^1 \widetilde{E}_6 \end{pmatrix}. \tag{5.95}$$

By explicitly computing $b_2^4 L_7^1 \widetilde{E}_4 + b_2^6 L_7^1 \widetilde{E}_6$ at the four possible points $u \in W$ one can show this is non-zero and hence there is a solution of the form $w^6(hu, h) = 0$, and $w^7(hu, h)$ depends only on u. $\qquad\square$

Chapter 6

APPLICATIONS AND FUTURE DIRECTIONS

We have proved that Loop's subdivision functions approximate functions in a second-order Sobolev space. Our main theorem, which proves this, gives the order of decay of the approximation error when approximating a sufficiently smooth function. In this final chapter we summarize the theory developed in this thesis and discuss possible directions for extensions and an application of the theory.

In Section 6.1 we summarize this work and discuss the methods and results. In Section 6.2 we discuss possible extensions to the theory which could be explored in future investigations. Section 6.3 briefly outlines how subdivision surfaces can be used with the finite element method to solve a differential equation, and we show how our approximation theory can be applied to derive convergence rates for this application.

6.1 Summary

6.1.1 Smooth Structures on Simplicial Surfaces

In Chapter 2 we developed the basic theory of stationary subdivision surfaces defined on simplicial surfaces. In Section 2.1 we reviewed simplicial complexes, and defined subdivision schemes and subdivision (limit) functions. The definitions are similar to, and motivated by, the definitions in Chapter 2 of Zorin [25], although I think the definitions here are cleaner.

In Section 2.2 we defined an affine atlas on a simplicial surface K, excluding extraordinary vertices. Loop's subdivision functions are piecewise polynomial in affine coordinates.

In Section 2.3 we defined the n-regular complexes K_n and the contraction map which acts on K_n. The n-regular complexes are the model neighborhoods. The analysis of a subdivision scheme reduces to considering the scheme on the n-regular neighborhoods. The symmetries of K_n and the scaling provided by the contraction map play an important

role. The symmetries are generated by a rotation and a reflection. But notice, there is no translation symmetry on K_n. Recall translation symmetry underlies many of the analysis techniques on $\mathbb{R}^n$, e.g., the Fourier transform.

In Section 2.4 we introduced the key tools for analysis of subdivision surfaces, the subdivision map (or subdivision matrix) and the characteristic map. Since subdivision surfaces were introduced by Catmull, Clark, Doo and Sabin in [4, 7] the analysis of these surfaces has been based on the subdivision matrix. The characteristic map, derived from the eigenvectors of the subdivision matrix, was introduced by Reif in [16]. He proved that if the characteristic maps are regular and injective, then subdivision surfaces are C^1 for almost every control net.

We defined characteristic coordinate charts on a simplicial complex in Section 2.5. Suppose $\mathcal{S}$ is a stationary subdivision scheme with characteristic maps for every valence. For any finite simplicial complex K without boundary and a vertex $v \in K$, the interior of the 1-neighborhood $|N_1(v)|^\circ$ is a characteristic coordinate neighborhood. We used a characteristic map to define a characteristic coordinate chart on this neighborhood.

This is primarily a different interpretation of the work of Reif. In the parallel theory, we showed that if subdivision functions are C^r away from extraordinary vertices and the characteristic maps are regular and injective, then characteristic charts on a complex K form a C^r atlas on $|K|$. We call such a scheme a C^r-subdivision scheme. We then showed in Proposition 17 that subdivision functions are C^1 with respect to this atlas. As a corollary we have the following.

Corollary 78. *Given an $\mathbb{R}^3$-valued control net on a simplicial surface without boundary, if the generated subdivision function is injective and everywhere of full rank, then its image is a C^1-surface embedded in $\mathbb{R}^3$.*

A subdivision surface is the image of a C^1-differentiable map $f : |K| \to \mathbb{R}^3$. This map is the natural generalization of a parameterization for the surface. Actually, the surface is covered by local parameterizations. This framework is advantageous since, given a C^r-subdivision scheme, we have a class of C^r functions on a subdivision surface, even though the surface itself is only C^1. We exploited these smooth functions in our approximation

theorem.

In Section 2.6 we extended the degree of smoothness by introducing the class $C_{\text{loc}}^{r,1}$, consisting of functions that are of class C^r and for which all the derivatives of order r are locally Lipschitz. For instance, since Loop's subdivision scheme is a $C_{\text{loc}}^{2,1}$-subdivision scheme it induces a $C_{\text{loc}}^{2,1}$-smooth structure on a simplicial surface.

In the final section, Section 2.7, we prove a number of geometric facts in characteristic coordinates about the simplicial neighborhoods N_dT for faces $T \in K_n^k$.

6.1.2 Approximation in Sobolev Spaces

In Chapter 3 we defined Sobolev norms on a simplicial surface using the $C_{\text{loc}}^{R,1}$-atlas induced by a stationary subdivision scheme. A function defined on a simplicial surface $|K|$ is in the Sobolev space $H^s(K)$ for $s \leq R + 1$ if its characteristic coordinate representations are in H^s.

The Sobolev norm is especially well suited to studying approximation properties of stationary subdivision schemes such as Loop's. We discuss three benefits. First, Sobolev norms measure derivatives as well as function values. In geometric modeling applications it is important to at least approximate first derivatives accurately, since, for instance, the first derivatives are explicitly used in the form of normal directions for lighting calculations.

Secondly, a Sobolev norm is more forgiving at isolated points than the corresponding supremum norm of derivatives. For instance, Loop's subdivision functions are not C^2 at extraordinary vertices [17]. However, as we showed in Theorem 42, Loop's subdivision functions on a simplicial surface $|K|$ are in $H^2(K)$. Furthermore, Theorem 2 shows that subdivision functions can approximate the curvature of a surface to arbitrary accuracy as measured in the L^2-norm.

Thirdly, in certain applications such as the finite element method it is approximation in the Sobolev norm which ensures convergence of the method. Applications to the finite element method are discussed in Section 6.3.

In Section 3.3, we constructed approximations to a function f on $|K|$ using local ap-

proximations in characteristic coordinates. Suppose a family of local, linear operators

$$Q^k : L^1_{\Lambda,\mathrm{loc}}(K_n) \to \mathcal{S}(K_n^k) \tag{6.1}$$

satisfies the approximation error bound

$$\|Q^k f - f\|_{H^s_\Lambda} = O(\lambda_{\max}^{(r-s-\epsilon)k}) \tag{6.2}$$

for a function $f \in H^r(K_n)$. We used a partition of unity $\{\rho_v : v \in \mathsf{Vertex}(K)\}$ subordinate to the covering $\{|N_1(v)|^\circ : v \in \mathsf{Vertex}(K)\}$ to construct cut-off local representatives $f_v = \rho_v \cdot f$ in each coordinate chart. We used the local operators Q^k to approximate the cut-off representatives and then added the approximations to get a global approximation

$$P^k f = \sum_{v \in K} Q^k(\rho_v \cdot f) \in \mathcal{S}(K^k).$$

Theorem 43 gives a bound for the global approximation error,

$$\|P^k f - f\|_{H^s(K)} = O(\lambda_{\max}^{(r-s-\epsilon)k}).$$

6.1.3 The Main Theorem

The main result of this thesis gives the approximation order of Loop's subdivision functions. We restate the main theorem and two immediate corollaries.

Theorem 2. *Let $\mathcal{S}(K^k)$ be the space of Loop's subdivision functions on the k-times subdivided complex K^k. For integers $0 \le s < r \le 3$ and any $\epsilon > 0$ we have the following bound on the minimal $H^s(K)$-approximation error of a function $f \in H^r(K)$:*

$$\mathrm{dist}(f, \mathcal{S}(K^k))_{H^s(K)} \le C_\epsilon \lambda_{\max}^{(r-s-\epsilon)k} \|f\|_{H^r(K)}, \tag{6.3}$$

where the constant $C_\epsilon = C(\epsilon, K)$ is independent of k and f.

Corollary 3. *Given a smooth function $f \in H^3(K)$ we have*

$$\mathrm{dist}(f, \mathcal{S}(K^k))_{L^2(K)} = O(\lambda^{(3-\epsilon)k}) \tag{6.4}$$

for any $\epsilon > 0$, where $\lambda = \lambda_{\max}(K)$.

Corollary 4. *The space of Loop's subdivision functions at all levels $\bigcup_0^\infty \mathcal{S}(K^k)$ is dense in $H^2(K)$.*

The second corollary is proved by approximating a function $f \in H^2(K)$ by smoother functions in $H^3(K)$ then applying Theorem 2. We say the approximation power of Loop's subdivision functions is order 3 because of the exponent in (6.4).

The comparable theorem for quartic triangular splines shows that quartic splines have order 4 approximation power.

Theorem 79. *Suppose $S^k = \text{span}\{\phi(2^k(\cdot - v)) : v \in 2^{-k}\mathbb{Z}^2\}$ are nested spaces of splines where $\phi = M_{222}$ is the quartic triangular box spline basis function. Then for integers $0 \le s < r \le 4$ we have the following approximation error bound*

$$\text{dist}(f, S^k)_{H^s} \le C(\tfrac{1}{2})^{(r-s)k}\|f\|_{H^r} , \tag{6.5}$$

where C is independent of f and k.

The two main differences between the estimates (6.3) and (6.5) are the base and the maximum exponent in the rate of decay for the approximation error. The maximum factor of k in exponent is the approximation order. The approximation order for Loop's subdivision functions is one less than the approximation order for quartic triangular splines. The decrease in approximation power is caused by the behavior of subdivision surfaces at extraordinary vertices. The natural strengthening of (6.3) to a higher-order of approximation would involve $\|f\|_{H^4(K)}$, but this space is not even well defined for a $C_{\text{loc}}^{2,1}$-subdivision scheme such as Loop's. To prove that the bound in Theorem 2 is sharp is open.

The base $\lambda_{\max}$ in the decay rate of (6.3), as compared to the base $\frac{1}{2}$ in (6.5), is due to the characteristic coordinate chart. In the introduction to Chapter 4, we reviewed a standard application of the quasi-interpolant method to prove Theorem 79. The base $\frac{1}{2}$ in (6.5) came from the equality $\text{diam} \, N_d T = C(\frac{1}{2})^k$ for any face T in the subdivided regular grid K_6^k. The base $\lambda_{\max}$ in (6.3) was similarly derived from $\text{diam} \, N_d T$. In this case T is a face in the subdivided n-regular complex K_n^k, and the diameter is measured in characteristic coordinates. In Section 2.7 we proved that $\text{diam}_\Lambda(N_d T) < C\lambda_{\max}^k(K_n)$. For Loop's subdivision scheme, the sub-dominant eigenvalue of the subdivision map for a vertex of valence n is given by

$$\lambda = \tfrac{3}{8} + \tfrac{1}{4}\cos\left(\tfrac{2\pi}{n}\right) .$$

So for valences greater than 6, the rate of error decay in (6.3) for $r = 3$ and $s = 0$ is less than $(\frac{1}{2})^{3k}$.

6.1.4   Approximation via Quasi-Interpolants

We defined the properties of a quasi-interpolant on K_n in Chapter 4, and proved a bound on its approximation error.

We could directly apply the quasi-interpolant technique to the problem of approximation on K_n. By constructing a family of maps $Q^k : L^1_{\Lambda,\text{loc}}(K_n) \to S(K_n^k)$ that is local, bounded, and reproduces quadratic polynomials in characteristic coordinates, we could prove that the approximation error of the quasi-interpolant would satisfy the estimate (6.2). However, for Loop's subdivision scheme, and most other subdivision schemes that produce C^1-smooth surfaces, the quadratic polynomials on K_n are not subdivision functions. It is necessary to have quadratic polynomials on K_n as subdivision functions in order for a scheme to produce C^2-surfaces with non-zero curvature at extraordinary points (see Zorin [23]). So direct application of the quasi-interpolant method on K_n would involve constructing a quasi-interpolant that reproduced linear polynomials in characteristic coordinates. Such a quasi-interpolant would only demonstrate second-order approximation power.

The key to showing Loop's subdivision functions have third-order approximation power is to also take advantage of polynomial reproduction in affine coordinates. By Definition 44 in Section 4.1, an order r quasi-interpolant on K_n must reproduce polynomials in characteristic coordinates of degree less than $r - 1$, and must locally reproduce affine coordinate polynomials of degree less than r away from the central vertex. Theorem 45 is the main theorem of Chapter 4. It shows that a quasi-interpolant on K_n of order r results in an order r approximation error as in (6.2).

The approximation theorem on K_n (Theorem 45) is proved in Section 4.3. The proof follows the same outline as the proof for the approximation theorem for quartic splines which we proved in the introduction to Chapter 4. We proved the result by estimating the approximation error $\|Q^k f - f\|_{H^s_\Lambda(T)}$ on a single face T of K_n^k. We approximated f on a neighborhood of T by a quadratic polynomial $g \in \mathbb{P}_2(K_n)$, and we used the triangle

inequality and the boundedness and locality of Q^k to get

$$\|Q^k f - f\|_{H^s_\Lambda(T)} \leq \|Q^k(f-g)\|_{H^s_\Lambda(T)} + \|Q^k g - g\|_{H^s_\Lambda(T)} + \|f-g\|_{H^s_\Lambda(T)}$$

$$\leq (1+C)\|f-g\|_{H^s_\Lambda(N_d T)} + \|Q^k g - g\|_{H^s_\Lambda(T)}. \tag{6.6}$$

Now since Q does not reproduce $\mathbb{P}_2(K_n)$, we had to estimate the last term in (6.6). In Section 4.3.2 this term was bounded using the polynomial reproducing properties of the quasi-interpolant in affine coordinates.

We presented an alternative boundedness criteria for a quasi-interpolant on K_n in Section 4.4. This new criteria is simpler to verify as it only concerns the level $k = 0$ instance of Q^k. The bounds are in affine coordinates, where the subdivision functions are translation invariant, making them easier to analyze.

6.1.5   Construction of a Quasi-Interpolant

The main theorem of Chapter 5 is Theorem 57, showing that we can construct an order 3 quasi-interpolant on K_n for Loop's subdivision functions. By Lemma 55 we had to construct a rotation equivariant map $Q : L^1_{\Lambda,\mathrm{loc}}(K_n) \to \mathcal{S}(K_n)$ that is local with support width d and that satisfies the semi-norm estimates in affine coordinates

$$|Qf|_{H^m_A(T)} \leq |Q| \, |f|_{H^m_A(N^c_d T)} \tag{6.7}$$

on each face $T \in K_n$ and $m = 0, 1$ or 2. Additionally, Q has to reproduce linear polynomials in characteristic coordinates and locally reproduce functions that are locally quadratic polynomials in affine coordinates.

We constructed the quasi-interpolant as a composition of three maps

$$Q : L^1_{\Lambda,\mathrm{loc}}(K_n) \xrightarrow{\mathcal{R}} \mathsf{CN}(K_n) \xrightarrow{\mathcal{A}} \mathsf{CN}(K_n) \xrightarrow{\mathcal{S}^\infty} \mathcal{S}(K_n), \tag{6.8}$$

namely, the restriction map $\mathcal{R}$, the averaging operator $\mathcal{A}$, and the subdivision limit operator $\mathcal{S}^\infty$. Properties of box splines are catalogued in Section 5.1. In Section 5.2, we introduced Sobolev norms and semi-norms on control net spaces, which are the intermediate spaces in the composition (6.8). We then analyzed the boundedness of each component separately. The restriction operator produces a control net from a function on $|K_n|$. The control value

at a vertex is calculated by averaging the function over a small ball centered at the vertex. Propositions 61 and 63 showed that $\mathcal{S}^\infty$ and $\mathcal{R}$ satisfy semi-norm estimates that were used, together with similar estimates on $\mathcal{A}$, to prove (6.7).

In Section 5.3 we constructed an averaging operator $\mathcal{A}$, to achieve the desired polynomial reproducing properties. Locally, in the neighborhood of a face $T \in K_n$, the quasi-interpolant Q must reproduce an 8-dimensional space of functions, consisting of the 6-dimensional space of affine quadratic polynomials and the 2-dimensional space spanned by the components of a characteristic map. For each vertex $v \in K_n$ and control net $u \in \mathsf{CN}(K_n)$, we defined $(\mathcal{A}u)_v$ to be a weighed average of control values of u in the 2-neighborhood of v. In particular, we defined an 8 parameter vector space of possible weights. Then at each vertex $v \in K_n$ we wrote an 8×8 linear system that the weights had to satisfy so that a basis of polynomials would be reproduced. This system was simplified to a 2×2 system called the polynomial reproducing system. Lemma 69 claims that the polynomial reproducing system has solutions for all vertices sufficiently far from the central vertex. We used this Lemma to prove Theorem 57, the main theorem of Chapter 5. In Sections 5.4-5.7 we proved Lemma 69 by considering three different classes of vertices $v \in K_n$. The classes are distinguished by how many polynomial pieces the characteristic map has on the $N_3(v, K_n)$ neighborhood of v.

6.2 Future Extensions of the Theory

It would be useful to have a more general theory. In particular, the main theorem only applies to Loop's subdivision scheme. Chapters 2, 3 and 4 do however apply to general stationary subdivision schemes defined on simplicial surfaces. For instance, one might be able to use a modified Loop's subdivision scheme, as defined in Section 2.1.4, to get the subdominant eigenvalues λ of the subdivision map closer to $\frac{1}{2}$ for valences greater than 6. The new scheme is a $C^{2,1}_{\text{loc}}$ subdivision scheme if the characteristic maps are injective and regular. Then if we could construct an order 3 quasi-interpolant, we could get a rate of decay for the approximation error that is faster than that for Loop's subdivision functions. Zorin, Peters and Reif [23, 14] have presented general methods to determine if the characteristic maps for general subdivision schemes are injective and regular.

Extending the approximation result to Catmull-Clark's subdivision scheme and its generalizations is also desirable. Catmull-Clark's subdivision schemes generate C^1-smooth surfaces so the characteristic maps could be used to construct a $C^{2,1}_{\text{loc}}$-manifold on an initial mesh. Chapter 4 is still applicable with small changes to accommodate rectangles instead of triangles. The existence of a high-order quasi-interpolant for these schemes would have to be shown.

Numerical corroboration of the result and the sharpness of the result are two other issues which should be explored. The estimate is sharp if the rate of decay of the approximation error $\lambda_{\max}^{(3-s)k}$ is the fastest possible for all $f \in H^3(K)$.

6.3 *Application: Finite Element Method on Surfaces*

Approximation theory has been a powerful tool in the analysis of applications which rely on producing approximations to some ideal result. Examples include signal processing, image processing, numerical solution of differential equations, numerical integration and data fitting. In particular, we discuss the application of subdivision surfaces to the problem of approximately solving partial differential equations on surfaces.

The finite element method (FEM) is a numerical technique to approximate the solution to a partial differential equation. We show how to construct FEM approximations, for a partial differential equation on a surface, that are subdivision functions. Cirak et al., [9], describes how to use subdivision functions to solve elasticity problems on thin shell structures. As an example, we discuss a simpler problem, the Poisson problem on a surface. Then we use the standard error analysis technique for the FEM (see Brenner and Scott [3]) together with our approximation theorem, to show the convergence rate of the finite element method.

Let $\psi : |K| \to \mathbb{R}^3$ be a Loop's subdivision surface on a base simplicial surface K without boundary. We represent a function on the surface $\psi(K)$ as a function on $|K|$. There are differential operators Δ and ∇ acting on functions defined on the surface, which generalize the Laplacian and gradient on $\mathbb{R}^2$ (see Taylor [21] for their definitions). The Poisson problem

is to find a function u on $|K|$ such that

$$\Delta u = f \qquad \text{on } |K| \quad \text{and} \tag{6.9}$$

$$\int u \, ds = 0 \,, \tag{6.10}$$

where $\int f \, ds = 0$ and ds represents integration with respect to surface area. This equation models, for instance, the steady-state temperature distribution on a surface due to a steady source f. Since there is no heat lost through a boundary, the net effect of the source must be zero to ensure a steady state.

The first step in applying the FEM is to rewrite the differential equation as a variational problem. The operators Δ and ∇ satisfy a Green's identity

$$\int \Delta u \cdot v \, ds = -\int \langle \nabla u, \nabla v \rangle \, ds \,, \tag{6.11}$$

where $\langle \cdot, \cdot \rangle$ is the dot product of vectors on the surface. Multiplying both sides of (6.9) by $v \in H^1(K)$ and integrating over the surface, we get the variational form of Poisson's equation: Find $u \in H_0^1(K) = \{f \in H^1(K) : \int f \, ds = 0\}$ such that

$$\int \langle \nabla u, \nabla v \rangle \, ds = -\int f v \, ds \qquad \text{for all } v \in H^1(K) \,. \tag{6.12}$$

We construct an approximate solution from the trial space $\mathcal{S}_0(K^k)$ of Loop's subdivision functions at level k, i.e. $v \in \mathcal{S}(K^k)$, such that $\int v \, ds = 0$. Notice that for any $v \in \mathcal{S}(K^k)$, we have $v_0 = v - \frac{1}{\text{area}(K)} \int v \, ds \in \mathcal{S}_0(K_k)$, where $\text{area}(K)$ is the surface area of $\psi(K)$. The Galerkin FEM approximation $u_k \in \mathcal{S}_0(K^k)$ is defined as the solution of the finite-dimensional version of the variational problem

$$\int \langle \nabla u_k, \nabla v \rangle \, ds = -\int f v \, ds \qquad \text{for all } v \in \mathcal{S}_0(K^k) \,. \tag{6.13}$$

This is a linear problem with a unique solution.

By standard FEM error analysis, we now show that the Galerkin approximation u_k is, in a sense, the best approximation to the true solution u from the space $\mathcal{S}_0(K^k)$. We define a symmetric positive definite bilinear form a by

$$a(f, g) = \int \langle \nabla f, \nabla g \rangle \, ds \,, \tag{6.14}$$

and the induced norm $\|f\|_E = a(f, f)^{1/2}$ on $H_0^1(K)$ called the energy norm.

From (6.12) and (6.13) we see that $a(u - u_k, v) = 0$ for all $v \in \mathcal{S}(K^k)$. Therefore, for any $v \in S^k$ we have

$$\|u - u_k\|_E^2 = a(u - u_k, u - u_k)$$
$$= a(u - u_k, u - v) + a(u - u_k, v - u_k).$$

The second term is zero since $v - u_k \in \mathcal{S}^k$, and therefore, by the Cauchy-Schwarz inequality, we have $\|u - u_k\|^2 \le \|u - u_k\|_E \|u - v\|_E$. Dividing by $\|u - u_k\|_E$ we get

$$\|u - u_k\|_E \le \|u - v\|_E \qquad \text{for all } v \in \mathcal{S}^k.$$

The energy norm on $H_0^1(K)$ is equivalent to the $H^1(K)$-norm as we have defined it in Section 3. Therefore there is a constant C such that

$$\|u - u_k\|_{H^1} \le C \operatorname{dist}(u, \mathcal{S}(K^k))_{H^1}.$$

Assuming that $u \in H^3(K)$, we apply our main approximation theorem to get

$$\|u - u_k\|_{H^1} \le C \lambda^{2k}$$

for any $\lambda < \lambda_{\max}$.

6.4   Conclusion

Subdivision surfaces are a smooth geometric modeling primitive that can model surfaces of arbitrary topology. We have proved an approximation theorem for Loop's subdivision surfaces in Sobolev spaces. Loop's subdivision functions have order 3 approximation power. This is a higher-order of approximation power than for piecewise linear functions or triangular meshes, which have order 2 approximation power. However, the approximation power of Loop's subdivision functions is not as high as the approximation power of the quartic triangular splines from which they are derived. These have order 4 approximation power.

This result can be used to give convergence rates for finite element methods using subdivision functions.

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