Notes on Differential Equations

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Preface

Here's an outline of the course:

Part 1 For the first three and a half weeks, we'll study first order differential equations. We'll begin with more general first order differential equations and end by concentrating on first order linear differential equations.

Part 2 For the next three and a half weeks, we'll study second order linear differential equations, beginning with *homogeneous differential equations* of the form

$$ay'' + by' + cy = 0$$

and some applications, including the *harmonic oscillator* and concluding with *non-homogeneous differential equations* of the form

$$ay'' + by' + cy = f(t)\,.$$

The special case where $f(t) = \cos(\omega t)$ is particularly important.

Part 3 During the final three weeks, we'll study how to solve differential equations using Laplace transforms.

By the end of the course, you should know how to do the following:

- Model simple systems involving first order differential equations.
- Visualize solutions using direction fields.
- Use Euler's method to find approximate solutions to first order differential equations.
- Solve first order linear differential equations and initial value problems via integrating factors.
- Solve constant coefficient second order linear initial value problems using the method of undetermined coefficients.
- Calculate with complex numbers and the complex exponential function, compute derivatives and integrals of the complex exponential function.
- Express the function $y(t) = (a\cos(\omega t) + b\sin(\omega t))e^{ct}$ in the forms

$$y(t) = Ae^{ct}\cos(\omega t - \varphi) \text{ and } y(t) = \mathrm{Re}\left(Ce^{(c+i\omega)t}\right)$$

- Model simple mechanical and electrical systems with linear second order differential equations.
- Compute amplitude gain and phase shift with sinusoidal forcing function.
- Compute resonant frequency.
- Compute Laplace transforms and inverse Laplace transforms of commonly occurring functions.
- Solve constant coefficient linear initial value problems using the Laplace transform together with tables of Laplace transforms.
- Use Laplace transforms to solve initial value problems when the forcing function is piecewise continuous or involves the Dirac delta function.
- Express the solution of constant coefficient second order differential equations in terms of the convolution integral.

These notes differ from the book by Boyce and DiPrima in a number of ways.

- The notes give more emphasis on applications and less on theory than Boyce and DiPrima.
- Complex numbers are used more extensively than in Boyce and DiPrima. There are two reasons for this increased emphasis:
  (1) Using complex-valued functions often simplifies computations.
  (2) They are routinely used in applications involving periodic behavior such as electrical circuits, control theory, signal processing (including image processing), crystallography, etc.

This is the first quarter where these notes are bing used, so I welcome feedback! Please let me know of any mistakes, including typographical errors, and of any places where the text is confusing. Suggestions should be sent to me via email at `duchamp@uw.edu`.

Acknowledgments. Much of the material in these notes is based on material from my colleagues. Several problems on first order differential equation were written by Neal Koblitz. The material on complex numbers is based on notes written by Bob Phelps. The chapter on Laplace transforms is based on notes written by John Palmieri. I also made heavy use of John Sylvester's lecture notes for Math 307.

Contents

CHAPTER 1

Introduction

Perhaps the most famous differential equation, dating back to 1686, is Newton's Second Law of Motion: "Force equals mass times acceleration." In the special case of an object of mass m moving along a straight line, it can be written in the form

$$my'' = F(t, y, y') , \tag{1.1}$$

where $y = y(t)$ is the position of the object at time t and where $F(t, y, y')$ is the force exerted on the object at time t, which may also depend on the position y and velocity y' of the object. Equation (1.1) gives a relation between the function $y(t)$ and its first and second derivatives, but it does not give an explicit formula for $y(t)$. "Solving" this differential equation means finding the formula for $y(t)$. Because (1.1) expresses the second derivative y'' in terms of t, y, and y', it is called a *second order* differential equation.

A good part of this course (more than half!) will be devoted to the study of the solutions of (1.1) in the special case where the force is of the special form

$$F(t, y, y') = f(t) - ky - \gamma y' ,$$

and Newton's second law assumes the special form

$$my'' + \gamma y' + ky = f(t) . \tag{1.2}$$

As we shall see, this differential equation applies to all sorts of mechanical problems such as free-fall with drag taken into account, the simple pendulum, and struts on cars and airplanes. We will also see that the same differential equation models certain electrical circuits (called RLC-circuits), where it is usually written in the form

$$LV'' + RV' + \frac{1}{C}V = V_S(t) , \tag{1.3}$$

where $V = V(t)$ is a voltage, L, R, and C are parameters associated with electronic components, and $V_S(t)$ is an applied voltage (say from a battery or from radio waves).

Newton's Law of Cooling, which Isaac Newton published in 1701, is another important differential equation:

$$T' = -k (T - T_A(t)) .$$

Here, $T = T(t)$ denotes the temperature of an object at time t, $T_A(t)$ denotes the "ambient temperature" (the temperature of the environment of the object), and $k > 0$ is a constant that measures how well the object is insulated from its environment. A nicer way to write the law of cooling is

$$T' + kT = kT_A(t) . \tag{1.4}$$

If you took Math 125 here at the UW, you've already studied this differential equation, and you may have noticed that (apart from changes in symbols) the same differential equation appears in other modeling problems, such as those involving radioactive decay, exponential population growth, repayment of loans, and some "mixing problems."

Equations (1.2),(1.3), and (1.4) are all examples of *linear differential equations*, which are the most common differential equations. From a purely mathematical point of view, we are going to spend most of the time in this course studying only two differential equations:

$$ay' + by = f(t) \text{ and } ay'' + by + cy = f(t)\,,$$

where a, b, and c are constants.

1.1. What does it mean to "solve" a differential equation?

Suppose that t and y are two quantities where y depends on t. A *first order ordinary differential equation* or simply a *first order differential equation*[1] is an equation of the form

$$y' = F(t, y)$$

where $F(t, y)$ denotes a function of t and y. A *solution* of the differential equation is a function $y = y(t)$ that satisfies the identity

$$y'(t) = F(t, y(t))\,.$$

on some interval.

EXAMPLE 1.1. The function $y(t) = e^{-t^2/2}$ is a solution of the differential equation $y' = -ty$ because

$$y'(t) = e^{-t^2/2}\left(\frac{-2t}{2}\right) = -te^{-t^2/2} = -ty(t)\,.$$

Another solution solution is $y_1(t) = 5e^{-t^2/2}$.

Similarly, a *second order differential equation* is an equation of the form

$$y'' = F\left(t, y, y'\right)\,,$$

and a *solution* is a function $y = y(t)$ that satisfies the equation

$$y''(t) = F(t, y(t), y'(t))$$

EXAMPLE 1.2. The function $y(t) = \sin(2t)$ is a solution of the differential equation

$$y'' + 4y = 0$$

because $\sin''(2t) + 4\sin(2t) = -4\sin(2t) + 4\sin(2t) = 0$. The function $\cos(2t)$ is also a solution. In fact, any function of the form

$$y = C_1\cos(2t) + C_2\sin(2t)\,, \text{ for } C_1 \text{ and } C_2 \text{ constants,}$$

is a solution.

EXAMPLE 1.3. Differential equations can often be solved by guessing. Past experience shows that the exponential function $y = e^{rt}$ is the solution of many differential equations. Therefore, to solve the differential equation

$$y'' + 3y' + 2y = 0\,,$$

one might guess that $y(t) = e^{rt}$ is a solution. Substituting this into the differential equation gives

$$(e^{rt})'' + 3(e^{rt})' + 2(e^{rt}) = r^2e^{rt} + 3re^{rt} + 2e^{rt} = (r^2 + 3r + 2)e^{rt} = (r-2)(r-1)e^{rt} = 0\,.$$

This implies that $r = 2$ or $r = 1$. Therefore, $y(t) = e^{2t}$ and $y(t) = e^t$ are both solutions. Armed with these two solutions, one then finds that

$$y(t) = C_1e^t + C_2e^{2t}$$

[1] *Partial differential equations* involve functions of more than one variable and partial derivatives. The word "ordinary" refers to differential equations involving functions of only one variable. Since we do not consider partial differential equations in these notes, we will usually drop the word "ordinary."

is also a solution for any two constants C_1 and C_2.

1.2. What is an Initial Value Problem?

These examples show that differential equations such as those above have many solutions. To narrow the possibilities, additional information is necessary. A *(first order) initial value problem* or *IVP* is given by a differential equation together with the value of the solution at point:

$$y' = F(t, y) \text{ and } y(t_0) = y_0 \,. \tag{1.5}$$

A *solution* of the initial value problem is a solution of the differential equation that, in addition, satisfies the *initial condition* $y(t_0) = y_0$.

EXAMPLE 1.4. Consider the initial value problem

$$y' + y = 0\,, \quad y(1) = 4$$

One checks that the function $y(t) = Ce^{-t}$ for C a constant is a solution of the differential equation. The initial condition $y(1) = Ce^{1} = 4$ implies that $C = 4e^{-1}$. Consequently,

$$y(t) = 4e^{-1}e^{t} = 4e^{t-1}$$

is the solution to the initial value problem.

Similarly, a *(second order) initial value problem* consists of the following data:

$$y'' = F(t, y, y')\,, \quad y(t_0) = y_0\,, \quad y'(t_0) = y_0'\,. \tag{1.6}$$

A *solution* of the initial value problem (1.6) is a function $y = y(t)$ satisfying both the differential equation and the *initial conditions* $y(t_0) = y_0$, $y'(t_0) = y_0'$.

EXAMPLE 1.5. Find the solution of the initial value problem

$$y'' + 3y' + 2y = 0\,, \quad y(0) = 2\,, \quad y'(0) = 3\,.$$

SOLUTION. From Example 1.3, we know that the solution of the differential equation is

$$y(t) = C_1e^{t} + C_2e^{2t}\,,$$

where C_1 and C_2 are constants. The initial conditions are then

$$y(0) = C_1 + C_2 = 2 \text{ and } y'(0) = C_1 + 2C_2 = 3\,,$$

which can be solved for C_1 and C_2 to give $C_1 = 1$ and $C_2 = 1$. The solution to the initial value problem is, therefore,

$$y(t) = e^{t} + e^{2t}\,.$$

EXAMPLE 1.6. Solve the initial value problem

$$y'' + 4y = 0\,, \quad y(0) = 2\,, \quad y'(0) = 6\,.$$

SOLUTION. By Example 1.2, the function $y(t) = C_1\cos(2t) + C_2\sin(2t)$ is the solution of the differential equation. The initial conditions then give

$$y(0) = C_1 = 2 \text{ and } y'(0) = 2C_2 = 6\,,$$

which implies $C_2 = 3$. Therefore, the function $y(t) = 2\cos(2t) + 3\sin(2t)$ is the solution of the initial value problem.

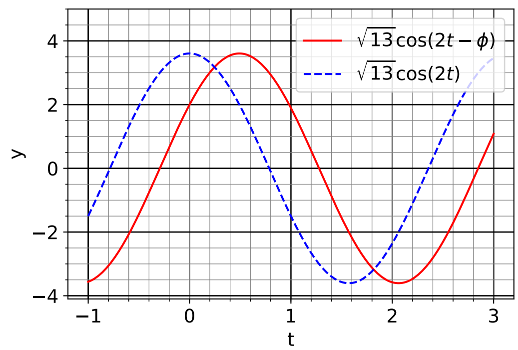


FIGURE 1.1. The graph of $y = \sqrt{13}\cos(2t - \arctan(3/2))$.

REMARK 1.1. When written in the form $y(t) = 2\cos(2t) + 3\sin(2t)$, it isn't clear what the graph of the function $y(t)$ looks like. A better way to visualize it is to express it in the form $y(t) = A\cos(2t - \phi)$. To do this use the "phase-shift formula" in Appendix A) to write $y(t)$ as follows:

$$y(t) = 2\cos(2t) + 3\sin(2t) = \sqrt{2^2 + 3^2}\cos\left(2t - \arctan(3/2)\right) \approx 3.6\cos(2(t - \sqrt{13}\cos\left(2(t - 0.49)\right)\ .$$

In this form, the graph of $y(t)$ is easily sketched (see Figure 1.1).

1.3. Some Examples: Falling Bodies, the Harmonic Oscillator, and Electrical Circuits

We have already mentioned two examples where a physical system can be described by a differential equation (Newton's Law of Cooling and Newton's Second Law of Motion).

EXAMPLE 1.7. Consider an object of mass m that falls or rises under the influence of gravity. Let y denote the height of object above ground level, and let $v = dy/dt$ denote its velocity. Observe that v is positive when the object is rising and negative when it is falling.

The gravitational field of the Earth exerts a force $F_{grav} = -mg$ on the object, where $g \approx 9.8\text{m/sec}^2$ denotes the acceleration due to gravity. (In the British system, $g \approx 32\text{ft/sec}^2$). The negative sign is necessary because the gravitational force points down.

Ignoring forces exerted on the object due to air resistance, Newton's second law of motion ("$F = ma$") shows that $v = v(t)$ satisfies the differential equation

$$m\frac{dv}{dt} = -mg \text{ or } \frac{dv}{dt} = -g\,.$$

Integration then yields the formula

$$v(t) = \int -g\,dt = -g\,t + C_1\,.$$

Since $\dfrac{dy}{dt} = v(t)$, another integration leads to a formula for $y(t)$:

$$y(t) = \int v(t)\,dt = -\frac{1}{2}\,g\,t^2 + C_1\,t + C_2\,.$$

If $y(0) = y_0$ and $y'(0) = v(0) = v_0$, then

$$y(0) = C_2 = y_0 \text{ and } y'(0) = C_1 = v_0\,,$$

yielding the well-known formulas

$$y(t) = -\frac{1}{2}\, g\, t^2 + v_0\, t + y_0 \text{ and } v(t) = -g\, t + v_0\,.$$

REMARK 1.2. (A NOTE ON UNITS) In the *British system,* the unit of mass is the *slug* and the unit of force is the *pound*. A mass of one slug has a weight of about 32 pounds. In general, if a body weighs w lbs. then its mass m is w/g slugs (this follows from the formula $w = mg$). In the *mks (meter-kilogram-second) system,* the unit of mass is the *kilogram (kg)* and the unit of force is the *Newton (N)*. In the *cgs (centimeter-gram-second) system* , the unit of mass is the *gram* and the unit of force is the *dyne*.

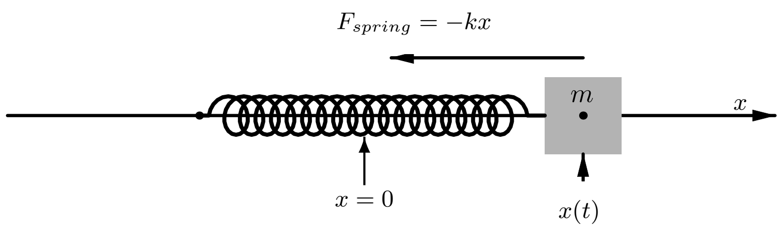


FIGURE 1.2. Hooke's Law states that the force a spring exerts on an object is proportional to the amount that the spring is stretched or compressed. The x axis is positioned so that when $x = 0$ the spring is in its equilibrium position and exerts no force on the object. For $x > 0$ the force points to the left (negative) and for $x < 0$ the force points to the right.

EXAMPLE 1.8. (THE HARMONIC OSCILLATOR) Figure 1.2 illustrates a mechanical system consisting of an object of mass m attached to a spring and free to move to the right and left without friction. Hooke's Law states that the force F_{spring} that the spring exerts on the object is proportional to the amount that the spring is stretched (or compressed) relative to its equilibrium position. If $x = x(t)$ denotes the position of the object relative to its rest position, Hooke's law can be expressed as

$$F_{spring} = -kx\,.$$

The constant $k > 0$ is called the *spring constant* and measures the strength of the spring. The units of k in the mks system are are N/meter and lbs/ft and in the British system. In this situation, Newton's second law of motion takes the form

$$m\frac{d^2x}{dt^2} = -kx \text{ or } \frac{d^2x}{dt^2} + \frac{k}{m}x = 0\,. \tag{1.7}$$

In the mks system, the units of $\dfrac{d^2x}{dt^2}$ are meters/sec^2. It follows that the units of $\dfrac{k}{m}$ are 1/sec^2.

As you would suspect, the object will oscillate back and forth along the x-axis, which is why this mechanical system is called the *harmonic oscillator*. In fact, it's easy to check directly that, for any constants C_1 and C_2, the function

$$x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) \text{ with } \omega_0 = \sqrt{\frac{k}{m}}$$

is a solution of (1.7). Notice that the units of ω_0 are 1/sec, so the product $\omega_0 t$ is dimensionless.

By the phase-shift formula (see Appendix A), the solution can also be written as

$$x(t) = A\cos(\omega_0 t - \phi)\,,$$

where $A = \sqrt{C_1^2 + C_2^2}$, and $\tan(\phi) = C_2/C_1$.

Clearly, to determine C_1 and C_2 more information is needed. Suppose both the position and the velocity of the object at a given time, say $t = t_0$, are known. This initial data is sufficient to uniquely determine the function $x(t)$. In other words, the data

$$mx'' + kx = 0\,, \quad x(t_0) = x_0\,, \quad x'(t_0) = x_0'\,,$$

comprised of a differential equation together with the position $x(t_0) = x_0$ and the velocity $x'(t_0) = v_0$ of the object at time $t = t_0$, are sufficient to determine the position of the object for all t. This is easy to see, for the initial conditions are

$$x(t_0) = C_1 \cos(\omega_0 t_0) + C_2 \sin(\omega_0 t_0) = x_0 \text{ and } x'(t_0) = -C_1\omega_0 \sin(\omega_0 t_0) + C_2\omega_0 \cos(\omega_0 t_0) = v_0\,.$$

These are two equations in the unknowns C_1 and C_2, which can be solved via high-school algebra.

Example 1.9. (Electrical Circuits) An electrical circuit is a collection of electronic components connected by wires through which an electrical current flows. The main components of electrical circuits are *resistors*, *capacitors*, and *inductors*, together with external *voltage sources*, such as batteries, electric generators, and antennas (which detect electromagnetic radiation, e.g. radio signals).

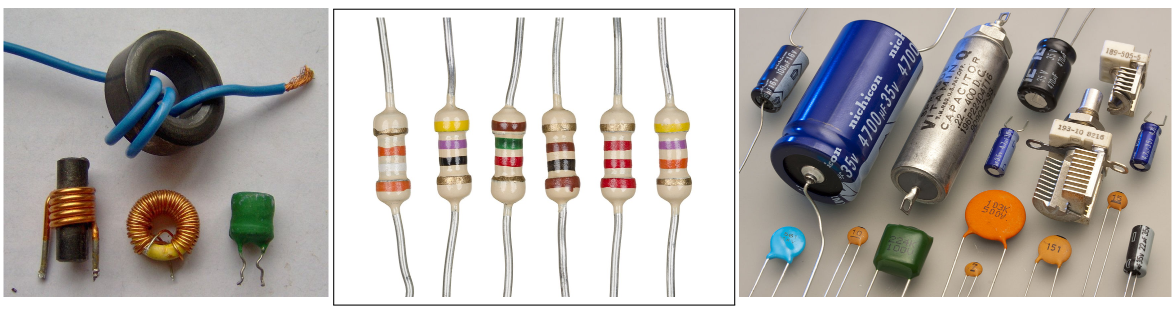


Figure 1.3. An assortment of inductors (left), resistors (center), and capacitors (right). (Photograph of inductors by F. Dominec. Photograph of capacitors by Eric Schrader.)

The website https://www.electronics-tutorials.ws/accircuits/passive-components.html has a nice description of these :

- Resistors regulate, impede or set the flow of current through a particular path or impose a voltage reduction in an electric circuit as a result of this current flow. Resistance is denoted by R and is measured in *Ohms* (denoted by Ω).
- The capacitor is a component that has the ability or capacity to store energy in the form of an electric charge like a small battery. Capacitance is denoted by C and is measured in *Farads* (denoted by F) or micro[2] Farads (denoted by μF).
- An inductor is a coil of wire that induces a magnetic field within itself or within a central core as a direct result of current passing through the coil. Inductance[3] is denoted by L and is measured in *Henries*. (denoted by H) or in *micro Henries* (denoted by μH)

Figure 1.4 illustrates how a resistor (R), inductor (L), and capacitor (C) might be assembled to form an electrical circuit called an *RLC*-circuit. By convention, if electrons are flowing counterclockwise in the wires, then the current $I(t)$ (measured in *amperes*) is considered to be flowing in the opposite

[2]Micro, denoted by μ, means 10^{-6}. For instance, 100μF denotes a capacitance of $100 \times 10^{-6} = 10^{-4}$ Farads.

[3]The symbol 'L' is used in the name of the physicist Heinrich Lenz, who studied inductance.

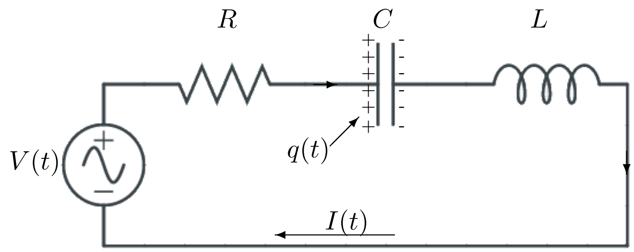


FIGURE 1.4. A schematic diagram of an RLC-circuit with an external (time dependent) voltage source $V(t)$.

direction, i.e. clockwise in the figure. Negative charge will accumulate on one "side" of the capacitor and a positive charge $q(t)$ (measured in *coulombs*) will collect on the other side at the rate

$$q'(t) = I(t)\,, \tag{1.8}$$

as illustrated in the schematic diagram shown in Figure 1.4.

The current in the circuit is related to the voltage source $V(t)$, which acts as a pressure causing current to flow along the wires of the circuit. Moving clockwise around the circuit, the voltage ("pressure") drops across each component in a manner determined by the construction of the component. More precisely, denoting the voltage drops across components by $V_R(t)$, $V_C(t)$ and $V_L(t)$, respectively:

$$V_R(t) = RI(t) \quad V_C(t) = \frac{1}{C}q(t) \quad V_L(t) = L\,\frac{dI(t)}{dt} \tag{1.9}$$

Kirchhoff's law states that the voltage drops around a closed circuit sum to zero:

$$\text{Kirchhoff's law:} \quad V_L(t) + V_R(t) + V_C(t) + (-V(t)) = 0\,, \tag{1.10}$$

where, because $V(t)$ is a voltage <u>increase</u>, it becomes $-V(t)$ when viewed as a drop.

Combining Equations (1.8), (1.9), and (1.10) leads to the following second order differential equation for $q(t)$:

$$L\,q'' + R\,q' + \frac{1}{C}\,q = V(t)\,, \tag{1.11}$$

modeling an RLC-circuit. Because $V_C = q/C$, the differential equation (1.11) can be rewritten as a differential equation for voltage across the capacitor:

$$(LC)\,V_C'' + (RC)\,V_C' + V_C = V(t)\,,$$

Notice that if we remove the resistor and voltage source from the circuit, then the differential equation for V_C reduces to

$$V_C'' + \frac{1}{LC}\,V_C = 0\,,$$

which (apart from the change of symbols) is the same as Equation (1.7) for the harmonic oscillator. This implies that, just as the position of the mass in the mass-spring system oscillates, so does the charge on the capacitor in an LC-circuit.

Exercises. The prerequisites for Math 307 include high school algebra and trigonometry as well as Calculus at the level of Math 125. Here are a few problems that you should do to ensure that you are ready to take Math 307.

Algebra Review Problems.

(1) Write down all values of r that are solutions of the quadratic equation $mr^2 + k = 0$, where m and k are both positive real numbers.

(2) Write down all values of r that are solutions to the quadratic equation $mr^2 + cr + k = 0$, where m, c and k are real numbers and $m \neq 0$.

(3) Simplify the expression $\dfrac{\frac{1}{AB}}{\frac{1}{A} + \frac{1}{B}}$, where A and B are non-zero real numbers.

(4) Simplify the expression $\dfrac{x^{1/3}x^{-1/5}}{x^{-a}}$, where x is a positive real number and a is any real number.

(5) Solve the equation $\dfrac{\frac{1}{K} + a}{\frac{1}{K} + b} = e^{-rt}$ for K in terms of the other quantities. Simplify as much as possible.

(6) Solve the equation $\ln(x) - \ln(y) = -2\ln(y) + ax + b$ for y, where x, a and b are positive real numbers. Simplify as much as possible.

(7) Express some basic properties of the natural logarithm and its inverse, the exponential function, by completing the following:

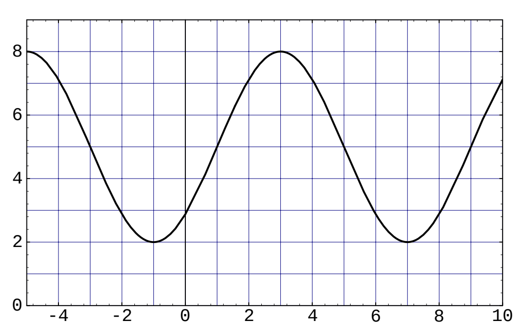
(a) $\ln xy = ?$ (b) $e^x e^y = ?$ (c) $e^{\ln x} = ?$ (d) $\ln e^x = ?$ (e) $\ln 1 = ?$

(8) Find counterexamples to each of the following "identities"; that is, find specific numbers $x = a, y = b$ such that equality **FAILS** for a and b. (For instance, to show that $(x+y)^2 \neq x^2 + y^2$, it would suffice to take $x = 1, y = 1$, since $(1+1)^2 = 4 \neq 2 = 1^2 + 1^2$.)
(a) $\ln(x+y) = \ln x + \ln y$, where $x > 0, y > 0$.
(b) $\dfrac{1}{x} + y = \dfrac{1}{x} + \dfrac{1}{y}$ $(x, y \neq 0)$.
(c) $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ $(x \geq 0, y \geq 0)$.
(d) $e^{x+y} = e^x + e^y$.

Trigonometry Review Problems.

(9) Use the formula for the cosine of the sum of two angles to express $f(t) = \sqrt{3}\sin(7\,t) - \cos(7\,t)$ in the form $f(t) = a\cos(bt + c)$, where a, b, and c are real numbers.

(10) The graph below is the graph of a function of the form $y = A\cos(\omega t - \phi) + b$. Find the specific values of A, ω, ϕ and b.



(11) Sketch the curves in the plane given by each of the following two parametric equations:

(a) $\begin{cases} x = 4\cos(3\,t) \\ y = 3\sin(3\,t) \end{cases}, 0 \le t \le \pi/2;$

(b) $\begin{cases} x = e^{-0.5t}\cos(2\pi\,t+\pi) \\ y = e^{-0.5t}\sin(2\pi\,t+\pi) \end{cases}, 0 \le t \le 4.$

Calculus Review Problems.

You'll need to understand the chain rule for differentiation and several basic methods of integration, like substitution, integration by parts, trigonometric substitutions, integration by parts, and rational functions with quadratic denominator. Remember the necessity of adding a constant of integration for indefinite integrals.

(12) Evaluate each of the following:

(a) $\int_0^1 dx/(4-x^2)$; (b) $\int_1^\infty \frac{dx}{(x^2+bx)}$, $b>0$; (c) $\int \frac{dx}{(x-a)(x-b)}$, $a \ne b$; (d) $\int \frac{dx}{1+x^2}$;
(e) $\int \frac{1}{1-x^2}\,dx$; (f) $\int x\cos x\,dx$; (g) $\int_0^\infty \cos(t)e^{-at}\,dt$, $a>0$; (h) $\int \frac{x-a}{x-b}\,dx$;
(i) $\int_0^\infty te^{-at}\,dt$, $a>0$; (j) $\int \frac{dx}{\sqrt{x^2-r^2}}$, $r>0$; (k) $\int \frac{dx}{\sqrt{r^2-x^2}}$, $r>0$; (l) $\int \frac{dx}{x^2+6x+25}$.

Part 1

First Order Differential Equations

CHAPTER 2

The Geometry of First Order Differential Equations

Chapter begins a more detailed study first order differential equations. It introduces the direction field of a differential equation, giving a way to visualize solutions of first order differential equations and initial value problems. It ends with Euler's method for finding approximate solutions of the initial value problem

$$y' = F(t, y)\,, \quad y(t_0) = y_0$$

in cases where we can't find an explicit solution.

In the following two chapters focus on two particularly classes of first order differential equations (separable and linear differential equations) where explicit methods for finding solutions are known.

A *(first order)* separable *differential equation* is one of the form[4]

$$h(y)y' = g(t)\,, \tag{2.1}$$

where $g(t)$ and $h(y)$ are continuous functions, is called a *separable* differential equation..

A *(first order)* linear *differential equation* is one of the form

$$y' + p(t)y = f(t)\,, \text{where } p(t) \text{ and } f(t) \text{ are continuous} \tag{2.2}$$

In the special case when $f(t) = 0$, the differential equation is called a *homogeneous linear differential equation* ; otherwise, it is said to be a *nonhomogeneous* differential equation. The function $f(t)$ on the right-hand side of a differential equationis called the *forcing function*

Example 2.1. Here's an alarming example of an initial value problem that doesn't have a unique solution: consider the initial value problem

$$\frac{dy}{dt} = F(y)\,, \quad y(0) = 0\,,$$

where $F(y) = \sqrt{2|y|}$. For any real number $a > 0$, consider the function $y_a(t)$ defined as follows:

$$y_a(t) = \begin{cases} (t-a)^2/2 & \text{for } t \geq a \\ 0 & \text{for } t \leq a\,. \end{cases}$$

By construction, $y_a(t)$ satisfies the initial condition $y_a(0) = 0$. It also satisfies the differential equation

$$y_a'(t) = F(y_a(t)) \text{ for all } t\,.$$

This is clear because

$$y_a'(t) = 0 = F(0) = F(y_a(t)) \text{ for } t \leq a\,,$$

and

$$\frac{d(t-a)^2/2}{dt} = (t-a) = \sqrt{2(t-a)^2/2} = F(y_a(t)) \text{ for } t \geq a\,.$$

[4]If you took Math 125 here at the UW, then you've already worked with separable differential equations, and much of the material in the next section will be a review for you.

REMARK 2.1. The previous example naturally leads to the following question:

What conditions on $F(t,y)$ are sufficient to guarantee that the initial value problem

$$y' = F(t,y)\,, \quad y(t_0) = y_0$$

has a unique solution?

This question will be addressed in more detail later in these notes. For now, suffice it to say that if the function $F(t,y)$ is sufficiently nice[5] (which is the case in virtually all differential equations encountered in practice) then the initial value problem has a unique solution.

2.1. The Direction Field of a Differential Equation

In this section, we present a geometric description of the differential equation

$$\frac{dy}{dt} = F(t,y)$$

that is useful for understanding the behavior of its solutions.

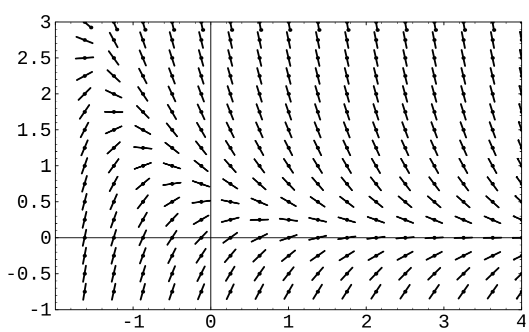
FUNDAMENTAL OBSERVATION: Suppose we already know that $y = y(t)$ is a solution to this differential equation. We can evaluate the derivative $y'(a)$ without differentiating:

$$y'(a) = F(a, y(a))\,.$$

Therefore, if $y(a) = b$, then the slope m of the tangent line to the curve $y = y(t)$ at (a,b) is $m = y'(a) = F(a,b)$; and the equation of the tangent line to $y = y(t)$ at $t - a$ is

$$y = m(t-a) + b \text{ where } m = F(a,b).$$

We can encode this by drawing a short line segment of slope $m = F(a,b)$ through the point (a,b). This line segment is called a *direction element* or *line element* of the differential equation.



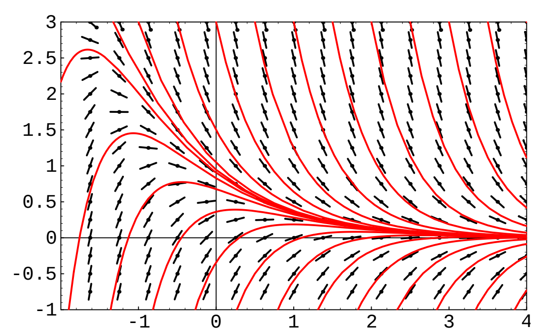


FIGURE 2.1. The direction field of a differential equation is shown on the left. The figure on the right shows the same direction field, together with several integral curves.

EXAMPLE 2.2. For instance, suppose that $y = y(t)$ is a solution of the differential equation $y' = te^y$ and we know that $y(1) = 2$. Then $y'(1) = (1)e^2 = e^2$. The direction element for this differential equation is, therefore, a line segment through the point $(1,2)$ of slope e^2.

[5]The function $F(t,y)$ in Example 2.1 is not differentiable with respect to y for $y = 0$; it is not "nice."

The *direction field* of the differential equation is the picture obtained by drawing a direction element through each point in the (t, y)-plane. The effect of drawing lots of direction elements is a picture that resembles a collection of iron filings in a magnetic field (the filings line up parallel to the magnetic field).

By construction, if $y = y(t)$ is a solution of the differential equation, then at every point $(a, y(a)$ the slope of the tangent line to the curve agrees with the slope of the line element $F(a, b)$. This forces the graphs of solutions to conform with the direction field of the differential equation (see Figure 2.1),

The graphs of solutions of a differential equation are called *integral curves* of the differential equation. Recall that we claimed that initial value problems for "nice" differential equations have unique solutions. This has a geometric interpretation: *the integral curves of a first order differential equation never cross*; and *there is a unique integral curve through each point* (t_0, y_0) *in the* (t, y)*-plane.*

2.2. Euler's Method

Although there are a number of techniques for solving special classes of differential equations, there is no general algorithm for solving <u>all</u> differential equations. Consequently, mathematicians have developed a number of numerical methods for finding approximate solutions of many differential equations that appear in applications. Finding better numerical methods remains an active area of research. Of these, *Euler's method* is the simplest and the easiest to describe.

The method is based on the tangent line approximation. Suppose that $y = y(t)$ is a differentiable function of t and we know both the value of $y(t)$ and the slope of the tangent line to $y = y(t)$ at $t = t_0$. The *tangent line approximation* of $y(t)$ at $t = t_0$ is the linear function

$$y = y_0 + y'(t_0)\,(t - t_0)\,,$$

which approximates $t = y(t)$ for values of t near t_0 (see Figure 2.2).

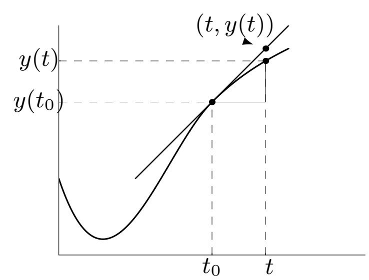


FIGURE 2.2. The tangent line approximation

EXAMPLE 2.3. To understand how to use the tangent line approximation to approximate the solution of a differential equation, consider the following initial value problem:

$$y' = y\,, \quad y(0) = 1\,.$$

Ignore for the moment that the solution is $y(t) = e^t$. Choose a small step size, say $h = 0.1$. We will use the tangent line approximation to find approximate values for $y(h) = y(0.1)$, $y(2h) = y(0.2)$, $y(3h) = y(0.3)$, etc.

Notice that because $y'(t) = y(t)$, we know that $y'(0) = y(0) = 1$. The tangent line approximation then gives the approximation

$$y(0.1) \approx y(0) + y'(0)(0.1 - 0) = 1 + (1)(0.1) = 1.1\,.$$

$n =$	0	1	2	3	4	5	6	7	8	9	10
$t =$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$y_n =$	1.000	1.100	1.210	1.331	1.464	1.611	1.772	1.949	2.144	2.358	2.594
$y(t) = e^t =$	1.000	1.105	1.221	1.349	1.492	1.649	1.922	2.014	2.226	2.460	2.718

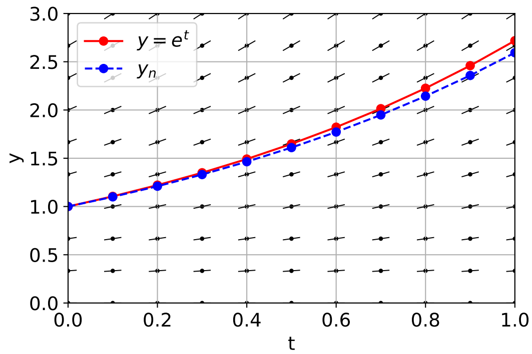


FIGURE 2.3. Euler's method applied to the initial value problem $y' = y$, $y(0) = 1.0$, with step size $h = 0.1$.

The tangent line approximation can be used again to approximate $y(0.2)$: From the differential equation, $y'(0.1) = y(0.1) \approx 1.1$. Therefore,

$$y(0.2) \approx y(0.1) + y'(0.1)(0.1) = 1.1 + (1.1)(0.1) = 1.21\,.$$

We can repeat this as many times as we like. Figure 2.3 summarizes the result for the first 10 iterations of this process.

The general method proceeds along the same lines as the example. To find an approximate solution of the initial value problem

$$y' = F(t, y), \quad y(t_0) = y_0\,,$$

proceed as follows:

(0) Choose a step size $h > 0$.
(1) Set $n = 0$.
(2) Set $y'_n = F(t_n, y_n)$.
(3) Set $t_{n+1} = t_n + h$.
(4) Set $y_{n+1} = y_n + y'_n \cdot h$
(5) Increase n by one and go to step (2).

EXAMPLE 2.4. Figure 2.4 shows the result of applying Euler's method with a step size of $h = 0.1$ to the initial value problem $y' + y = 10\cos(\pi t)\,, \quad y(0) = 0$.

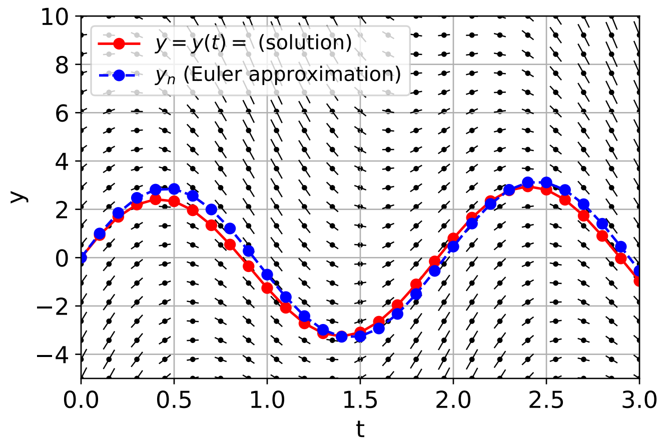


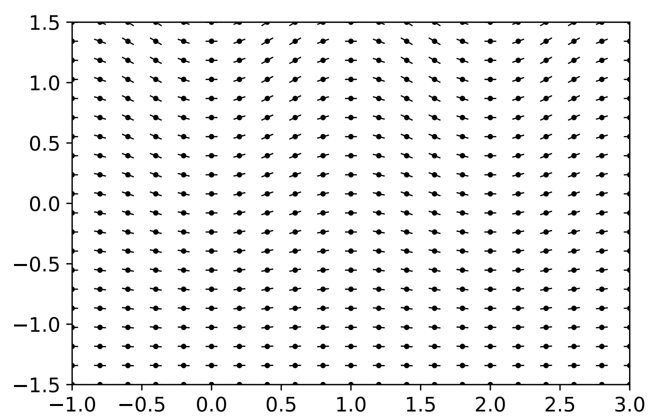
FIGURE 2.4. Euler's method applied to the initial value problem $y' + y = 10\cos(\pi t)$, $y(0) = 0.0$, with step size $h = 0.1$.

EXERCISES 1.

(1) The direction field of the differential equation $v' = F(t, v)$ is shown below.
  (a) On the figure below, sketch the solution to the initial value problem

$$v' = F(t, v)\,, \quad v(0) = 0.5\,,$$

  and label your sketch "(a)".



  (b) Let $y = y_1(t)$ be the solution of the initial value problem in part (a). What is the approximate value of $y_1(2.5)$?
  (c) On the figure above, carefully sketch the solution of the initial value problem $v' = F(t, v)\,, \quad v(0) = 0$, and label your sketch "(c)". Let $y = y_2(t)$ be the solution of this initial value problem.
  (d) Assume that the solutions $y_1(t)$ and $y_2(t)$ are defined for all values of t between 0 and 5. Is it possible for the graphs of $y_1(t)$ and $y_2(t)$ to cross at $t = 5$? Explain your answer.

(2) Suppose that $y = y(t)$ is the solution of the initial value problem

$$y' = t + \cos(y)\,, y(0) = 1\,.$$

Use Euler's method with a step size of $h = 0.2$ to estimate $y(1)$.

CHAPTER 3

Solving First Order Differential Equations

Euler's method only gives approximate solutions to differential equations. In this chapter, we discuss two clases of differential equations in which it is possible to obtain exact solutions: *separable differential equations* and *linear differential equations*.

3.1. Separable Differential Eqauations

A *separable* differential equation is one that can be written in the form

$$h(y)y' = g(t)\,,$$

where $g(t)$ and $h(y)$ are continuous functions. Such differential equations are of interest because they are easily solve and, as the examples in this chapter demonstrate, they arise in a number of modeling problems,

Assume that the function $y = y(t)$ is a solution of the differential equation. Then, by definition, it satisfies the equation

$$h(y(t))y'(t) = g(t)\,,$$

for all values of t in some interval, say $a < t < b$. It follows that the integral of the left-hand side differs from the integral of the right-hand side by a constant:

$$\int h(y(t))y'(t)\,dt = \int g(t)\,dt + C\,.$$

Let[6]

$$H(y) = \int^y h(z)\,dz \text{ and } G(t) = \int^t g(s)\,ds$$

be anti-derivatives of $h(y)$ and $g(t)$, respectively. Thus, the function $y = y(t)$ is implicitly defined by the equation

$$\boxed{H(y) = G(t) + C}\,. \tag{3.1}$$

This process can be reversed. Suppose that $y = y(t)$ satisfies (3.1). The computation

$$\frac{dH(y(t))}{dt} = H'(y(t))\,y'(t) = h(y(t))\,y'(t) = G'(t) = g(t)\,.$$

then shows that it is $y(t)$ a solution of the differential equation (2.1), called the *general solution* of the differential equation.

[6]We use the notation $\int^x f(s)\,ds$ to denote a specific anti-derivative of the function $f(x)$. For instance, $\int^x \cos(s)\,ds = \sin(x)$ rather than $\sin(s) + C$.

REMARK 3.1. The function $y = y(t)$ is said to be *implicitly defined* by (3.1). Unfortunately, Equation (3.1) cannot always be *explicitly solved* for y in terms of t. It is, however, usually important to obtain an explicit solution when possible.

EXAMPLE 3.1. *Solve the differential equation* $\dfrac{dy}{dt} = (1 + y^2)e^{t/2}$.

SOLUTION. Rewrite the differential equation in the form $\dfrac{1}{1+y^2}\dfrac{dy}{dt} = e^{t/2}$.
Integrate

$$\int^y \frac{dz}{1+z^2} = \int^t e^{s/2}\,ds + C$$

to arrive at the *implicit solution* $\tan^{-1}(y) = 2e^{t/2} + C$, where C is an arbitrary constant.
Lastly, solve for y to obtain the *explicit solution*

$$y(t) = \tan\left(2e^{t/2} + C\right) .$$

3.1.1. Solving Initial Value Problems. The solution of the initial value problem

$$h(y)y' = g(t) \quad y(t_0) = y_0 \,,$$

can be found by first finding the general solution:

$$H(y) = G(t) + C \,,$$

where $H'(y) = h(y)$ and $G'(t) = g(t)$; and then setting $y = y_0$ and $t = t_0$ to solve for C:

$$H(y_0) = G(t_0) + C \text{ or } C = H(y_0) - G(t_0)$$

to obtain the solution

$$\boxed{H(y) - H(y_0) = G(t) - G(t_0)\,.} \tag{3.2}$$

EXAMPLE 3.2. *Solve the initial value problem* $y' = (1 + y^2)e^t \,, \quad y(0) = 1$.

SOLUTION. Use separation of variables to find the general solution of the differential equation: $\tan^{-1}(y) = e^t + C$.

Next, set $t = 0$ and $y = 1$ to compute C as follows:

$$\tan^{-1}(1) = e^0 + C \implies \pi/4 = 1 + C \implies C = \pi/4 - 1\,.$$

Finally, substitute the value of C into the general solution:

$$\tan^{-1}(y) - \tan^{1}(1) = e^t - e^0 \text{ or } \tan^{-1}(y) - \pi/4 = e^t - 1\,.$$

Solving for y yields the final result:

$$y(t) = \tan\left(e^t - 1 + \pi/4\right) .$$

REMARK 3.2. An alternate way to solve initial value problems that avoids having to solve for the constant C is to use definite integrals from the start. Specifically, the solution of the initial value problem

$$h(y)y' = g(t) \quad y(t_0) = y_0$$

is easily seen to be

$$\int_{y_0}^{y} h(z)\,dz = \int_{t_0}^{t} g(s)\,ds\,.$$

Indeed, by the Fundamental Theorem of Calculus, this is just

$$H(y) - H(y_0) = G(t) - G(t_0)\,,$$

which is precisely Equation (3.2).

3.2. Linear First Order Differential Equations

Recall that a *linear first order differential equation* is a differential equation that can be written in the form

$$y' + p(t)y = f(t)\,,$$

where (usually) $p(t)$ and $f(t)$ are continuous or piecewise continuous on some interval. When $f(t) = 0$, the equation is said to be a *homogeneous* differential equation, otherwise, it is said to be *nonhomogeneous*.

EXAMPLES 3.3. The following differential equations are all linear:

$$\begin{aligned}
&\frac{dy}{dt} + 2y = 0 && y' + 2y = e^t\\
&\frac{dy}{dt} + ty = 0 && y' + ry = k, \quad r,k \quad \text{constant}\\
&(1+t^2)y' + y = 0 && ty' + y = te^t, \quad t > 0\\
&y' + t^{-1}\,y = 1, \quad t > 0 && y' + y = \sin^{-1}(t), \quad |t| < 1\\
&y' + \sqrt{1-t^2}\,y = t, \quad |t| < 1 &&
\end{aligned}$$

Note: the equation $(1+t^2)y' + y = 0$ is linear because it can we rewritten as $y' + \dfrac{1}{1+t^2}\,y = 0$.

If a differential equation is not linear we say that it is a *nonlinear differential equation*. Here are some examples of nonlinear differential equations:

$$\begin{aligned}
&\frac{dy}{dt} + 2y^2 = 0 && y' + 2\frac{1}{y} = e^t && y\frac{dy}{dt} + ty = 2\\
&(y')^2 + y = t && y' + \sqrt{y} = 0 && y' + ty = (1+y^2)
\end{aligned}$$

3.2.1. The Constant Coefficient Case. When $p(t) = k$, for k a constant, the differential equation has the form

$$y' + ky = f(t)\,.$$

REMARK 3.3. The models for radioactive decay, exponential population growth, and Newton's law of cooling are all of this form. For instance, Newton's law of cooling (4.2) can be written in the form

$$T' + kT = kT_A(t)\,.$$

When, in addition, $f(t)$ has certain simple forms, it's often possible to guess a solution. The easiest case is when $f(t) = 0$, where general solution is the exponential function

$$y(t) = Ce^{-kt}$$

The next easiest case is when $f(t) = a$, where a is a (non-zero) constant:

$$y' + ky = a\,.$$

Let's try a constant solution $y(t) = A$. Substituting this into the differential equation gives

$$y'(t) + ky(t) = 0 + kA = kA = a$$

Therefore $A = a/k$. Adding to this solution the solution of the homogeneous equation yields another solution:

$$y(t) = Ce^{-kt} + \frac{a}{k}\,,$$

where C is an arbitrary constant.

Another important case consists of linear differential equations of the form

$$y' + ky = e^{-at} \text{ for } a \neq k \text{ a constant.}$$

Substituting $y(t) = Ae^{-at}$ into the differential equation and computing as follows:

$$(Ae^{at})' + k(Ae^{at}) = (-a + k)A_e^{at} = e^{-at}$$

shows that $A = \frac{1}{k-a}$ and, therefore $y(t) = \dfrac{1}{k-a}e^{at}$. As in the previous example, the function

$$y(t) = Ce^{-kt} + \frac{1}{k-a}e^{at}, \quad \text{for } C \text{ a constant,}$$

is also a solution.

Notice that this trick won't work for $a = k$ because Ae^{-kt} is a solution of the homogeneous differential equation $y' + ky = 0$. In this case, one seeks a solution of the form $y(t) = Ate^{-kt}$. Then

$$(Ate^{-kt})' + k(Ate^{-kt}) = (1 - kt)Ae^{-kt} + k(Ate^{-kt}) = Ae^{-kt} = e^{-kt}.$$

This forces $A = 1$. Therefore, $y(t) = te^{-kt}$ is a solution, as is the function

$$y(t) = Ce^{-kt} + te^{-kt} = (t + C)e^{-kt}.$$

Another commonly occurring class of equations consists of differential equations of the form

$$y' + ky = b\cos(\omega t),$$

where k, b, and ω are positive numbers. The following example illustrates this case in an applied setting.

Example 3.4. Let t be time in hours (with $t = 0$ at noon on Jan 1. Suppose further that during a particularly cold month of January, the outside temperature in Seattle varies between $-10°$C and $10°$C according to the formula $T_A(t) = 10\cos\left(\dfrac{2\pi}{24}t\right)$. Let $T(t)$ be the temperature (in degrees C) inside a container that was left outside. Then according to Newton's law of cooling, the temperature inside the container satisfies the differential equation[7]

$$\frac{dT}{dt} = -k\,(T - T_A(t)) \text{ or } \frac{dT}{dt} + kT = kT_A(t),$$

where k is a measure of how well the container is insulated. Suppose further that $T(0) = 0$. Find a formula for $T(t)$. Next, find the largest value of k so that for large values of t, $T(t)$ will stay between $-2°$C and $2°$C.

Solution. To simplify the computations, set $\omega = \pi/12$ and write the differential equation in the form

$$T' + kT = 10k\cos(\omega t).$$

The function $T(t) = A\cos(\omega t) + B\sin(\omega t) + Ce^{-kt}$ is a solution for appropriate values of A,B, and C. To see this, substitute $T(t)$ into the differential equation and compute as follows:

$$\begin{aligned} T'(t) + kT(t) &= \omega\{-A\sin(\omega t) + B\cos(\omega t)\} + k\{A\cos(\omega t) + B\sin(\omega t)\} \\ &= (kB - \omega A)\sin(\omega t) + (\omega B + kA)\cos(\omega t) = 10k\cos(\omega t) \end{aligned}$$

For equality to hold, A and B must satisfy the equations

$$kB - \omega A = 0 \text{ and } \omega B + kA = 10k.$$

Solving for A and B gives

$$A = \frac{10k^2}{k^2 + \omega^2} \text{ and } B = \frac{10\omega k}{k^2 + \omega^2}.$$

[7]Because $T_A(t)$ is the temperature around the container, it's the *ambient temperature* or the temperature of the environment around the container. That's why I chose to use the symbol T_A—"A" for "ambient."

Hence,

$$T(t) = \frac{10k^2}{k^2+\omega^2}\cos(\omega t) + \frac{10\omega k}{k^2+\omega^2}\sin(\omega t) + Ce^{-kt}\,.$$

Since $T(0) = 0$, it follows that that $\frac{10k}{k^2+\omega^2} + C = 0$, so

$$C = -\frac{10k^2}{k^2+\omega^2}\,.$$

But it isn't clear what the solution looks like! To obtain a better formula for $T(t)$, employ the "phase-shift" formula from Appendix A: Notice that

$$\frac{10k^2}{k^2+\omega^2}\cos(\omega t) + \frac{10\omega k}{k^2+\omega^2}\sin(\omega t) = \left\{\frac{10k}{\sqrt{k^2+\omega^2}}\right\}\left(\frac{k}{\sqrt{k^2+\omega^2}}\cos(\omega t) + \frac{\omega}{\sqrt{k^2+\omega^2}}\sin(\omega t)\right)$$

Setting $\phi = \arctan(B/A) = \arctan(\omega/k) = \arctan\left(\frac{\pi}{12k}\right)$ gives the formula

$$T(t) = \left\{\frac{10k}{\sqrt{k^2+\omega^2}}\right\}\cos(\omega t - \phi) - \frac{10k^2}{k^2+\omega^2}e^{-kt}\,.$$

To determine the value of k, notice that "in the long run" (i.e. for t large) $Ce^{-kt} \approx 0$. Ignoring the exponential term gives the approximation

$$T(t) \approx \left\{\frac{10k}{\sqrt{k^2+\omega^2}}\right\}\cos(\omega t - \phi)\,.$$

Because the exponential term vanishes quickly, it is called a *transient*; and remaining periodic term is called the *periodic solution* or the *stable solution* of the differential equation. Thus, for t large, $T(t)$ is approximately a shifted "sine-wave" of amplitude $\dfrac{10k}{\sqrt{k^2+\omega^2}}$, and $T(t)$ will stay within 2°C of 0°C provided that we choose k to satisfy the inequality

$$\frac{10k}{\sqrt{k^2+\omega^2}} \le 2\,.$$

Clearly, the maximum value of k is a solution of the equation $\frac{10k}{\sqrt{k^2+\omega^2}} = 2$. Squaring and clearing denominators gives

$$100k^2 = 4(k^2+\omega^2) \quad\Longrightarrow\quad 96k^2 = 4\omega^2 \quad\Longrightarrow\quad k = \sqrt{\frac{4}{96}}\omega = \frac{\pi}{24\sqrt{6}} \approx 0.0534$$

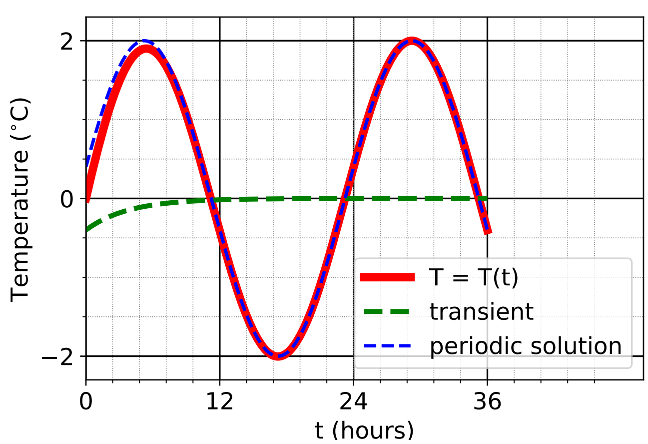


FIGURE 3.1. The graphs of the actual solution $T(t)$ when $k = 0.0534$, together with the transient Ce^{-kt} and the periodic solution.

3.2.2. The General Case. Rather than guessing solutions, there is a more systematic approach that applies to all initial value problems of the form

$$y' + p(t)y = f(t)\,, \quad y(t_0) = y_0\,, \tag{3.3}$$

where $p(t)$ and $f(t)$ are continuous on an interval $a < t < b$.

In this approach, multiplying the differential equation by a so-called *integrating factor* transforms the differential equation into one that can be solved by Riemann integration.

The computations are easier to understand in the *constant coefficient case*, where $p(t) = k$ and where $t_0 = 0$:

$$y' + ky = f(t)\,, \quad y(0) = y_0\,. \tag{3.4}$$

Begin by assuming that the function $y = y(t)$ is a solution of (3.4). Then, by definition, it satisfies the equation

$$y'(t) + ky(t) = f(t)\,.$$

Multiply both sides of this equation by e^{kt} (this is the "integrating factor") to obtain the equation

$$e^{kt}y'(t) + ke^{kt}y(t) = e^{kt}f(t)\,;$$

and notice that, by the product rule for differentiation,

$$\left(e^{kt}y(t)\right)' = e^{kt}y'(t) + ke^{kt}y(t)\,.$$

This shows that the solution satisfies the equation

$$\left(e^{kt}y(t)\right)' = e^{kt}f(t)\,,$$

which can be integrated to obtain the equality

$$\int_0^t \left(e^{ks}y(s)\right)'\,ds = \int_0^t e^{ks}f(s)\,.$$

By the Fundamental Theorem of Calculus, the integral on the left can be explicitly evaluated to yield the equation

$$e^{kt}y(t) - y(0) = \int_0^t e^{ks}f(s)\,ds\,, \tag{3.5}$$

which, in turn, can be solved for $y(t)$ to yield a formula for $y(t)$:

$$y(t) = e^{-kt}\left(\int_0^t e^{ks}f(s)\,ds + y(0)\right)\,. \tag{3.6}$$

Finally, recall that $y(t)$ satisfies the initial condition $y(0) = y_0$ and simplify to obtain the formula

$$\boxed{y(t) = e^{-kt}\int_0^t e^{ks}f(s)\,ds + y_0e^{-kt}\,.} \tag{3.7}$$

Remark 3.4. To can check directly that $y(t)$ is a solution of the initial value problem, first notice that the computation

$$y(0) = e^{-k0}\int_0^0 e^{ks}f(s)\,ds + y_0e^{-k0} = 0 + y_0 = y_0\,,$$

shows that $y(t)$ satisfies the initial condition. Next notice that (by the Fundamental Theorem of Calculus):

$$\begin{aligned}
y'(t) &= -ke^{-kt}\left(\int_0^t e^{ks}f(s)\,ds\right) + e^{-kt}\left(e^{kt}f(t)\right) - ky_0e^{-kt} \\
&= -\,k\left(e^{-kt}\left(\int_0^t e^{ks}f(s)\,ds\right) + y_0e^{-kt}\right) + f(t) \\
&= -ky(t) + f(t)\,.
\end{aligned}$$

Consequently, $y'(t) + ky(t) = (-ky(t) + f(t)) + ky(t) = f(t)$, showing that $y(t)$ is, indeed, a solution of the differential equation.

This analysis accomplished three goals:

(i) It shows that the initial value problem (3.4) has a solution.
(ii) Because it started with an unknown solution $y(t)$ and arrived at the formula (3.7), it shows that there is only one solution.
(iii) The formula (3.7) gives an explicit algorithm for solving the initial value problem: to find $y(t)$, one need only evaluate one definite integral.

A similar trick applies to the general case,

$$y' + p(t)y = f(t)\,,$$

but with one change: the function e^{kt} must be replaced by a more complicated expression.

Let $P(t) = \displaystyle\int^t p(s)\,ds$, and replace the "integrating factor" e^{kt} by the function $e^{P(t)}$. Then, as above

$$e^{P(t)}\left(y'(t) + p(t)y(t)\right) = \left(e^{P(t)}y(t)\right)' = e^{P(t)}f(t)$$

which can be integrated to yield the formula

$$e^{P(t)}y(t) = \int^t e^{P(s)}f(s)\,ds + C\,.$$

This, in turn, can be solve for $y(t)$ to obtain the formula

$$\boxed{y(t) = y_p(t) + Cy_h(t)\,,} \tag{3.8a}$$

where

$$\boxed{y_h(t) = e^{-P(t)}\,, y_p(t) = e^{-P(t)}\int^t e^{P(s)}f(s)\,ds\,,\ \text{and}\ P(t) = \int^t p(s)\,,ds.} \tag{3.8b}$$

The value of the constant C is determined from the initial condition $y(t_0) = y_0$ by solving the equation

$$y(t_0) = y_p(t_0) + Cy_h(t_0) = y_0$$

for C.

Remark 3.5. Notice that if $f(t) = 0$, then $y_p(t) = 0$, and so $y = Cy_h(t)$ is the general solution of the homogeneous differential equation $y' + p(t)y = 0$. The function $y_p(t)$ is called a *particular solution* of the nonhomogeneous differential equation, since it does not involve any arbitrary constants. The function $y_h(t)$ is a particular solution of the homogeneous differential equation.

Notice that having found an algorithm for the solution of the initial value problem implies that a solution exists.

Moreover, the computation began with the assumption that a solution $y(t)$ existed and then *solved* for $y(t)$. This showed that every solution of the differential equation is of the form (3.8a). Since the initial condition $y(t_0) = y_0$ is sufficient to determine the constant C, it follows that there is only one solution to the initial value problem. The only assumption that made in the computations was that the two integrals

$$\int^t p(s)\,ds \text{ and } \int^t e^{P(s)}g(s)\,ds$$

made sense. This is certainly the case if $p(t)$ and $g(t)$ are continuous (or even piecewise continuous) functions. The following theorem summarizes the above discussion.

THEOREM 1. *Let $p(t)$ and $g(t)$ be continuous functions defined on the interval $a < t < b$ and suppose that $a < t_0 < b$. Then there is one and only one solution of the initial value problem*

$$y' + p(t)\,y = f(t)\,, \quad y(t_0) = y_0\,.$$

EXAMPLE 3.5. *Solve the initial value problem $y' + 3ty = te^{t^2}$, $y(2) = 5$.*

SOLUTION. First solve the homogeneous equation: $y' + 3ty = 0$. Since $P(t) = 3t^2/2$ is an antiderivative of $3t$, the function

$$y_h(t) = e^{-3t^2/2}$$

is a solution of the homogeneous equation. Next set $y_p = h(t)y_1(t) = h(t)e^{-3t^2/2}$ and plug into the nonhomogeneous differential equation to get

$$h'(t)e^{-3t^2/2} = te^{t^2} \text{ or } h'(t) = te^{5t^2/2}$$

Integration gives $h(t) = \frac{1}{5}e^{5t^2/2}$, so the general solution is

$$y(t) = \left(\frac{1}{5}e^{5t^2/2} + C\right)e^{-3t^2/2} = \frac{1}{5}e^{t^2} + Ce^{-3t^2/2}\,.$$

The initial condition $y(2) = 5$ then determines C:

$$\frac{1}{5}e^{(2)^2} + Ce^{-3(2)^2/2} = 5\,.$$

Thus,

$$C = e^{3(2)^2/2}\left(5 - \frac{1}{5}e^{(2)^2}\right) = 5e^6 - \frac{e^{10}}{5} \approx -2388$$

and

$$y(t) \approx \frac{1}{5}e^{t^2} - 2388e^{-3t^2/2}\,.$$

CHAPTER 4

Modeling with First Order Differential Equations

4.1. Linear Models

Example 4.1. (Radioactive decay) *Suppose that a certain quantity of a radioactive substance is placed in a container and that after 10 years the amount has decreased by 0.01%. What percent of the original quantity will remain after 25 years?*

Solution. Begin by assigning labels to the various quantities related to the problem:

- Let t denote the time (in years) after the substance is placed in the container.
- Let Q_0 be the quantity (say in grams) of the radioactive substance initially placed in the container.
- Let $Q(t)$ denote the quantity remaining after t years.

Next, recall that the quantity of radioactive substance in the container decays at a rate proportional to the quantity remaining. Letting $k > 0$ denote the constant of proportionality. This, together with the condition $Q(0) = Q_0$ means that $Q(t)$ is a solution of the initial value problem

$$\frac{dQ}{dt} = -kQ\,, \quad Q(0) = Q_0\,.$$

This is a linear differential equation with solution

$$Q(t) = Q_0 e^{-kt}\,.$$

The decay rate k can be determined from the amount of radioactive substance remaining after 10 years:

$$Q(10) = (1 - 0.0001)Q_0 = 0.9999\, Q_0\,.$$

Hence,

$$\ln\left(\frac{Q(10)}{Q_0}\right) = \ln(0.9999) = -k(10) \implies k = -\frac{1}{10}\ln(0.9999) \approx 0.00001\,.$$

Substituting this value of k into the equation $\ln(Q/Q_0) = -kt$ and exponentiation yields the formula

$$Q(t) = Q_0 e^{-0.00001t}\,.$$

It remains only to compute $Q(25)$ as a percent of Q_0:

$$\frac{Q(25)}{Q_0} \times 100\% = 100 e^{-(0.00001)25} \approx 99.975\%\,.$$

Remark 4.1. Each radioactive element has a characteristic decay rate k. However, rather than specifying k directly, it is traditional to express it indirectly in term of the *half-life* t_h, which is a more intuitive measure of the rate of decay than the parameter k: The half-life of a radioactive element is the time required for half of the element to decay into other (lighter) elements.

The decay rate k can be computed from the half-life as follows. Let Q_0 be the amount of radioactive material at some time, set to $t = 0$, and let $Q(t)$ denote the amount of a radioactive material remaining t years later. Then, by definition, $Q(t_h)/Q_0 = 1/2$. On the other hand, $Q(t_h) = Q_0 e^{-kt_h}$.

Therefore, $Q_0 e^{-kt_h} = \dfrac{1}{2} Q_0$. Canceling the term Q_0, taking the natural logarithm, and solving for k yields the formula

$$\boxed{k = \frac{\ln(2)}{t_h}\,.} \tag{4.1}$$

EXAMPLE 4.2. *You find a frozen animal and you determine by experiment that the concentration of C^{14} in it is 22% of the amount found in live animals. How old is the animal?*

SOLUTION. The concentration of C^{14} in live animals depends on the concentration of C^{14} in the atmosphere, and is roughly independent[8] of time. When an animal dies, it stops adsorbing carbon from its environment, and so the concentration of C^{14} decreases as the C^{14} decays into Nitrogen 14. This is the key fact behind C^{14} dating.

To compute the age of the animal, let $Q(t)$ denote the amount of C^{14} in the animal t years after its death and let $t = T$ denote the value of t when you determined the concentration of C^{14}. Solving the differential equation $Q' = -kQ$, gives

$$Q(t) = Q_0 e^{-kt}\,,$$

where Q_0 is the amount of C^{14} in the animal at the time of death and

$$k = \frac{\ln(2)}{5568} \approx 0.0001245\,.$$

Because $Q(T) = 0.22 Q_0$ and $Q(t) = Q_0 e^{-kt}$, it follows that

$$0.22 Q_0 = Q_0 e^{-kT}\,,$$

which can be solved for T:

$$T = -\frac{\ln(0.22)}{k} = -\frac{\ln(0.22)}{\ln(2)/5568} \approx 12,163 \quad \text{years.}$$

EXAMPLE 4.3. (NEWTON'S LAW OF COOLING)

Recall that Newton's law of cooling states that if an object is brought into an environment, then the rate of cooling is proportional to the difference between the temperature of the object and the temperature of the environment (the *ambient temperature*). This law can be reformulated as an initial value problem as follows.

Let

$$\begin{aligned} T(t) &= \text{temperature of the object at time } t, \\ T(0) = T_0 &= \text{initial temperature of the object}, \\ T_A &= \text{ambient temperature, assumed constant here.} \end{aligned}$$

Then Newton's law of cooling takes the form:

$$\frac{dT}{dt} = -k\,(T - T_A)\,, \quad T(0) = T_0\,, \tag{4.2}$$

[8] In fact, it is necessary to make allowance for changes in the concentration of C^{14} in the atmosphere over time by incorporating tree-ring data into the calculation of Q_0.

where $k > 0$ is the constant of proportionality, which depends on the thermal properties of the object. Equation

The initial value problem (4.2) can be solved by separation of variables:

$$\int_{T_0}^{T} \frac{du}{u - T_A} = -\int_0^t k\,ds$$

Therefore,

$$\ln(T - T_A) - \ln(T_0 - T_A) = \ln\left(\frac{T - T_A}{T_0 - T_A}\right) = -kt \tag{4.3}$$

Exponentiating and solving for T results in the formula

$$T(t) = (T_0 - T_A)e^{-kt}) + T_A\,.$$

Notice that the differential equation(4.2) is linear, so the solution can alos be otained by the method of integrating factors.

REMARK 4.2. Newton's law of cooling is only a rough model of how objects cool. It assumes that the temperature of the object is the same at all points in the object. In most cases, this is not the case and a more complex model, involving partial differential equations is needed.

QUESTION. *A cup of coffee at 200° F is brought out into a room that is kept at 60° F. Two minutes later you measure the coffee's temperature to be 180°. Find a formula for the coffee's temperature at any time t.*

SOLUTION. Substituting $T(0) = 200$ and $T_A = 60$ into Equation (4.3) gives rise to the equation

$$\ln\left(\frac{T - 60}{200 - 60}\right) = -kt\,.$$

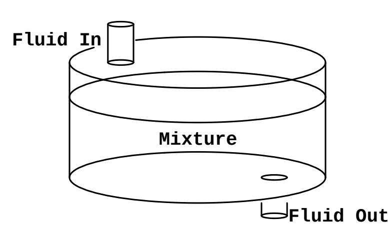
To find k, substitute the values $t = 2$ and $T = 180$ in this formula to find

$$\ln\left(\frac{180 - 60}{200 - 60}\right) = \ln\left(\frac{120}{140}\right) = -2k \implies k = -\frac{1}{2}\ln(6/7) = \frac{1}{2}\ln(7/6) \approx 0.077$$

Consequently,

$$T(t) \approx 60 + 140e^{-0.077t}\,.$$

EXAMPLE 4.4. (A MIXING PROBLEM) *Suppose that a tank with a capacity of 300 gallons initially contains 100 gallons of pure water. A salt solution containing 3 pounds of salt per gallon is allowed to run into the tank at a rate of 8 gal/min, and the mixture is then removed at a rate of 6 gal/min, as shown in the figure. The process is continued until the tank is filled. Determine the concentration of the salt solution in the tank at the end of the process (see figure below).*



Solution. First label variables and then formulate an appropriate initial value problem:

$$\begin{aligned}
t &= \text{time in mins}\\
Q(t) &= \text{the total quantity of salt in the tank}\\
V(t) &= \text{the volume of the fluid in tank} = 100 + (8-6)\, t = 100 + 2\, t \text{ gal}\\
C(t) &= \frac{Q(t)}{V(t)} = \text{concentration of solution in the tank}
\end{aligned}$$

The net rate at which salt is entering the tank is the difference between the rate that it enters the tank and the rate that it leaves the tank:

$$\begin{aligned}
\frac{dQ_{in}}{dt} &= \text{rate salt entering tank} = (8 \cdot 3) = 24\\
\frac{dQ_{out}}{dt} &= \text{rate salt leaving tank} = 6\, C(t) = 6\frac{Q(t)}{V(t)} = \frac{6}{100+2t}\, Q(t)\\
\frac{dQ}{dt} &= \frac{dQ_{in}}{dt} - \frac{dQ_{out}}{dt} = 24 - \frac{6}{100+2t}\, Q(t)\,.
\end{aligned}$$

Since the tank is initially full of pure water, $Q(0) = 0$. Thus, $Q(t)$ is the solution of the initial value problem

$$\frac{dQ}{dt} + \frac{6}{100+2t} Q = 24\,, \quad Q(0) = 0\,.$$

The integrating factor for this differential equation is

$$\mu(t) = \exp\left(\int^t \frac{6ds}{100+2s}\right) = \exp(3\ln(50+t)) = (50+t)^3\,.$$

Thus,

$$(50+t)^3 Q(t) = \int 24(50+t)^3 dt = 6(50+t)^4 + c\,.$$

Using initial condition $Q(0) = 0$ (initially, the water is pure), gives $c = -6(50)^4$. Solving for Q, yields the formula

$$Q(t) = 6(50+t) - 300\left(\frac{50}{50+t}\right)^3 \quad \text{lbs.}$$

To find the time when the tank is full, solve the equation

$$V(t) = 100 + 2t = 300$$

for t to find that $t = (300-100)/2 = 100$ min. The concentration of the salt solution at end of the filling procedure is, therefore,

$$C(100) = \frac{Q(100)}{V(100)} = \frac{6(150) - 300(50/150)^3}{300}\,\text{lb/ gal} = (80/27)\,\text{lb/ gal} \approx 2.963\,\text{lb/ gal}.$$

Example 4.5. (Falling bodies with air resistance)

The case of a falling body where air resistance is taken into account is more complicated than the simple case discussed in the introduction where the forces due to air resistance were ignored. For slowly moving bodies, the force caused by moving through air (called *drag*) is proportional to the speed of the object and points in the direction <u>opposite</u> the motion:

$$\text{drag } = -kv\,, k > 0\,,$$

where k is a positive constant called the *drag coefficient*. In this situation, Newton's second law of motion assumes the form

$$m\frac{dv}{dt} = -mg - kv \;\text{ or }\; \frac{dv}{dt} + \frac{k}{m}\, v = -g\,,$$

a linear differential equation. The integrating factor is $\mu(t) = e^{kt/m}$. Therefore,

$$e^{kt/m} v = -\int e^{kt/m} g = -\frac{mg}{k} e^{kt/m} + C\,,$$

or $v(t) = -\dfrac{gm}{k} + C\,e^{-kt/m}$ The initial condition $v(0) = v_0$ determines C, resulting in the formula

$$v(t) = -\frac{gm}{k} + \left(v_0 + \frac{gm}{k}\right) e^{-kt/m}\,.$$

Notice that, as the speed increases, so does the drag. At a certain velocity the force of gravity will exactly cancel with the drag. This velocity is called the *terminal velocity*:

$$v_\infty = \lim_{t\to\infty} v(t)\,.$$

To find v_∞, solve the equation $-mg - kv_\infty = 0$ for v_{infty} to obtain the formula

$$v_\infty = -\frac{mg}{k}\,.$$

Integrating the formula for $v(t)$ yields an expression for $y(t)$:

$$y(t) = -\frac{gm}{k}t - \frac{m}{k}\left(v_0 + \frac{gm}{k}\right) e^{-kt/m} + C\,.$$

The initial condition $y(0) = y_0$ determines C:

$$C = \frac{m}{k}\left(\frac{gm}{k} + v_0\right) + y_0\,,$$

giving rise to the formula

$$y(t) = -\frac{gm}{k}t - \frac{m}{k}\left(v_0 + \frac{gm}{k}\right) e^{-kt/m} + \frac{m}{k}\left(\frac{gm}{k} + v_0\right) + y_0\,,$$

which simplifies to

$$y(t) = y_0 - \frac{gm}{k}t + v_0\frac{m}{k}\left(1 - e^{-kt/m}\right) + \frac{gm^2}{k^2}\left(1 - e^{-kt/m}\right)\,,$$

where y_0 and v_0 are the position and velocity, respectively, of the particle at time $t = 0$.

Remark 4.3. In the special case $v_0 = 0$, the formula for $y(t)$ simplifies further to

$$y = y_0 + \frac{gm^2}{k^2}\left(1 - e^{-kt/m}\right)$$

This can be thought of as a modification of the formula $y = y_0 - \frac{1}{2}gt^2$, as cam be seen by substituting the approximation,

$$e^{-\frac{k}{m}t} \approx 1 - \frac{k}{m}t + \frac{k^2}{2m^2}t^2 - \frac{k^3}{6m^3}t^3$$

(which is valid for $\frac{k}{m}t$ small) into the formula for $y(t)$ to obtain the approximate formula

$$y \approx y_0 - \frac{1}{2}gt^2 + \frac{kg}{6m}t^3.$$

Example 4.6. *Suppose that a bag weighing 120 lb and having a coefficient of air resistance of 1 lb-sec/ft falls out of an airplane. How close to the terminal velocity will it be after 30 seconds? How many feet will the bag have fallen at that time?*

Solution. First compute mass

$$m = \frac{120}{32}\text{ slugs} = 3.75\text{ slugs}$$

The terminal velocity is, therefore, $v_\infty = -\frac{mg}{k} = -120\ \text{ft/sec}$.

Substitution of these numerical values into the formulas for $v(t)$ and $y(t)$ gives

$$v(t) = -120(1 - e^{-0.2667\,\mathrm{sec}^{-1} t})\frac{\mathrm{ft}}{\mathrm{sec}}$$
$$y(t) = -\left(120\frac{\mathrm{ft}}{\mathrm{sec}}\right) t + 450(1 - e^{-(0.2667\,\mathrm{sec}^{-1})t})\,\mathrm{ft}$$

The following table shows the difference between free fall with and without air resistance:

t (sec)	no air v (ft/sec)	air v (ft/sec)	no air y (ft)	air y (ft)
0	0.0	0.0	0.0	0.0
1	-32.0	-28.1	-16.0	-14.67
2	-64	-49.6	-144.0	-112.2
5	-160.0	-88.4	-400.0	-268.6
10	-320.0	-111.7	-1600.0	-781.3
20	-640.0	-119.4	-6400.0	-1952.2
30	-960.0	-119.96	-14,400.0	-3150.2

After 30 seconds, the velocity will be -119.96 feet/sec, almost indistinguishable from the terminal velocity; and the bag will have fallen 3150.2 feet.

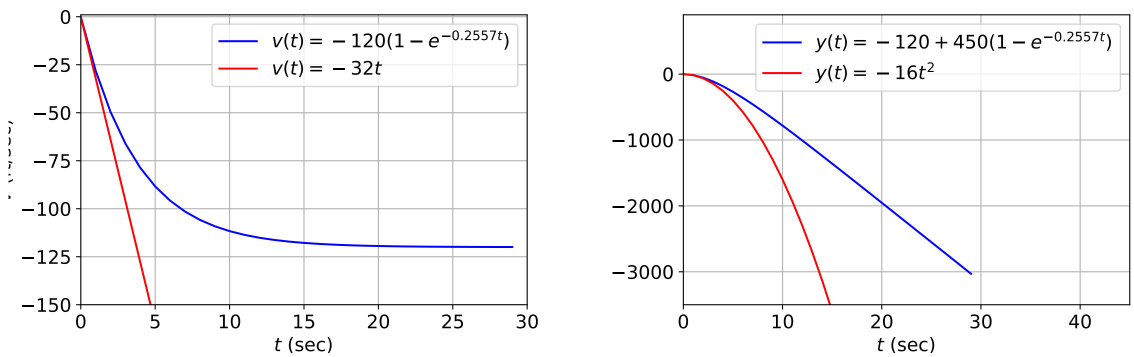


Figure 4.1. Free fall with air resistance taken into account. Notice that the velocity asymptotically approaches a constant $v(t) \approx -120$ ft/sec, and the height approaches a linear function $y(t) \approx -120t + 450$.

4.2. Stability Analysis of Autonomous First Order Differential Equations

A differential equation of the form

$$\frac{dy}{dt} = F(y) \quad (F \text{ is independent of } t) \tag{4.4}$$

is called an *autonomous* differential equation. As illustrated in Figure 4.2, the direction elements of autonomous differential equations have constant slope along horizontal lines. A point $y = y_e$ with $F(y_e) = 0$ is called an *equilibrium point* of the differential equation 4.4 (Such points are also called *critical points* or *fixed points*.) Notice that if y_e is a fixed point then the constant function $y(t) = y_e$ is a solution of the differential equation.

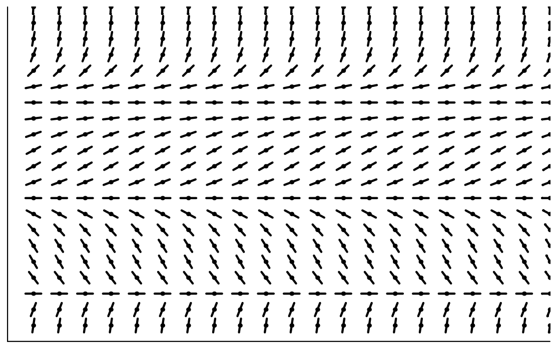


FIGURE 4.2. The direction field of an autonomous differential equation.

EXAMPLE 4.7. (a) The fixed points of the differential equation $y' = (1-y^2)$ are the points $y = 1$ and $y = -1$. (b) The fixed points of the differential equation $y' = \sin(\pi y)$ are the integers $y = 0, \pm 1, \pm 2, \ldots$.

A fixed point of an autonomous differential equation that acts like an attractor is called a *stable equilibrium point*. That is, "*y_e is a stable equilibrium point if the solution of the initial value problem approaches y_e whenever the initial condition is sufficiently close to y_e.*"

Criterion for stability: *If for some $\epsilon > 0$, the function $F(y)$ is negative for y in the interval $(y_e, y_e + \epsilon)$ and positive for all y in the interval $(y_e - \epsilon, y_e)$, then y_e is a stable equilibrium point.*

The equilibrium point y_e is called semi-stable if y_e is not a stable equilibrium point, but for some $\epsilon > 0$ the function $F(y)$ is negative for y in the interval $(y_e, y_e + \epsilon)$ or $F(y)$ is positive for all y in the interval $(y_e - \epsilon, y_e)$.

If the equilibrium point y_e is neight stable nor semi-stable, then it is said to be unstable.

EXAMPLE 4.8. (EPIDEMICS) Here is an example of how stability analysis can be used to study the spread of a disease in a population (under a number of simplifying assumptions). Assume that the following conditions are satisfied:

- The population is divided into two sub-populations:

  $x =$ the proportion susceptible to infection ("well")

  $y =$ the proportion infected ("sick")

  Note: Under these assumptions $x + y = 1$.
- The disease spreads through contact between sick individuals and well individuals, and the rate of the spread dy/dt is proportional to the number of such contacts per unit of time.
- Members of both groups move about freely among each other, so the number of contacts per unit time is proportional to the product of x and y.

QUESTION: *What proportion of the population will ultimately become infected if only a small proportion y_0 is initially infected?*

SOLUTION. It follows from the assumptions made that

$$\frac{dy}{dt} = \alpha x \cdot y \,,$$

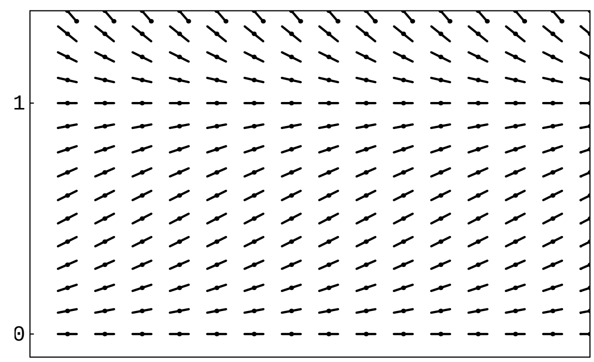


FIGURE 4.3. The direction field of the differential equation $y' = \alpha(1-y)y$.

where α is a positive constant related to the frequency of contact and the probability of infection upon contact. Because everyone is assumed to be either well or sick, it follows that $x + y = 1$ (100% of the population). Consequently, $x = 1 - y$ and $\dfrac{dy}{dt} = \alpha\,(1-y)y$.

If $y(0) = y_0$ is the fraction of the total population that is initially infected, the spread of the epidemic can be predicted by solving the initial value problem

$$\frac{dy}{dt} = \alpha\,(1-y)y\,, \quad y(0) = y_0\,.$$

The critical point $y = 1$ is stable because $\alpha(1-y)y$ is positive for $0 < y < 1$ and negative for $1 < y$:

Hence, $\lim_{t\to\infty} y(t) = 1$, provided $0 < y(0) = y_0 < 1$.

It follows that (under the assumptions of the model) everyone will eventually become infected, even if initially only a small proportion of the population is infected.

EXAMPLE 4.9. (THE LOGISTIC EQUATION FOR POPULATION GROWTH)

In the late 1920's Raymond Pearl analyzed data collected by to determine how well the logistic equation predicted the population growth of yeast.[9] In Carlson's experiments a small number of yeast cells were placed into a jar containing sugar, and the approximate number of yeast cells were counted each hour. Here is some of the data from the experiment:

$t =$	0	1	2	3	4	5	6	7	8	9
$Y(t) =$	10	18	19	47	71	119	175	257	351	441
$t =$	10	11	12	13	14	15	16	17	18	
$Y(t) =$	513	560	595	629	641	651	656	660	661	

Pearl conjectured that the number of yeast $Y(t)$ after t hours obeyed a differential equation of the form

$$\frac{1}{Y}\frac{dY}{dt} = R(Y)\,,$$

where $R(Y)$ is a function involving only the number of yeast. Notice that the left hand side is the "fractional rate of growth" (or the *logarithmic growth rate*). The right hand side is called the *reproduction function*. The object of Pearl's analysis was to determine $R(Y)$.

[9] "The growth of populations" , Raymond Pearl, Quarterly Review of Biology, 2 (1927) 532–548.

At any t, the derivative can be approximated by a "difference quotient" $\Delta Y/\Delta t$. A better estimate can be obtained by averaging successive quotients. For example, $Y'(5)$ can be estimated by

$$Y'(5) \approx \frac{1}{2}\left(\frac{Y(6)-Y(5)}{1}+\frac{Y(5)-Y(4)}{1}\right) = \frac{Y(6)-Y(4)}{2}\,.$$

The logarithmic growth rate is then

$$\frac{1}{Y(5)}\frac{dY(5)}{dt} \approx \frac{175-71}{2\cdot 119} = 0.437\,.$$

Applying this approach to the data in the table above, and fitting a straight line to the data (see figure below) gives the formula

$$R(Y) = 0.53 - 0.00079Y = 0.53\left(1-\frac{Y}{671}\right)$$

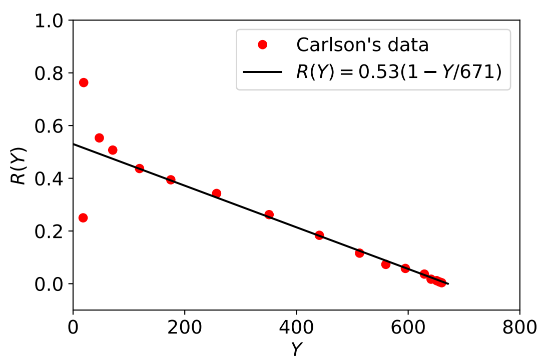


FIGURE 4.4. Carlson's data (horizontal axis: Y, vertical axis: approximation of $\frac{1}{Y}\frac{dY}{dt}$), together with best fitting line.

Given the initial population $Y(0) = 10$ yeast buds, the yeast population for $t > 0$, can be predicted by solving the initial value problem

$$\frac{dY}{dt} = 0.53\left(1-\frac{Y}{671}\right)Y\,,\ Y(0) = 10\,.$$

Figure 4.5 illustrates excellent agreement between Carslon's data and the solution of the initial value problem.

A number of conclusions can be drawn from this model:

(i) For Y is small, $R(Y) \approx r = 0.53$ (*intrinsic rate of growth*) $Y(t) \approx Y_0 e^{rt}$ for Y small.

(ii) For $Y > K \approx 671$, $R(Y)$ is negative and the population decreases with time. The parameter $K = 671$ is called the *carrying capacity* of the environment.

(iii) For any initial value of Y, $\lim_{t\to\infty} Y(t) = K \approx 671$.

The above differential equation is a special case of the *logistic equation*:

$$\frac{dP}{dt} = r\left(1-\frac{P}{K}\right)P\,.$$

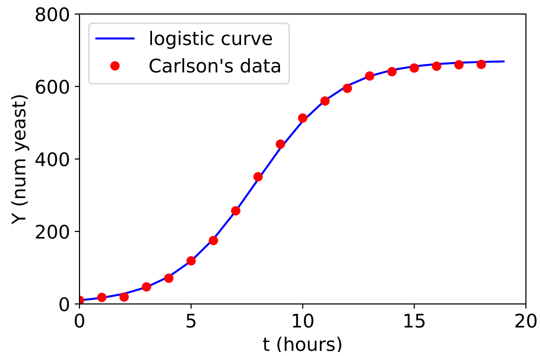


FIGURE 4.5. Logistic curve fitted to the growth of the yeast. The data is from a 1927 experiment by G.F. Gause.

where P is the population of some organism. The book by G. Evelyn Hutchinson, *An Introduction to Population Ecology*, New Haven and London, Yale University Press, 1978, pages 23–32, presents data suggesting that many animal and plant populations obey the logistic model of growth.

Because the logistic equation. is separable, it can be solved by separation of variables:

$$\int \frac{dP}{(1-P/K)P} = \int r dt\,.$$

The left hand side can be integrated by partial fractions:

$$\frac{1}{(1-P/K)P} = \left(\frac{1}{K-P} + \frac{1}{P}\right) \implies \int \frac{dP}{P(1-P/K)} = \int \frac{dP}{K-P} + \int \frac{dP}{P}$$

So

$$\int \frac{dP}{P(1-P/K)} = \int \frac{dP}{K-P} + \int \frac{dP}{P} = -\ln|K-P| + \ln|P| + C = \ln\left|\frac{P}{K-P}\right| + C$$

Thus $\ln\left|\frac{P}{K-P}\right| = rt + C \implies \left|\frac{P}{K-P}\right| = e^{rt+C} \implies \frac{P}{K-P} = Ae^{rt}$. where $A = \pm e^C$. Solving for P yields ans explicit formula for $P(t)$:

$$P = Ae^{rt}(K-P) \implies P(t) = \frac{AKe^{rt}}{1+Ae^{rt}} = \frac{K}{1+e^{-kt}/A}\,.$$

This function is called the *logistic function*. Its graph is the “S”-shaped curve sketched in Figure 4.6.

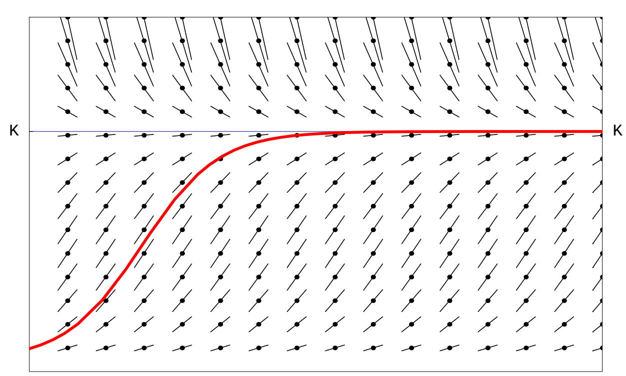


FIGURE 4.6. The logistic curve.

Exercises 2.

(1) Solve each of the following first order differential equations and initial value problems.
   (a) $y' = (1-x)(2-x)$, $y(0) = 0$
   (b) $y' = ay(b-y)$, $y(0) = y_0$, where $a > 0$ and $b > 0$ are constants.
   (c) $y' = 1 - y^2$, $y(0) = 0$
   (d) $y' = \cos(x)(y^2+1)$
   (e) $\dfrac{dy}{dx} = \sqrt{L - y^2}$, for $L > 0$ and $-L < y < L$.
   (f) $y' = 1 - y^2$, $y(0) = 0$
(2) Solve each of the following first order differential equations and initial value problems.
   (a) $y' + 2y = e^{-3x}$
   (b) $(1+t^2)\dfrac{dy}{dt} + 2ty = t(1+t^2)\,, \quad y(1) = 1$
   (c) $y' - 2ty = e^{t^2}$
   (d) $y' + 2y = \cos(3t)$.
   (e) $y' + ky = ae^{bt}$, $a \neq 0$, $b \neq -k$.
   (f) $y' + ky = ae^{-kt}$, $a \neq 0$
(3) Suppose that the population of a certain species grows at the instantaneous rate of 2% per year (i.e., its instantaneous rate of increase in number of individuals per year is 2% of the population at the moment). Let $y(t)$ stand for the population after t years.
   (a) Write a differential equation for $y(t)$.
   (b) Solve the differential equation by separation of variables.
   (c) If the present population is 1,000,000, what will the population be in 1 year? In 20 years? How long will the population take to double?
(4) Assume that the population of the Earth changes at an instantaneous rate proportional to the population. Assume further that at time $t = 0$ (A.D.1650) its population was 250 million and at time $t = 300$ (A.D. 1950) its population was 2.5 billion. Find an expression giving the population of the Earth at any time. If the greatest population that the Earth can support is 25 billion when will this limit be reached?
(5) Suppose that interest on money in the bank accumulates at an annual rate of 7% compounded continuously. Let $y(t)$ stand for the amount of money in the bank after t years.
   (a) Write a differential equation for $y(t)$.
   (b) Solve the differential equation by separation of variables to show that $y(t) = P\exp(0.7t)$, where $P = y(0)$ is the initial deposit in the bank.
   (c) How much money should be invested today so that 20 years from now it will be worth \$20,000?
(6) At time $t = 0$ you buy a house, using a fixed-rate, fixed payment mortgage to pay for most of it. Let $y(t)$ be the amount you owe after t years. Thus, $y(0) = y_0$ is the cost of the house minus the down-payment.
   (a) Write the differential equation for the amount $y(t)$ you owe after t years.
   (b) Solve the differential equation to find a formula for $y(t)$
   (c) Suppose that the bank will give you a 30-year mortgage at 12% interest, but they only consider you an acceptable credit risk if your monthly payments do not exceed one fourth of your \$1,600 monthly salary. Compute the maximum mortgage they'll give you.
   (d) Suppose you can come up with \$15,000 for a down-payment. What is the most expensive house you can buy? What will be the actual amount you'll end up paying for that house by the time the mortgage is payed off?
   (e) Now suppose you find a house for \$50,000 and make a \$15,000 down-payment, using a mortgage for the rest. What will your monthly payments be for the 30-year mortgage at 12% interest?

(f) What will your monthly payments be for the house in part (e) if the interest rate is 15% instead of 12% ? 10% instead of 12% ?
(g) How much will your monthly payment in part (e) increase if you get a 25-year mortgage instead of the 30-year mortgage?
(h) Suppose you get the mortgage in part (e). How much of the amount you pay during the first year is tax-deductible (interest payment), and how much is principal payment? In the fifth year? In the twenty-fifth year? How much total interest do you end up paying?
(i) Sketch a graph showing the proportion of your payments for the mortgage in part (e) that goes toward interest and the proportion that goes toward principal during the 30 years.

(7) Suppose you make a pudding for a dinner party and put it in the refrigerator at 6 p.m. ($t = 0$). Your refrigerator maintains a constant temperature of 40°F. The pudding will gel when it cools to 45°. If the pudding's temperature when you put it in the refrigerator is 110°F and when your first guest arrives at 7 p.m. you measure the temperature and get a reading of 50°F. Use the information about the two temperature readings to determine the earliest time you can serve dessert/ I.e. when will the temperature reach 45°F?

(8) Newton's law of cooling also applies when a colder object heats up in a warmer environment. Suppose water at 55°F is pumped into a swimming pool on a 90° summer day. After 2 hours the temperature of the water is 60°. In how many hours (assuming the outside temperature remains 90°) will the water reach a comfortable swimming temperature of 70°?

(9) Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 gal of a dye solution with a concentration of 1 gm/gal. To prepare for the next experiment, the tank is to be rinsed with <u>fresh</u> water flowing in at a rate of 2 gal/min, the well-stirred solution flowing out at the same rate.
(a) Follow the procedure of the previous problem to write a differential equation for y, the amount of dye in the tank after t minutes. Your differential equation should be a differential equation for exponential decay. Because the water flowing in is fresh, there is no constant term in the differential equation, and as a result, the amount of dye decreases exponentially.
(b) Find $y(t)$ as a function of t.
(c) What is the "half-life of $y(t)$?
(d) How much time elapses before the concentration of dye in the tank reaches 1

(10) Consider a tank with a 50 gallon capacity that initially contains 50 gallons of brine (salt water) with an initial salt concentration of 0.10 pounds per gallon. The tank is equipped with stirrers that keep the mixture will stirred at all times. Starting at time $t = 0$ minutes a brine solution with with a salt concentration of 0.50 pounds per gallon flows into the tank at a rate of 4 gallons per minute and the well-stirred mixture flows out at the same rate. Let $Q = Q(t)$ be the amount of salt in the tank at time t (measured in minutes).
(a) Find a differential equation satisfied by Q.
(b) Carefully sketch its field of line elements.
(c) Find the formula for $Q(t)$ and sketch it (on the sketch of the field of line elements.

(11) Suppose that a room containing 1200 cubic feet of air is originally free of carbon monoxide. Beginning at time $t = 0$ cigarette smoke containing 4 percent carbon monoxide is introduced into the room at a rate of $0.1\text{ft}^3/\text{min}$ and the well-circulated mixture is allowed to leave the room at the same rate.
(a) Find an expression for the concentration of carbon monoxide in the room at any time $t > 0$.
(b) Extended exposure to a carbon monoxide concentration as low as 0.00012 (i.e. 0.012%) is harmful to the human body. Find the time at which this concentration is reached.

(12) Unlike the case of an object moving at low speed, at high speed, the drag on an object traveling in the atmosphere is proportional to the *square* of the speed. Suppose that such a projectile is initially falling at 5 times the speed of sound (mach 5) and that 30 seconds later it is traveling at only mach 2.

(a) Write down the appropriate initial value problem.
(b) Solve it.
(c) How fast is the projectile traveling after one minute?
(d) How long will it take for before it is traveling at exactly the speed of sound?
(e) How far will it travel in the time it takes to slow down to the speed of sound?

(13) Consider the logistic population model

$$\frac{dP}{dt} = 2\left(1 - \frac{P}{1000}\right)P$$

for a species of fish in a lake, were t is measured in years and $P(t)$ is the number of fish in the lake at time t. Suppose that it is decided that fishing will be allowed in the lake, but it is unclear how many fishing licenses should be issued. Suppose the average catch of a fisherman with a license is 5 fish per year.

(a) What is the largest number of licenses that can be issued if the fish are to have a chance to survive in the lake?
(b) Suppose the number of fishing licenses in part (a) are issued. What will happen to the fish population—that is, how does the behavior of the population depend on the initial population?
(c) The simple population model above can be thought of as a model of an ideal fish population that is not subject to many of the environmental problems of an actual lake. For the actual fish population, there will be occasional changes in the population that were not considered in the building of the model. If the water level were high because of a heavy rainstorm, a few extra fish might be able to swim down a usually dry stream bed to reach the lake; or the extra water might wash toxic waste into the lake, killing a few fish. Given the possibility of unexpected perturbation of the population, not included in the model, what do you think will happen to the actual fish population if we fix the fishing level at the one determined in part (b)?

(14) A boater and a motor boat together weigh 640 lbs. Suppose that the thrust of the motor is equal to a constant force of 20 lb. in the direction of motion, and that the resistance of the water to the motion is equal numerically to twice the speed in feet per second and that the boat is initially at rest. Denote the speed of the boat at time t by $v = v(t)$.

(a) Write down an initial value problem for v.
(b) Find the formula for $v(t)$.
(c) What is the limiting velocity?

(15) The velocity $v(t)$ of a falling body meeting air resistance proportional to its velocity satisfies the differential equation

$$m\,\frac{dv}{dt} = -kv - mg\,,$$

where $g = 32\text{ft/sec}^2$ is the magnitude of gravitational acceleration and k is a constant that depends on the shape of the object and m is the mass of the object. When a 120 lb (= mg) parachutist drops from a plane (with zero initial velocity), she first falls with a small coefficient of air resistance $k = 1.2$lb-sec/ft. Five seconds later her parachute opens and k jumps to 12.0lb-sec/ft.

(a) Write a differential equation for $v(t)$ for during the first five seconds.
(b) Solve the differential equation to find am expression for $v(t)$ during the first five seconds.
(c) Graph $v(t)$ for the first 5 seconds and show by a dotted line what $v(t)$ would be over the next ten seconds if the parachute didn't open.
(d) Write the differential equation for $v(t)$ after the parachute opens.
(e) Solve the differential equation for $v(t)$ after the parachute opens. Use the value of $v(5)$ from part (b) as the initial data.
(f) Extend the graph of $v(t)$ you constructed in part (c) for the five seconds <u>after</u> the parachute opens.

(g) (g) Again on the same sheet of graph paper draw two horizontal lines indicating the terminal velocity of the parachutist in the case where the parachute opens and in the case where is does not open.

(16) The velocity $v(t)$ of a falling body meeting air resistance proportional to its velocity satisfies the differential equation

$$m\frac{dv}{dt} = -kv + mg\,,$$

where $g = 32\text{ft/sec}^2$ is the magnitude of gravitational acceleration, k is a constant that depends on the object (called the **coefficient of air resistance**) and m is the mass of the object. When a 120 lb ($= mg$) parachutist drops from a plane (with zero initial velocity), she first falls with a small coefficient of air resistance $k = 1.2$lb-sec/ft. Five seconds later her parachute opens and k jumps to 12.0lb-sec/ft.

Note: The convention we are using here is that v is positive when the body is descending and negative when it is ascending.

(a) Write a differential equation for $v(t)$ for during the first five seconds.
(b) Solve the differential equation to find am expression for $v(t)$ during the first five seconds.
(c) Graph $v(t)$ for the first 5 seconds and show by a dotted line what $v(t)$ would be over the next ten seconds if the parachute didn't open.
(d) Write the differential equation for $v(t)$ after the parachute opens.
(e) Solve the differential equation for $v(t)$ after the parachute opens. Use the value of $v(5)$ from part (b) as the initial data.
(f) Graph (**on the same sheet of graph paper**) $v(t)$ for the five seconds after the parachute opens.
(g) Again on the same sheet of graph paper draw two horizontal lines indicating the terminal velocity of the parachutist in the case where the parachute opens and in the case where is does not open.

(17) Consider the differential equation $y' = F(y)$ where the graph of $F(y)$ is shown in Figure 4.7 *Locate the equilibrium points in Figure 4.7 and determine which ones are stable and which ones are not stable. It may help to sketch a few solution curves.*

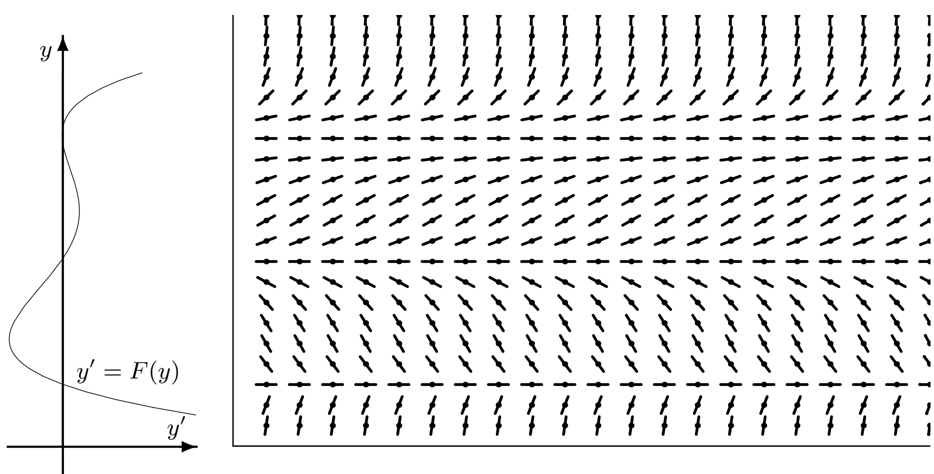
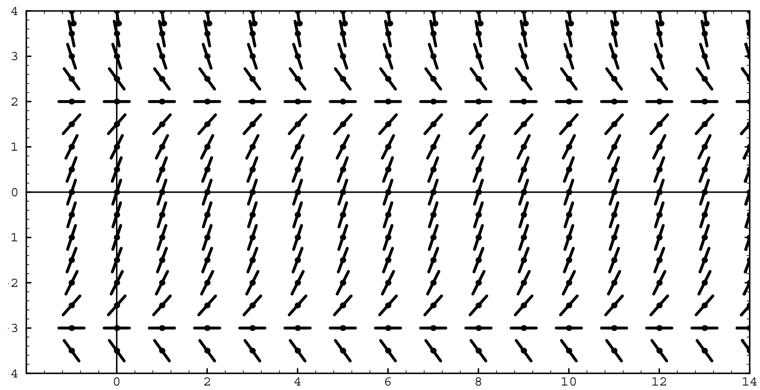


FIGURE 4.7. Flipping the graph of $F(y)$ along the diagonal and aligning it with the direction field illustrates the relation between the graph of $F(y)$ and the direction field of the differential equation.

(18) The direction field for a differential equation of the form $y' = F(y)$ is shown below. Sketch the solution of the initial value problem

$$y' = F(y)\,, \quad y(2) = 1.5\,.$$



If $y = y(t)$ is the solution, give your best estimate of $y(10)$. Approximately, what is the value of $y(100)$? Suppose $y(0) = y_0$, for $0.1 < y_0 < 2$, what is your best estimate of $y(100)$?

Part 2

Second Order Differential Equations

CHAPTER 5

Complex Numbers

5.1. Complex Numbers

A *complex number* z is given by a pair of real numbers x and y and is written in the form[10] $z = x+yi$, where i satisfies $i^2 = -1$. [11] If $z = x + yi$, then the term x is called the *real part* of z and written $x = \operatorname{Re} z$. The term y is called the *imaginary part* of z and written $y = \operatorname{Im} z$. Thus,

$$\operatorname{Re}(4+5i) = 4 \text{ and } \operatorname{Im}(4+5i) = 5\,.$$

Remember: $\operatorname{Im} z$ is a *real* number!

Complex numbers are added in a natural way: If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, then

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i \tag{5.1}$$

For example, $(4 + i) + (2 + 3i) = (6 + 4i)$. Complex numbers are also multiplied in a natural way:

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i \tag{5.2}$$

Note that the product behaves exactly like the product of any two algebraic expressions, keeping in mind that $i^2 = -1$. Thus,

$$(2+i)(-2+4i) = 2(-2) + 8i - 2i + 4i^2 = -8 + 6i\,.$$

There is only one way to satisfy the equality $z_1 = z_2$, namely, if $x_1 = x_2$ <u>and</u> $y_1 = y_2$. An equivalent statement (one that is important to keep in mind) is that $z = 0$ if and only if $\operatorname{Re} z = 0$ and $\operatorname{Im} z = 0$. If a is a real number and $z = x + iy$ is complex, then $az = ax + iay$ (which is exactly what one would get from the multiplication rule above if z_2 were of the form $z_2 = a + i0$).

Division is more complicated. To find z_1/z_2 it suffices to find $1/z_2$ and then multiply by z_1. The rule for finding the reciprocal of $z = x + yi$ is given by:

$$\frac{1}{x+yi} = \frac{1}{x+yi} \cdot \frac{x-yi}{x-yi} = \frac{x-yi}{(x+yi)(x-yi)} = \frac{x-yi}{x^2+y^2} \tag{5.3}$$

For instance,

$$\frac{1}{3+4i} = \frac{3-4i}{25} = \frac{3}{25} - \frac{4}{25}i\,.$$

Notice that using the formula for the product of complex numbers gives

$$(3+4i)\left(\frac{3}{25} - \frac{4}{25}i\right) = \frac{9+16}{25} + \frac{(3)(-4)+4(3)}{25} = 1 + 0i = 1\,,$$

[10]At times, it is more convenient to write $z = x + iy$, rather than $z = x + yi$. Both forms are used in these notes.

[11]Electrical engineers (who make heavy use of complex numbers) reserve the letter i to denote electric current and they use j for $\sqrt{-1}$.

as one would expect!

The expression $x - iy$ appears so often and is so useful that it is given a name. It is called the *complex conjugate* of $z = x + iy$ and a shorthand notation for it is $\overline{z}$; that is, if $z = x + iy$, then $\overline{z} = x - iy$. For example, $\overline{3+4i} = 3 - 4i$, as illustrated in Figure 5.1(a). Note that $\overline{\overline{z}} = z$ and $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$. Exercise (3b) is to show that $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$.

Another important quantity associated with a given complex number z is its *modulus*

$$|z| = (z\overline{z})^{1/2} = \sqrt{x^2 + y^2} = \left((\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2\right)^{1/2}$$

Note that $|z|$ is a *real* number. For example, $|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

The modulus of a complex number is a generalization of the notion of the absolute value of a real number, as the following example illustrates:

$$|-3| = |(-3) + 0i| = ((-3)^2 + (0)^2)^{1/2} = (9)^{1/2} = 3\,.$$

5.2. The Complex Plane

The complex numbers, as well as various operations involving compeles numbers have elegant geometric descriptions. The complex numbers may be represented as points in the plane (sometimes called the Argand diagram) or as vectors. The real number 1 is represented by the point $(1, 0)$, and the complex number i is represented by the point $(0, 1)$. The x-axis is called the "real axis", and the y-axis is called the "imaginary axis".

Complex conjugation is given by reflection about the real axis, as illustrated in Figure 5.1(a). Addition of complex numbers is given by the *parallelogram rule*, as illustrated in Figure 5.1(b).

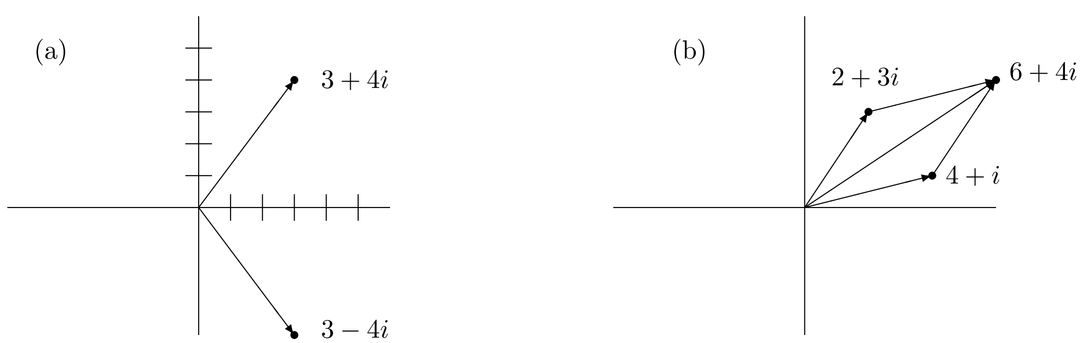


Figure 5.1. (a) The complex numbers $3 - 4i$ and $3 + 4i$ are complex conjugates of one another. (b) The complex number $6 + 4i$ is the sum of $2 + 3i$ and $4 + i$.

The geometric description of multiplication involves both a rotation and a stretch. As illustrated in Figure 5.2, to visualize the product $z_1 z_2$, construct a triangle with vertices 0, 1 and z_1 (red triangle at left of figure). Then construct a similar triangle where the "base" edge from 0 to 1 is replaced by the segment from 0 to z_2 (red triangle at right of figure). Then the vertex opposite the base is the product $z_1 z_2$. By high school geometry, one can show that the coordinates of the product are given by Equation (5.2).

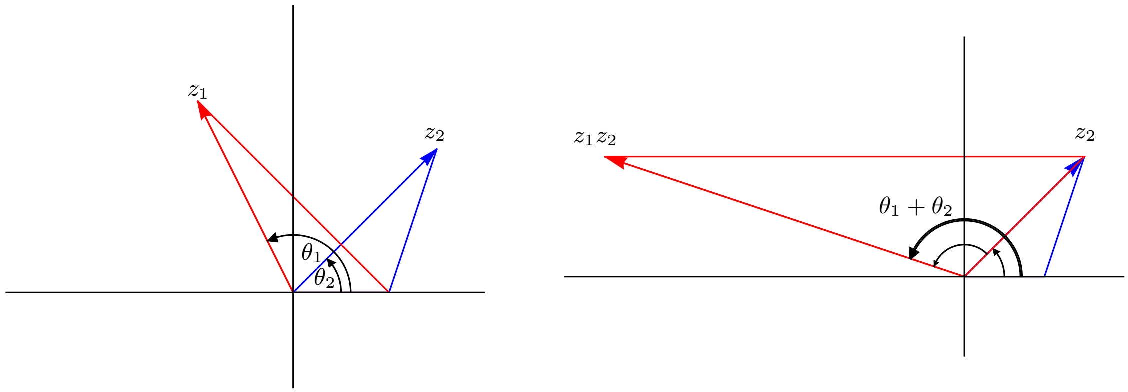


FIGURE 5.2. Geometric description of complex multiplication. The red triangles in the figure are similar, with "bases" of lengths 1 and $|z_2|$, respectively. By high school geometry, one can show that $|z_1 z_2| = |z_1||z_2|$. The angle that the product $z_1 z_2$ makes with the positive real axis is the sum of the angles that z_1 and z_2 make with the positive real axis.

EXERCISE 5.1.

(1) Prove that the product of $z = x + iy$ and the expression in (5.3) (above) equals 1.
(2) Verify each of the following:
   (a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$ (b) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i} = -\dfrac{2}{5}$
   (c) $\dfrac{5}{(1-i)(2-i)(3-i)} = \dfrac{1}{2}i$ (d) $(1-i)^4 = -4$
(3) Prove the following:
   (a) $z + \overline{z} = 2\mathrm{Re}\, z$ and z is a real number if and only if $\overline{z} = z$.
   (b) $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$.
(4) Prove that $|z_1 z_2| = |z_1||z_2|$ (Hint: Use (3b).)
(5) Find all complex numbers $z = x + iy$ such that $z^2 = 1 + i$.

5.3. Polar Representation of Complex Numbers

Recall that the plane has polar coordinates as well as rectangular coordinates. The relation between the rectangular coordinates (x, y) and the polar coordinates (r, θ) is

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta)$$
$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \tan(\theta) = \frac{y}{x}$$

Thus, (See Figure 5.3) the complex number $z = x + iy$ can we written in the form:

$$\boxed{z = r\left(\cos(\theta) + i\sin(\theta)\right) = re^{i\theta}\,,} \quad (\textit{Polar Representation}), \tag{5.4}$$

where

$$\boxed{r = \sqrt{x^2 + y^2} = |z| \text{ and } \tan(\theta) = \frac{y}{x}\,.} \tag{5.5}$$

The angle θ is called the *argument* of the complex number z. It is often denoted by $\arg(z)$.

EXAMPLE 5.1. The complex number $z = 8+6i$ may also be written as $re^{i\theta}$, where $r = \sqrt{8^2+6^2} = 10$ and $\theta = \arg(8+6i) = \arctan(6/8) \approx 0.64$ radians.

REMARK 5.1. In formula (5.4), we are defining $e^{i\theta}$ to be $\cos(\theta) + i\sin(\theta)$. We justify this definition later in these notes.

REMARK 5.2. (CAUTION) There is ambiguity in equation (5.5) about the inverse tangent, which can (and must) be resolved by looking at the signs of x and y, respectively, in order to determine the quadrant in which θ lies. If $x > 0$, then $\theta = \arctan(y/x)$. If $x < 0$, then $\theta = \arctan(y/x) \mp \pi$, depending on the sign of y. When $x = 0$, then $\theta = \pm\pi/2$, depending on the sign of y. (If $z = 0$, then $r = 0$ and θ can be anything.)

If $x = 0$, then the formula for θ makes no sense, but $x = 0$ simply means that z lies on the imaginary axis and so θ must be $\pi/2$ or $3\pi/2$ (depending on whether y is positive or negative).

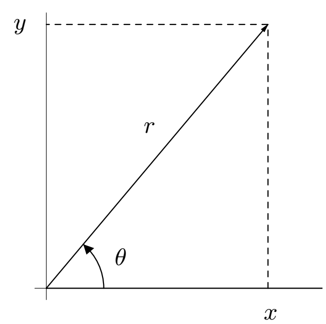


FIGURE 5.3. The Polar Representation: $x + yi = re^{i\theta}$, where $r = \sqrt{x^2+y^2}$ and $\tan(\theta) = y/x$.

REMARK 5.3. The conditions for equality of two complex numbers using polar coordinates are not quite as simple as they were for rectangular coordinates. If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then $z_1 = z_2$ if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi k,\ k = 0, \pm1, \pm2, \ldots$.

EXAMPLE 5.2. For instance, $i = e^{i\pi/2} = e^{i5\pi/2}$, $\quad -1 = e^{\pi i} = e^{3\pi i}$, and $+1 = e^{0i} = e^{(0+2\pi)i} = e^{2\pi i}$.

EXAMPLE 5.3. If $z = -4+4i$, then $r = \sqrt{4^2+4^2} = 4\sqrt{2}$ and $\theta = 3\pi/4$, therefore $z = 4\sqrt{2}e^{3\pi i/4}$. Any angle that differs from $3\pi/4$ by an integer multiple of 2π will give us the same complex number. Thus, $-4+4i$ can also be written as $4\sqrt{2}e^{11\pi i/4}$ or as $4\sqrt{2}e^{-5\pi i/4}$.

Recall (see Figure 5.2), that complex multiplication involves both a stretch and a rotation. The polar representation gives another particularly useful description of complex multiplication:

$$\text{if } z_1 = r_1e^{i\theta_1} \quad \text{and} \quad z_2 = r_2e^{i\theta_2}, \quad \text{then} \quad z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}\,. \tag{5.6}$$

For example, let

$$z_1 = 2+i = \sqrt{5}e^{i\theta_1}, \qquad \theta_1 \approx 0.464$$
$$z_2 = -2+4i = \sqrt{20}e^{i\theta_2}, \qquad \theta_2 \approx 2.034$$

Then $z_3 = z_1z_2$, where $z_3 = -8+6i = \sqrt{100}e^{i\theta_3}$ $\quad \theta_3 \approx 2.498$. (see Figure 5.4)

EXERCISE 5.2.

(1) Let $z_1 = 3i$ and $z_2 = 2-2i$
    (a) Plot the points z_1+z_2, z_1-z_2 and $\overline{z_2}$.

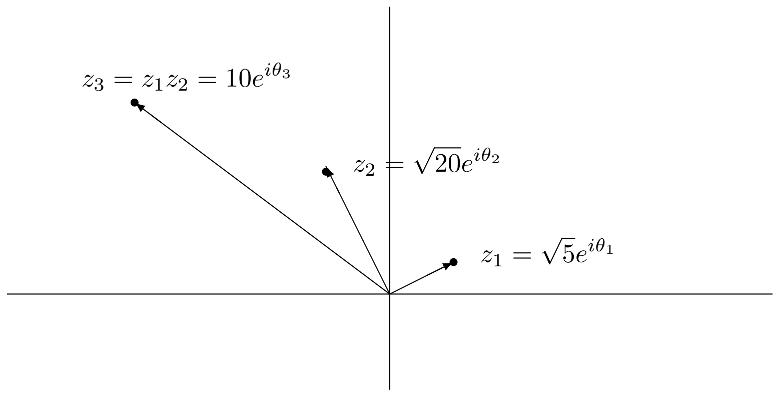


FIGURE 5.4. Complex multiplication in polar form. Notice that $|z_3| = |z_1||z_2|$ and $\theta_3 = \theta_1 + \theta_2$.

(b) Compute $|z_1 + z_2|$ and $|z_1 - z_2|$.
(c) Express z_1 and z_2 in polar form.

(2) Let $z_1 = 6e^{i\pi/3}$ and $z_2 = 2e^{-i\pi/6}$. Plot z_1, z_2, z_1z_2 and z_1/z_2.
(3) Let $z = re^{i\theta}$. Show that $\overline{z} = re^{-i\theta}$ and $z^{-1} = r^{-1}e^{-i\theta}$.

5.4. Complex-valued Functions

Now suppose that $w = w(t)$ is a complex-valued function of the real variable t. That is

$$w(t) = u(t) + iv(t)\,,$$

where $u(t)$ and $v(t)$ are real-valued functions. A complex-valued function can be thought of a defining a curve in the complex plane. The derivative of $w(t)$ with respect to t is *defined* to be the function

$$w'(t) = u'(t) + iv'(t) = \frac{dw(t)}{dt}$$

(This is just like the definition of the derivative of a vector-valued function—just differentiate the components.) The derivative $w'(t)$ can be thought of as the tangent to that curve $w(t)$.

It is easily checked (just expand the left and right hand sides of each identity) that, just as in the case of real-valued functions, the following formulas hold for complex-valued functions $z = z(t)$ and $w = w(t)$:

$$C' = 0\,, \text{ where } C = \text{constant} \tag{5.7a}$$
$$(Cz)' = Cz'\,, \text{ where } C = \text{constant} \tag{5.7b}$$
$$(z+w)' = z' + w' \qquad \text{(the sum rule)} \tag{5.7c}$$
$$(zw)' = z'w + zw' \qquad \text{(the product rule)} \tag{5.7d}$$
$$\left(\frac{z}{w}\right)' = \frac{z'w - zw'}{w^2} \qquad \text{(the quotient rule)} \tag{5.7e}$$

In other words, the derivatives of complex-valued functions behave the same as the derivatives of real valued functions.

EXAMPLE 5.4. The complex-valued function

$$f(t) = \cos(t) + i\sin(t)$$

is of particular interest. When viewed as a curve in the complex plane, it defines a circle. It has two important properties:

$$\text{(i)} f(t)f(s) = f(t+s)$$
$$\text{(ii)} f'(t) = if(t)\,.$$

To verify (i), compute as follows using the sum of angle formulas from trigonometry:

$$\begin{aligned} f(t)f(s) &= (\cos(t) + i\sin(t))(\cos(s) + i\sin(s)) \\ &= (\cos(t)\cos(s) - \sin(t)\sin(s)) + (\cos(t)\sin(s) + \sin(t)\cos(s))i \\ &= \cos(t+s) + \sin(t+s)i = f(t+s)\,. \end{aligned}$$

To verify (ii), compute as follows from the definition of the derivative:

$$z'(t) = -\sin(t) + i\cos(t) = i(\cos(t) + i\sin(t)) = iz(t)\,.$$

Because (i) and (ii) are also satisfied by the exponential function: e^{rt}:

$$e^{rt}e^{st} = e^{(r+s)t} \text{ and } (e^{rt})' = re^{rt}\,,$$

the same notation is used here:

$$\boxed{e^{it} = \cos(t) + i\sin(t)}\,. \tag{5.8}$$

This is called *Euler's Formula.*. With this notation (i) and (ii) assume the forms

$$\boxed{e^{it}e^{is} = e^{i(t+s)} \text{ and } \left(e^{it}\right)' = i\,e^{it}\,.} \tag{5.9}$$

5.5. The complex exponential function

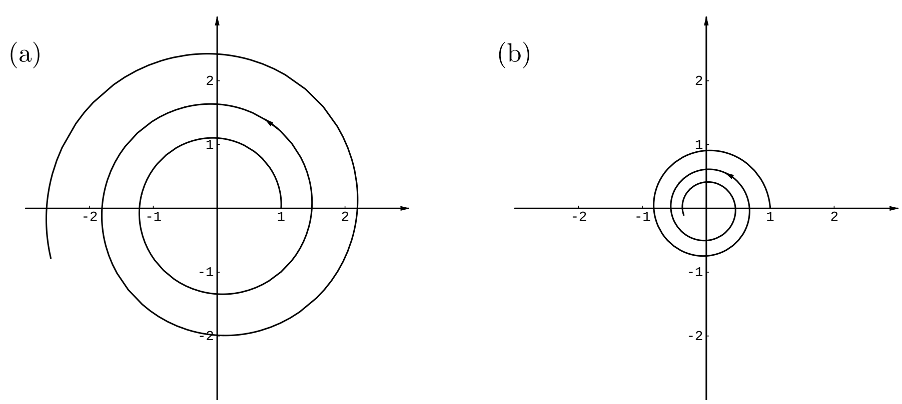


FIGURE 5.5. (a) $z(t) = e^{(\rho+i\omega)t}$, $\rho > 0$, $\omega > 0$. (b) $z(t) = e^{(\rho+i\omega)t}$, $\rho < 0$, $\omega > 0$.

One function is of particular interest to us: the *complex exponential function*. It is defined as follows:

$$\boxed{e^{(\rho+i\omega)t} = e^{\rho t}e^{i\omega t} = e^{\rho t}(\cos(\omega t) + i\sin(\omega t)) = e^{\rho t}\cos(\omega t) + ie^{\rho t}\sin(\omega t)\,.} \tag{5.10}$$

Thought of as a curve in the complex plane, the complex exponential is the formula for a spiral curve (Figure 5.5). The quantity ω is the angular velocity of the spiral ($\omega > 0$ corresponds to a counterclockwise spiral, $\omega < 0$ to a clockwise one). The quantity ρ measures the rate at which the spiral expands outward ($\rho > 0$) or contracts inward ($\rho < 0$).

As the following examples illustrate, functions of the form

$$f(t) = C_1 e^{\rho t}\cos(\omega t) + C_2 e^{\rho t}\sin(\omega t)$$

can be rewritten in terms of the complex exponential function.

EXAMPLE 5.5. Show that $5e^{-4t}\cos(3t) + 3e^{-4t}\sin(3t) = \text{Re}\left((5-3i)e^{(-4+3i)t}\right)$.

SOLUTION. By definition,

$$\begin{aligned}(5-3i)e^{(-4+3i)t} &= (5-3i)e^{-4t}(\cos(3t) + i\sin(3t)) \\ &= e^{-4t}\left((5\cos(3t) + 3\sin(3t)) + i(5\sin(3t) + 3\cos(3t))\right) .\end{aligned}$$

Hence, $\text{Re}\left((5-3i)e^{(-4+3i)t}\right) = e^{-4t}\left(5\cos(3t) + 3\sin(3t)\right)$.

REMARK 5.4. In polar form $5+3i = \sqrt{34}\exp(\arctan(3/5)i)$. Hence, we can compute as follows:

$$\begin{aligned}\text{Re}\left((5-3i)e^{(-4+3i)t}\right) &= \text{Re}\left(\overline{(5+3i)}e^{(-4+3i)t}\right) \\ &= \sqrt{34}e^{-4t}\text{Re}\left(\sqrt{34}e^{-\arctan(3/5)i}e^{3i)t}\right) \\ &= \sqrt{34}e^{-4t}\text{Re}\left(\exp((3t-\arctan(3/5))i)\right) \\ &= \sqrt{34}\,e^{-4t}\cos\left(3t - \arctan(3/5)\right) .\end{aligned}$$

EXAMPLE 5.6. Express $\text{Re}\left(\dfrac{1}{3+3i}e^{(6+4i)t}\right)$ in the form $Ae^{\rho t}\cos(\omega t - \phi)$.

SOLUTION. Since $3+3i = 3\sqrt{2}e^{(\pi/4)i}$, it follows that

$$\text{Re}\left(\frac{1}{3+3i}e^{(6+4i)t}\right) = \text{Re}\left(\frac{1}{3\sqrt{2}e^{(\pi/4)i}}e^{6t}e^{4it}\right)\ \frac{e^{6t}}{3\sqrt{2}}\text{Re}\left(e^{(4t-\pi/4)i}\right) = \frac{1}{3\sqrt{2}}e^{6t}\cos\left(4t - \pi/4\right)$$

To find the derivative of the complex exponential function, compute the derivatives of the real and imaginary parts and collecting terms to obtain the formula

$$\left(e^{(\rho+i\omega)t}\right)' = (\rho+i\omega)e^{(\rho+i\omega)t} .$$

In other words, *even for* $r = \rho + i\omega$, *the formula*

$$\boxed{\frac{d}{dt}e^{rt} = re^{rt}} \tag{5.11}$$

holds!

More generally, if $z(t) = x(t) + iy(t) = Ce^{(\rho+i\omega)t}$, where $C = C_1 + iC_2$, then clearly

$$z'(t) = C\cdot(\rho+i\omega)e^{(\rho+i\omega)t} \text{ and } z''(t) = C\cdot(\rho+i\omega)^2e^{(\rho+i\omega)t} ..$$

On the other hand, from the definition of the derivative

$$z'(t) = x'(t) + iy'(t) ,$$

gives a simple way to compute derivatives of

$$x(t) = \text{Re}\,(z(t)) = (C_1\cos(\omega t) - C_2\sin(\omega t))e^{\rho t} \tag{5.12a}$$

and

$$y(t) = \text{Im}\,(z(t)) = (C_1\sin(\omega t) + C_2\cos(\omega t))e^{\rho t} \tag{5.12b}$$

the real and imaginary parts of $z(t)$:

$$\boxed{x'(t) = \mathrm{Re}\left(C \cdot (\rho + i\omega)e^{(\rho+i\omega)t}\right) \text{ and } y'(t) = \mathrm{Im}\left(C \cdot (\rho + i\omega)e^{(\rho+i\omega)t}\right).} \tag{5.12c}$$

$$\boxed{x''(t) = \mathrm{Re}\left(C \cdot (\rho + \omega i)^2 e^{(\rho+i\omega)t}\right) \text{ and } y''(t) = \mathrm{Im}\left(C \cdot (\rho + \omega i)^2 e^{(\rho+i\omega)t}\right).} \tag{5.12d}$$

Example 5.7. *Consider the function*

$$x(t) = (5\cos(2t) + 4\sin(2t))e^{-t/5}.$$

Graph $x(t)$, then compute its first and second derivatives.

Solution. Observe that $x(t) = \mathrm{Re}\,(z(t))$ with $z(t) = (5-4i)e^{(-1/5+2i)t}$. To graph $x(t)$, write $z(t)$ in polar form: $(5-4i) = Ae^{-i\phi}$, with $A = \sqrt{5^2+4^2} = \sqrt{41} \approx 6.40$ and $\phi = \arctan(4/5) \approx 0.67$. Hence,

$$z(t) = Ae^{-i\phi}e^{((1/5)+i2)t} = Ae^{-t/5}e^{(2t-\phi)i}$$

From this, it follows that

$$x(t) = Ae^{-t/5}\cos(2t-\phi) = Ae^{-t/5}\cos(2(t-\phi/2)) \approx 6.40e^{-t/5}\cos(2(t-0.38)),$$

from which one can more easily visualize the graph (see Example A.1 in Appendix A).

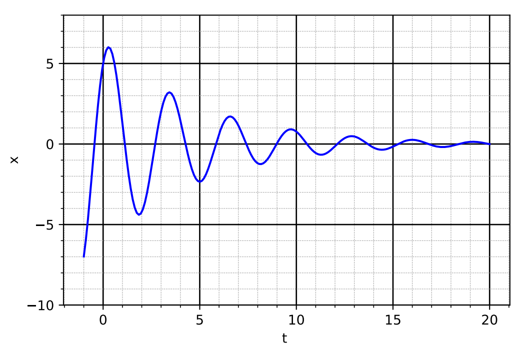


Figure 5.6. The graph of $x(t) = Ae^{-t/5}\cos(2t-\phi)$.

One could, of course, compute the first and second derivatives of $x(t)$ directly from the original formula, but that would be tedious. It's easier, however, to first compute the derivatives of $z(t)$ and then to take the real part to obtain the derivatives of $x(t)$. Here's the computation:

Since

$$z'(t) = (5-4i)(-1/5+2i)e^{(-1/5+2i)t} = \left(7 + \frac{54}{5}i\right)e^{(-1/5+2i)t},$$

$$x'(t) = \left(7\cos(2t) - \frac{54}{5}\sin(2t)\right)e^{-t/t}.$$

Since

$$z''(t) = (5-4i)(-1/5+2i)^2 e^{(-1/5+2i)t} = \left(-23 + \frac{296}{25}i\right)e^{(-1/5+2i)t},$$

$$x''(t) = -\left(23\cos(2t) + \frac{296}{25}\sin(2t)\right)e^{-t/t}.$$

EXAMPLE 5.8. Evaluate the definite integral $\displaystyle\int_0^1 (3\cos(2t) - 4\sin(2t))e^{5t}\,dt$.

SOLUTION. Observe that $(3\cos(2t) - 4\sin(2t))e^{5t} = \operatorname{Re}\left((3+4i)e^{(5+2i)t}\right)$. We can now compute as follows:

$$\begin{aligned}
\int_0^1 (3\cos(2t) - 4\sin(2t))e^{5t}\,dt &= \operatorname{Re}\left(\int_0^1 (3+4i)e^{(5+2i)t}\,dt\right) \\
&= \operatorname{Re}\left(\frac{3+4i}{5+2i}e^{(5+2i)t}\Big]_0^1\right) = \operatorname{Re}\left(\frac{3+4i}{5+2i}(e^{(5+2i)} - 1)\right) \\
&= \operatorname{Re}\left(\left(\frac{23}{29} + \frac{14}{29}i\right)\left(e^5(\cos(2) + i\sin(2)) - 1\right)\right) \\
&= \operatorname{Re}\left(\left(\frac{23}{29} + \frac{14}{29}i\right)\left((e^5\cos(2) - 1) + ie^5\sin(2)\right)\right) \\
&= \frac{23}{29}(e^5\cos(2) - 1) - \frac{14}{29}e^5\sin(2) \approx -114.9
\end{aligned}$$

EXERCISE 5.3.

(1) Sketch the graph of the curve $z(t) = (2+2i)e^{\left(\frac{1}{2}+\pi\, i\right)t}$ for $0 \le t \le 3$ in the complex plane.
(2) Write the function $x(t) = 3e^{-2t}\cos(4t) + 5e^{-2t}\sin(4t)$ in each of the forms $x(t) = \operatorname{Re}\left(Ce^{rt}\right)$ and $x(t) = Ae^{\rho\, t}\cos(\omega\, t - \phi)$, where A, ω and ϕ are real numbers and C and r are complex numbers.
(3) Using the complex exponential function, compute the second derivative of the function $x(t) = (2\cos(4t) - 3\sin(4t))e^{-t}$ Check your answer by also computing the second derivative directly.
(4) Evalutate the definite integral $\displaystyle\int_0^{\pi} e^{t/\pi}\sin(t)\,dt$.

CHAPTER 6

Introduction to Second Order Differential Equations

6.1. Introduction

Recall that a *second order ordinary differential equation* is one that can be written in the form

$$\frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right)$$

and that a *solution* is a function $y = y(t)$ satisfying the equation

$$y''(t) = F\left(t, y(t), y'(t)\right)$$

for all t in some interval. As was the case for the general first order differential equation, second order differential equations have many solutions. As mentioned in the introduction, differential equations have many solutions; additional information is needed to determine a unique solution. Usually, this is in the form of *initial conditions* of the form

$$y(t_0) = y_0 \text{ and } y'(t_0) = y'_0\,,$$

specifying the value of the solution and its derivative at a fixed time $t = t_0$. A function $y = y(t)$ satisfying the differential equation together with the initial conditions $y(t_0) = y_0$, $y'(t_0) = y'_0$ is a solution of the *initial value problem*

$$\frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right)\,, \quad y(t_0) = y_0\,, \quad y'(t_0) = y'_0\,.$$

Second order differential equations are important because most differential equations that arise in physics and engineering applications are second order differential equations. For example, the differential equations governing the motion of mechanical and electrical systems are generally cast as second order ordinary differential equations.

Newton's second law of motion, "$F = ma$," is perhaps the most famous differential equation. If x is the position of a particle of mass m moving along a straight line and subject to a total force $F(t, x, x')$ depending on time, the position of the particle, and the speed of the particle, then its position $x = x(t)$ at time t is a solution of the second order differential equation

$$mx'' = F(t, x, x')\,.$$

6.2. Special Cases

There are two cases in which a second order differential equation can be solved by reducing them to a pair of first order differential equations:

$$\frac{d^2y}{dt^2} = F\left(t, \frac{dy}{dt}\right) \quad \text{and} \quad \frac{d^2y}{dt^2} = F\left(y, \frac{dy}{dt}\right).$$

To solve the differential equation $\dfrac{d^2y}{dt^2} = F\left(t, \dfrac{dy}{dt}\right)$, let $u = dy/dt$. Then

$$\frac{du}{dt} = F(t, u)\,.$$

This is a first order differential equation. If $u = U(t, C_1)$ is a solution, then the differential equation

$$\frac{dy}{dt} = u = U(t, C_1)\,,$$

can be solved by integration:

$$y(t) = \int^t U(s, C_1)\, ds + C_2\,.$$

To solve the differential equation $\dfrac{d^2y}{dt^2} = F\left(y, \dfrac{dy}{dt}\right)$, let $u = dy/dt$, and use the chain rule as follows:

$$\frac{d^2y}{dt^2} = \frac{du}{dt} = \frac{du}{dy}\frac{dy}{dt} = \frac{du}{dy}\, u\,.$$

The original second order differential equation can now be rewritten as the first order differential equation in the independent variable y, rather than t:

$$u\frac{du}{dy} = F(y, u)\,.$$

Suppose that $u = U(y, C_1)$ is the solution, then the differential equation

$$\frac{dy}{dt} = U(y, C_1)\,,$$

can be solved by separation of variables:

$$\int \frac{dy}{U(y, C_1)} = t + C_2\,.$$

Example 6.1. To solve the differential equation $y'' + t(y')^2 = 0$, let $u = y'$. Then $u' + tu^2 = 0$, which can be solved by separation of variables:

$$\int \frac{du}{u^2} = -\int t dt \quad \Longrightarrow \quad -\frac{1}{u} = -\frac{t^2}{2} + C_1 \quad \Longrightarrow \quad y' == \frac{1}{t^2/2 + C_1} = \frac{2}{t^2 + 2C_1}\,.$$

Consequently,

$$y(t) = 2\int^t \frac{1}{t^2 + 2C_1}\, dt + C_2 = \frac{2}{\sqrt{2C_1}} \arctan\left(\frac{t}{\sqrt{2C_1}}\right) + C_2$$

Example 6.2. Recall the *harmonic oscillator* discussed in Example 1.8, consisting of an object of mass m attached to a spring and free to move to the right and left without friction. If x denotes the amount the spring has stretched relative to its equilibrium position, then by Hooke's law (see Figure 1.2) the force on the object is $-kx$; and Newton's second law "$F = ma$" takes the form

$$m\frac{d^2x}{dt^2} = -kx\,. \tag{6.1}$$

To solve the initial value problem

$$m\frac{d^2x}{dt^2} + kx = 0\,, \quad x(0) = x_0\,, \quad x'(0) = v_0\,,$$

set $v = dx/dt$. Then Equation (6.1) assumes the form $m\dfrac{dv}{dt} = -k\,x$. By the chain rule

$$\frac{dv}{dt} = \frac{dv}{dx}\,\frac{dx}{dt} = v\,\frac{dv}{dx}\,.$$

Substituting this into the equation $mv' = -kx$ and viewing v as a function of x rather than t results in the differential equation for $u = u(x)$

$$mv\frac{dv}{dx} = -kx\,,$$

which can be solved by separation of variables:

$$\int mv\,dv = \int -kx\,dx \implies \frac{m}{2}v^2 = -\frac{k}{2}\,x^2 + E\,, \implies \frac{m}{2}v^2 + \frac{k}{2}\,x^2 = E\,,$$

where E is a constant. Since $v(t_0) = x'(t_0) = v_0$ and $x(t_0) = v_0$,

$$E = \frac{m}{2}v_0^2 + \frac{k}{2}x_0^2.$$

Therefore, $x = x(t)$ and $v = v(t)$ satisfy the equation

$$\frac{m}{2}v^2 + \frac{k}{2}\,x^2 = \frac{m}{2}v_0^2 + \frac{k}{2}x_0^2.$$

Recall from high school physics that

$$\frac{1}{2}mv^2 = \textit{kinetic energy (of the mass)} \quad \text{and} \quad \frac{1}{2}kx^2 = \textit{potential energy (stored in the spring).}$$

The quantity E is, therefore, the total energy of the mass-spring system. This is a special case of the law of *conservation of energy*:

$$\text{kinetic energy} + \text{potential energy} = E = \text{a constant}\,.$$

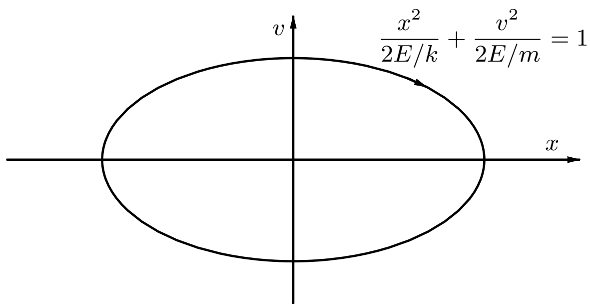


FIGURE 6.1. Conservation of energy implies that curve $t \mapsto (x(t), v(t))$ in the (x, v)-plane is an ellipse.

Having found the relation between $v = dx/dt$ and x, the position $x = x(t)$ can be found by solving the equation

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}x^2 = E\,.$$

for dx/dt and separating variables:

$$\sqrt{m}\,\frac{dx}{dt} = \pm\sqrt{2E - kx^2} = \pm\sqrt{k}\sqrt{2E/k - x^2} \implies \frac{1}{\sqrt{2E/k - x^2}}\,\frac{dx}{dt} = \pm\sqrt{\frac{k}{m}}\,.$$

Integration yields

$$\sin^{-1}\left(\sqrt{\frac{k}{2E}}x\right) = \pm\sqrt{\frac{k}{m}}t - \phi\,,$$

where ϕ is a constant of integration. Solving for x yields the equation

$$x(t) = \sqrt{\frac{2E}{k}}\sin\left(\pm\sqrt{\frac{k}{m}}\,t - \phi\right).$$

The most important conclusion to draw from this computation is the following:

The mass attached to the spring oscillates with frequency $\sqrt{\dfrac{k}{m}}$.

Note: *This is not the usual way the differential equation* $m\dfrac{d^2x}{dt^2} + kx = 0$ *is solved! In Chapter 8, an easier, less involved approach is given.*

Exercise 6.1. Let an annulus, i.e. the area between two concentric circles, be described in polar coordinates by $1 \le r \le 2$. If the inner boundary is held at temperature $T = 50°C$ and the outer boundary at $T = 100°C$ for a long time so that the annulus reaches thermal equilibrium, the temperature T of at a point in the annulus will depend only on the distance r from the center and it can be shown that it satisfies the second order differential equation

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0\,.$$

Find the temperature distribution $T = T(r)$.

Hint: *Let* $y(r) = T'(r)$ *and solve a first order differential equation for* $y(r)$.

6.3. Linear Second Order Differential Equations

The most important class of differential equations is the class of *linear second order differential equations*. These are differential equations that can be written in the form

$$\frac{d^2y}{dt^2} + p(t)\,\frac{dy}{dt} + q(t)\,y = f(t)\,,$$

where the functions $p(t)$, $q(t)$ and $f(t)$ are usually assumed to be continuous or piecewise continuous on an interval $a < t < b$. When $f(t) = 0$, the differential equation becomes

$$\frac{d^2y}{dt^2} + p(t)\,\frac{dy}{dt} + q(t)\,y = 0$$

and is called a *homogeneous differential equation*. When $f(t)$ is not zero the equation is called a *nonhomogeneous differential equation*. The function $f(t)$ is called a *forcing function*.

Notice the similarity between the form of linear second order differential equations and the form of linear first differential equations:

$$\frac{dy}{dt} + p(t)\,y = f(t)\,.$$

Remark 6.1. (Notation) Rather than writing out the full (and rather long) expression

$$\frac{d^2y}{dt^2} + p(t)\,\frac{dy}{dt} + q(t)\,y\,,$$

it is often convenient to use the shorthand notation $L[y]$ for the left-hand side:

$$\boxed{L[y] = \frac{d^2y}{dt^2} + p(t)\,\frac{dy}{dt} + q(t)\,y.}$$

For instance, if $L[y] = y'' + 4y$. Then
$$L[\sin(2t)] = (\sin(2t))'' + 4(\sin(2t)) = -4\sin(2t) + 4\sin(2t) = 0\,,$$
showing that the function $y(t) = \sin(2t)$ is a solution of the differential equation
$$y'' + 4y = 0\,.$$
Since, $L[\sin(t)] = -\sin(t) + 4\sin(t) = 3\sin(t)$, it follows that $y = \sin(t)$ is a solution of the differential equation
$$y'' + 4y = 3\sin(t)\,.$$
The object L is called a *linear operator* because
$$L[C_1 y_1(t) + C_2 y_2(t)] = C_1 L[y_1(t)] + C_2 L[y_2(t)]$$
for any two functions $y_1(t)$ and $y_2(t)$ and any constants C_1 and C_2.

Here is the main theoretical result about second order linear differential equations:

THEOREM 2. *Suppose that $p(t)$, $q(t)$ and $f(t)$ are continuous on the interval $a < t < b$. Suppose further that t_0 is between a and b. Then the initial value problem*
$$L[y] = y'' + p(t)y' + q(t)y = f(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$
has one and only one solution defined on the entire interval $a < t < b$.

Some bad news. This theorem does NOT tell us how to find the solution—it only confirms that a solution exists and that it is unique. Even worse, unlike the first order linear differential equation, there is no general formula for the solution of this second order initial value problem.

Some good news. When $p(t)$ and $q(t)$ are constants there are general techniques for finding a solution. This is the most important case, and the remainder of these notes are devoted to the study of initial value problems of the special form
$$\boxed{L[y] = ay'' + by' + cy = f(t)\,, \quad y(t_0) = y_0\,, \quad y'(t_0) = y_0'\,,} \tag{6.2}$$
where a, b, c are constants and $a > 0$.

CHAPTER 7

Solving Homogeneous Differential Equations

The goal of this chapter is to understand how to solve initial value problems of the form

$$L[y] = ay'' + by' + cy = 0\,, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'\,. \tag{7.1}$$

Suppose that we have succeeded in finding two solutions of the equation $L[y] = 0$, say $y_1(t)$ and $y_2(t)$. Then, by the *superposition principal* (mentioned in the previous lecture) the "linear combination"

$$y = C_1 y_1(t) + C_2 y_2(t)$$

is also a solution of $L[y] = 0$. Let's verify that this is the case:

$$\begin{aligned} L[y] &= L[C_1 y_1 + C_2 y_2] \\ &= a(C_1 y_1 + C_2 y_2)'' + b(C_1 y_1 + C_2 y_2)' + c(C_1 y_1 + C_2 y_2) \\ &= a(C_1 y_1'' + C_2 y_2'') + b(C_1 y_1' + C_2 y_2') + c(C_1 y_1 + C_2 y_2) \\ &= C_1(a y_1'' + b y_1' + c y_1) + C_2(a y_2'' + b y_2' + c y_2) \\ &= C_1 L[y_1] + C_2 L[y_2] \\ &= C_1 \cdot 0 + C_2 \cdot 0 = 0 \end{aligned}$$

Therefore, once we have found two solutions $y_1(t)$ and $y_2(t)$, we can construct lots of solutions by taking linear combinations.

Here is the general strategy for solving an initial value problem of the form

$$L[y] = 0\,, \quad y(t_0) = y_0\,, \quad y'(t_0) = y_0'\,.$$

(i) First find two specific solutions of the differential equation

$$L[y] = ay'' + by' + cy = 0\,.$$

Call them $y_1(t)$ and $y_2(t)$.

(ii) Form the *general solution* $y(t) = C_1 y_1(t) + C_2 y_2(t)$

(iii) Choose C_1 and C_2 so that the initial conditions are satisfied.

The last step involves solving the system of equations

$$\left\{ \begin{array}{l} C_1 y_1(t_0) + C_2 y_2(t_0) = y_0 \\ C_1 y_1'(t_0) + C_2 y_2'(t_0) = y_0' \end{array} \right. \tag{7.2}$$

for the unknowns C_1 and C_2. This is always possible, provided that the functions $y_1(t)$ and $y_2(t)$ are not scalar multiples of one another. In that case, $y_1(t)$ and $y_2(t)$ are said to be *independent solutions* or that they form a *fundamental basis* of solutions.

In examples, it is usually easy to solve for C_1 and C_2. The formulas:

$$C_1 = \frac{y_0\, y_2'(t_0) - y_0'\, y_2(t_0)}{y_1(t_0)\, y_2'(t_0) - y_2(t_0)\, y_1'(t_0)} \qquad C_2 = \frac{y_0'\, y_1(t_0) - y_0\, y_1(t_0)}{y_1(t_0)\, y_2'(t_0) - y_2(t_0)\, y_1'(t_0)}$$

are occasionally useful. The denominator is called the *Wronskian* of $y_1(t)$ and $y_2(t)$. The main theoretical result is contained in the following theorem.

THEOREM 3. *Let $y_1(t)$ and $y_2(t)$ be independent solutions of the differential equation*

$$L[y] = ay'' + by' + cy = 0\,.$$

Then for any value of t_0, the initial value problem

$$y(t_0) = y_0 \text{ and } y'(t_0) = y'_0$$

has a unique solution of the form $y(t) = C_1y_1(t) + C_2y_2(t)$ defined for all values of t.

REMARK 7.1. The fundamental basis is not unique! *There are MANY fundamental bases of solutions for a given homogeneous linear differential equation.*

EXAMPLE 7.1. Consider the differential equation

$$y'' - y = 0\,.$$

The functions e^t and e^{-t} form a fundamental basis of solutions. But so do the functions

$$\sinh(t) = \frac{e^t - e^{-t}}{2} \text{ and } \cosh(t) = \frac{e^t + e^{-t}}{2}\,.$$

Yet another is the pair of functions

$$\sinh(t-3) \text{ and } \cosh(t-3)$$

Note: These functions are all linear combinations of the solutions e^t and e^{-t}, hence they are solutions.

REMARK 7.2. Choosing the right fundamental system can often simplify the solution of initial value problems. For instance, consider the IVP

$$y'' - y = 0 \quad y(3) = 11 \quad y'(3) = 13\,.$$

The function $y(t) = C_1\cosh(t-3) + C_2\sinh(t-3)$ is the general solution of $y'' - y = 0$. From this, it is easy to determine C_1 and C_2:

$$y(3) = C_1\cosh(0) + C_2\sinh(0) = C_1 = 11$$

and

$$y'(3) = C_1\sinh(0) + C_2\cosh(0) = C_2 = 13\,.$$

Hence,

$$y(t) = 11\cosh(t-3) + 13\sinh(t-3)\,.$$

The function $y(t)$ can also be expressed in terms of the fundamental system e^t and e^{-t} by expanding:

$$\begin{aligned} y(t) &= 11\cosh(t-3) + 13\sinh(t-3) \\ &= \frac{11}{2}(e^{t-3} + e^{-t+3}) + \frac{13}{2}(e^{t-3} - e^{-t+3}) = 12e^{-3}e^t - e^3e^{-t}\,. \end{aligned}$$

Although this last way of writing $y(t)$ is correct, it hides the initial conditions.

7.1. The Characteristic Polynomial

One way to solve the homogeneous differential equation

$$L[y] = ay'' + by' + cy = 0$$

is to look for solutions of the form $y = e^{rt}$. Notice that

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + c(e^{rt}) = (ar^2 + br + c)\,e^{rt}\,.$$

Consequently, if r is a solution of

$$ar^2 + br + c = 0\,, \tag{7.3}$$

then e^{rt} is a solution of the differential equation. The polynomial $ar^2 + br + c$ is called the *characteristic polynomial* of the differential equation, and the equation (7.3) is called the *characteristic equation*.

The roots of the characteristic polynomial can be found by inspection, by factoring, or (in the worse case) from the quadratic formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,,$$

In any case, it appears that the problems of solving the differential equation has been reduced to a problem in high school algebra. Let r_1 and r_2 be the roots of the characteristic polynomial. There are three cases to consider:

(i) r_1 and r_2 are both real and $r_1 \neq r_2$ ($b^2 > 4ac$).
(ii) $r_1 = r_2$ ($b^2 = 4ac$).
(iii) r_1 and r_2 are complex ($b^2 < 4ac$).

In case (i), the two functions $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ clearly form a fundamental system of solutions. But this fails in case (ii), where the characteristic polynomial has only one root. In case (iii), where the roots are complex, the meaning of $e^{r_1 t}$ and $e^{r_2 t}$ is unclear.

7.2. Distinct Real Roots of the Characteristic Polynomial

This is the easiest case: the functions $e^{r_1 t}$ and $e^{r_2 t}$ are independent; and, therefore,

$$\boxed{y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}} \tag{7.4}$$

is the general solution of the differential equation.

Example 7.2. Solve the initial value problem

$$L[y] = y'' - 3y' + 2y = 0 \text{ and } y(0) = 0\,, \quad y'(0) = 1\,.$$

Solution. Substituting $y = e^{rt}$ into the equation $L[y] = 0$ gives

$$(r^2 - 3r + 2) \cdot e^{rt} = 0\,,$$

This implies that r satisfies the quadratic equation

$$r^2 - 3r + 2 = (r-2)(r-1) = 0\,.$$

Therefore, $y = e^t$ and $y = e^{2t}$ are two independent solutions of the differential equation and

$$y(t) = C_1 e^t + C_2 e^{2t}$$

is the general solution. The initial conditions are

$$y(0) = C_1 + C_2 = 0$$

and

$$y'(0) = C_1 + 2C_2 = 1\,.$$

It follows that $C_1 = -1$ and $C_2 = 1$. Therefore,

$$y(t) = -e^t + e^{2t}\,.$$

EXAMPLE 7.3. Solve the initial value problem

$$L[y] = y'' - 4y = 0\,, \quad y(0) = 7\,, \quad y'(0) = 8\,.$$

SOLUTION. First find the general solution of the differential equation by looking for solutions of the form $y = e^{rt}$. Since $L[e^{rt}] = (r^2 - 4) \cdot e^{rt}$, $r = \pm 2$ and the general solution is

$$y(t) = C_1 e^{2t} + C_2 e^{-2t}\,.$$

The initial conditions give

$$C_1 + C_2 = 7 \text{ and } 2C_1 - 2C_2 = 8\,,$$

which are easily solved to give

$$C_1 = \frac{11}{2} \text{ and } C_2 = \frac{3}{2}\,.$$

Hence, $y(t) = \dfrac{11}{2} e^{2t} + \dfrac{3}{2} e^{-2t}$.

EXAMPLE 7.4. Solve the initial value problem:

$$y'' - y = 0 \quad y(3) = 11, \quad y'(3) = 13\,.$$

SOLUTION. The roots of the characteristic polynomial $r^2 - 1$ are ± 1, hence the general solution is

$$y = C_1 e^t + C_2 e^{-t}\,.$$

The initial conditions give

$$y(3) = C_1 e^3 + C_2 e^{-3} = 11 \quad y'(3) = C_1 e^3 - C_2 e^{-3} = 13\,.$$

Solving for C_1 and C_2 gives:

$$C_1 = 12e^{-3} \text{ and } C_2 = -e^3\,.$$

The solution of the initial value problem is, therefore,

$$y = (12e^{-3})\, e^t - e^3\, e^{-t} = 12e^{(t-3)} - e^{-(t-3)}\,.$$

7.3. Repeated Roots of the Characteristic Polynomial

EXAMPLE 7.5. Consider the differential equation

$$y'' + 2y' + y = 0\,.$$

The characteristic polynomial is

$$r^2 + 2r + 1 = (r+1)^2\,,$$

which has only one root $r = -1$. Therefore, the function e^{-t} is a the only solution of the differential equation of the form e^{rt}.

Fortunately, one can check directly that te^{-t} is also a solution:

$$\begin{aligned} L[te^{-t}] &= (te^{-t})'' + 2(te^{-t})' + (te^{-t}) \\ & (t-2)e^{-t} + 2(1-t)e^{-t} + te^{-t} \\ & (t - 2t + t)e^{-t} + (-2+2)e^{-t} = 0\,. \end{aligned}$$

Since te^{-t} is not a constant multiple of e^{-t}, the general solution is

$$y(t) = C_1\, e^{-t} + C_2\, te^{-t} = (C_1 + C_2\, t)e^{-t}\,.$$

This idea works in all cases where the characteristic polynomial has a double root. For suppose that the characteristic polynomial factors has the double root r_0. Then

$$ar^2 + br + c = a(r - r_0)^2 = a(r^2 - 2r_0 r + r_0^2)\,.$$

One solution of the differential equation is $e^{r_0 t}$. To see that $te^{r_0 t}$ is another, compute as follows:

$$\begin{aligned} L[te^{r_0 t}] &= a\left\{(te^{r_0 t})'' - 2r_0(te^{r_0 t})' + r_0^2(te^{r_0 t}\right\} \\ &= a\left\{(2r_0 + r_0^2 t)e^{r_0 t} - a2r_0(e^{r_0 t} + r_0 te^{r_0 t}) + r_0^2 te^{r_0 t}\right\} \\ &= a\left\{(r_0^2 - 2r_0^2 + r_0^2)te^{r_0 t} + (2r_0 - 2r_0)e^{r_0 t}\right\} = 0 \end{aligned}$$

Thus, the general solution is

$$\boxed{y = C_1\, e^{r_0 t} + C_2\, te^{r_0 t} = (C_1 + C_2\, t)\, e^{r_0 t}\,.} \tag{7.5}$$

EXAMPLE 7.6. Find the solution of the initial value problem.

$$y'' - 6y' + 9y = 0\,, \quad y(2) = 3\,, \quad y'(2) = 0\,.$$

SOLUTION. The characteristic polynomial factors as

$$r^2 - 6r + 9 = (r - 3)^2\,.$$

So the general solution is $y = (C_1 + C_2 t)e^{3t}$. The initial conditions give

$$(C_1 + 2C_2)e^6 = 3\,, \quad (3C_1 + 7C_2)e^6 = 0$$

Solving for C_1 and C_2 gives $C_1 = 21e^{-6}$, $C_2 = -9e^{-6}$. Hence,

$$y(t) = (21 - 9t)e^{3t-6}$$

is the solution of the initial value problem.

7.4. Complex Roots of the Characteristic Polynomial

It remains to consider case (iii) where the characteristic polynomial of the differential equation

$$L[y] = ay'' + by' + cy = 0$$

has complex roots. Specifically, suppose $b^2 < 4ac$, then the two complex roots are

$$\rho \pm i\omega = \left(-\frac{b}{2a}\right) \pm i\left(\frac{\sqrt{4ac - b^2}}{2a}\right)\,.$$

Working formally, one expects the functions $e^{(\rho \pm i\omega)t}$ to be solutions the differential equation

$$ay'' + by' + cy = 0\,.$$

Recall that if $w(t)$ is a complex-valued function:

$$w(t) = u(t) + iv(t)\,,$$

where $u(t)$ and $v(t)$ are real-valued functions. Then

$$w'(t) = u'(t) + iv'(t)\,,$$

and by linearity of the operator L:

$$L[w] = (aw'' + bw' + cw'') = (au'' + bu' + cu) + i(av'' + bv' + cv) = L[u] + iL[v]\,.$$

Consequently,

$L[w] = 0$ *if and only if* $L[u] = 0$ *and* $L[v] = 0$.

In other words, *if $w(t)$ is a (complex-valued) solution of of the differential equation $L[y]=0$ then both $u(t)$ and $v(t)$ are solutions.*

This suggests looking for complex-valued solutions. Recall from Section 5 that

$$\frac{de^{rt}}{dt}=re^{rt}\,,$$

even for a complex number $r=\rho+i\omega$. Consequently, if r is a complex root of the characteristic polynomial ar^2+br+c, it follows that

$$L[e^{rt}]=(ar^2+br+c)e^{rt}=0\,;$$

and, therefore, so both the real and the imaginary parts of

$$e^{(\rho+i\omega)t}=e^{\rho t}\cos(\omega t)+ie^{\rho t}\sin(\omega t)$$

are solutions of $L[y]=0$. The functions $e^{\rho t}\cos(\omega t)$ and $e^{\rho t}\sin(\omega t)$ are clearly independent. This implies that

$$y(t)=e^{\rho t}(C_1\cos(\omega t)+C_2\sin(\omega t))=\mathrm{Re}\,\left((C_1-iC_2)e^{(\rho+i\omega)t}\right)$$

is the general solution of the differential equation.

Example 7.7. Find the solution of the initial value problem

$$y''-4y'+13y=0\,,\quad y(0)=1\,,\quad y'(0)=4\,.$$

Solution. By the quadratic formula, the roots of the characteristic polynomial are $2\pm3i$, the general solution is, therefore

$$y(t)=\mathrm{Re}\,\left((C_1-iC_2)e^{(2+i3)t}\right)=e^{2t}(C_1\cos(3t)+C_2\sin(3t))\,.$$

The initial conditions imply

$$\begin{aligned}y(0)&=\mathrm{Re}\,(C_1-iC_2)=C_1=1\\ y'(0)&=\mathrm{Re}\,((C_1-iC_2)(2+i3))=2C_1+3C_2=4\,.\end{aligned}$$

Hence,

$$C_2=\frac{4-2C_1}{3}=\frac{2}{3}$$

Therefore,

$$y(t)=\mathrm{Re}\,\left((1-\frac{2}{3}i)e^{(2+3i)t}\right)=\left(\cos(3t)+\frac{2}{3}\sin(3t)\right)e^{2t}\,.$$

Since

$$1+\frac{2}{3}i=\sqrt{(1)^2+(2/3)^2}e^{i\phi}=\frac{\sqrt{13}}{3}e^{i\phi}\,,\quad \text{where } \phi=\arctan(2/3)\approx0.675\,,$$

the solution can also be written in the form

$$y(t)=\mathrm{Re}\,\left(\frac{\sqrt{13}}{3}e^{-i\phi}e^{(2+i3)t}\right)=\mathrm{Re}\,\left(\frac{\sqrt{13}}{3}e^{2t}e^{i(3t-\phi)}\right)\approx1.20e^{2t}\cos\left(3(t-0.225)\right)\,.$$

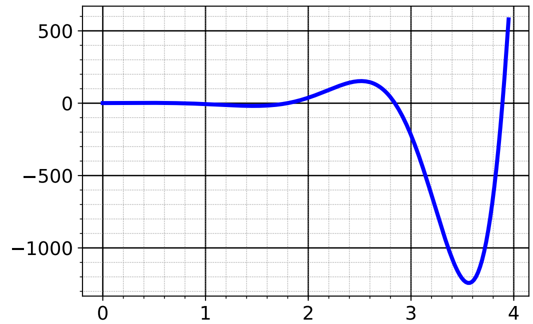


FIGURE 7.1. The graph of $y(t) = 1.20\, e^{2t} \cos(3(t - 0.225))$.

EXERCISES 3.

(1) Solve each of the following differential equations and initial value problems.
  (a) $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = 1$.
  (b) $y'' - 4y' + 3y = 0$, $y(1) = 0$, $y'(1) = 1$
  (c) $y'' - 3y' - 10y = 0$
  (d) $y'' - 3y = 0$.
  (e) $y'' - 4y' + 4y = 0$
  (f) $y'' + 6y' + 9y = 0$, $y(0) = 0$, $y'(0) = 1$.

(2) Consider the differential equation $ay'' + by' + cy = 0$.
  (a) Suppose that $y(t)$ is a solution. Show that for any real number t_0, the function $y(t - t_0)$ is also a solution.
  (b) Show that if the pair of solutions $y_1(t)$ and $y_2(t)$ is a fundamental system of solutions, then so is the pair $y_1(t - t_0)$ and $y_2(t - t_0)$.

(3) The function
$$y(t) = 2e^{-t} \cos(5t) + 3e^{-t} \sin(5t)$$
is the solution of the initial value problem
$$y'' + 2y' + 26 = 0 \quad y(0) = 2\,,\ y'(0) = 13$$
  (a) In the form above, it is difficult to graph. Rewrite it in each of the two forms
$$y(t) = \mathrm{Re}\,(Ce^{rt}) \text{ and } y(t) = Ae^{-at} \cos(\omega t - \phi)\,,$$
  where C and r are complex numbers that you have to determine, and A, ω and ϕ are real numbers that you also have to determine.
  (b) Sketch the solution.

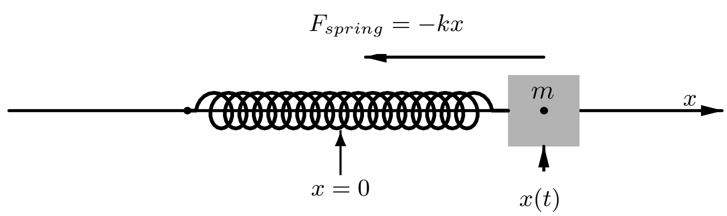
(4) Write the solution of the initial value problem
$$y'' + 25y = 0 \qquad y(0) = 1 \quad y'(0) = 2\,.$$
in the form $y(t) = \mathrm{Re}\left(Ce^{(\rho + i\omega)t}\right)$. By converting C to polar form write the solution in the form
$$y(t) = Ae^{\rho t} \cos(\omega t - \phi)\,,$$
where A and ϕ are real numbers determined by the initial conditions.

CHAPTER 8

The Harmonic Oscillator

Recall that the harmonic oscillator (see Examples 1.8 and 6.2) is the mechanical system consisting of an object of mass m attached to a spring with spring constant k. Recall also that if x denotes the amount that the spring is stretched with respect to its equilibrium position, then by Newton's second law of motion, the function $x = x(t)$, $t =$ time, is a solution of the linear differential equation $mx'' + kx = 0$. Because m and k are both positive, the equation can be rewritten in the form

$$\boxed{x'' + \omega_0^2\, x = 0\,, \text{ where } \omega_0 = \sqrt{\frac{k}{m}}\,.} \tag{8.1}$$



Since $\pm\omega_0\, i$ are the roots of the characteristic polynomial $r^2 + \omega_0^2$, the general solution can be written in the form

$$\boxed{x(t) = \mathrm{Re}\left(Ae^{i(\omega_0 t - \phi)}\right) = A\cos(\omega_0 t - \phi) = A\cos(\omega_0(t - t_0))\,, \text{ where } t_0 = \frac{\phi}{\omega_0}\,.} \tag{8.2}$$

Equation (8.1), therefore, predicts that an object attached to a spring will oscillate at frequency ω_0 and period $T = \dfrac{2\pi}{\omega_0}$. (See Figure 8.1.)

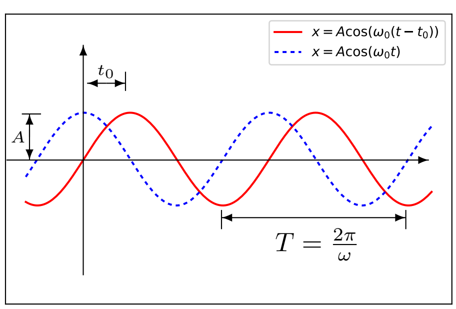


FIGURE 8.1. The graph of $x(t) = A\cos(\omega_0(t - t_0))$.

The mass-spring system is one of many systems modeled on the differential equation (8.2).

EXAMPLE 8.1. (LC-CIRCUITS) An *LC-circuit* is the electrical circuit (Figure 8.2) obtained by removing the resistor and voltage source from the electrical circuit of Example 1.9. Deleting the terms involving R and $V(t)$ from Equation (1.11), simplifies the differential equation to

$$Lq'' + \frac{1}{C}q = 0\,, \tag{8.3}$$

which governs the behavior of the charge $q(t)$ on the capacitor. Apart from a change of symbols, the differential equation (8.3) is the same as the differential equation (8.1). Therefore, the charge on the capacitor is also modeled by a function of the form

$$q(t) = A\cos(\omega_0 t - \phi)\,,$$

where in this case $\omega_0 = \sqrt{1/LC}$. Similarly, the voltage drop across the capacitor is

$$V_C(t) = \frac{q(t)}{C} = \frac{A}{C}\cos(\omega_0 t - \phi)\,.$$

Consequently, in an LC-circuit, the charge on the capacitor and the voltage drop across it both oscillate with frequency $\omega_0 = \sqrt{1/LC}$ and period $T = 2\pi\sqrt{LC}$.

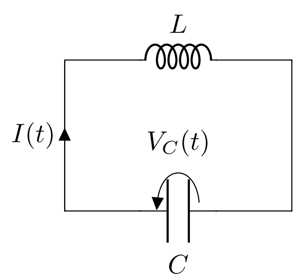


FIGURE 8.2. An LC-circuit.

Notice that because $I(t) = q'(t)$, the current also oscillates at the same frequency.

EXAMPLE 8.2. Suppose you want to design an LC-circuit in which the current oscillates at 60 Hertz (cycles per second). Suppose further that you have only one inductor with inductance of $9100\,\mu$H. (μH = micro Henries) What capacitance should you choose for the capacitor?

SOLUTION. If the frequency is 60 Hertz, then $\omega_0 = 2\pi\cdot 60 = 120\pi\,\mathrm{sec}^{-1}$. Since $\omega_0 = 1/\sqrt{LC}$,

$$C = \frac{1}{\omega_0^2 L} = \frac{1}{(120\pi)^2(9100\times 10^{-6})}\,\mathrm{F} \approx 773.2\mu\mathrm{F}\,,$$

(μF = micro Farads)

EXAMPLE 8.3. (A DAMPED MASS-SPRING SYSTEM) Since the mass-spring system and the LC-circuit are both modeled by the same differential equation, one might wonder if there is a mechanical system that exhibits the same behavior as the RLC-circuit described in the introduction:

$$L\,q'' + R\,q' + \frac{1}{C}\,q = V(t)\,.$$

There is one: the *damped harmonic oscillator*, which models a mechanical system consisting of an object of mass m suspended at the end of a spring, attached to a damping mechanism, and under the influence of gravity.

There are three forces acting on the object:

- $F_{gravity} = -gm$, $g = 9.8\text{m/sec}^2$.
- $F_{damping} = -\gamma \frac{dy}{dt}$, the damping force, where γ is the *damping coefficient*.
- $F_{spring} = -ky$, the spring force, where k is the *spring constant* and where we have set $y = 0$ at the rest position of the spring. Hence, for $y > 0$ the spring is compressed and the force spring exerts on the object is negative (pointing down); and for $y < 0$, the spring is stretched and the force is positive (pointing up).

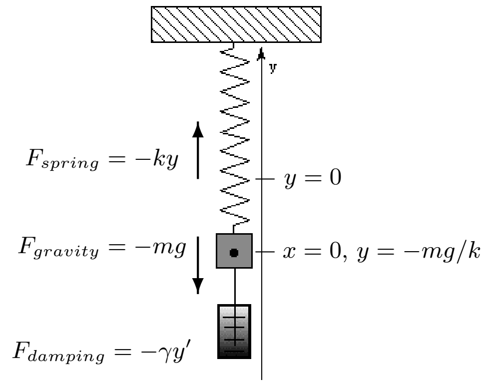


FIGURE 8.3. The damped harmonic oscillator

Let y denote the position of the object measured in meters along a vertical coordinate axis pointing up (see Figure 8.3) Align the y-axis so that $y = 0$ at the bottom end of the spring when no mass is attached. Applying Newton's second law of motion ("$F = ma$") results in the differential equation

$$m\,y'' = -g\,m - \gamma\,y' - k\,y$$

which is better written in the form

$$m\,y'' + \gamma\,y' + ky = -g\,m\,. \tag{8.4}$$

Observe that the mass is in equilibrium when the upward force of the spring cancels with the downward force of gravity, i.e. when $-ky - mg = 0$ or when $y = y_{eq} = -mg/k$ (the equilibrium configuration). This suggests writing y in the form

$$y = y_{eq} + x\,.$$

The quantity x is the displacement of the mass from its equilibrium position $y = y_{eq}$ (see Figure 8.3). Substitution of this formula into equation (8.4) gives

$$m\,x'' + \gamma\,x' + k\,x = 0\,. \tag{8.5}$$

The effect of expressing the position of the mass in terms of its displacement from equilibrium is to turn the nonhomogeneous equation (8.4) into (the equivalent) homogeneous equation (8.5).

Note: This shows that the motion of the original object under the influence of gravity is equivalent to its motion without gravity—that is, changing the origin of coordinate system has the effect of eliminating gravity from the equation of motion.

If, in addition an external force $F(t)$ (in addition to the gravitational force) is applied to the mass, then the homogeneous differential equation (8.5) becomes the nonhomogeneous differential equation

$$m\,x'' + \gamma\,x' + k\,x = F(t)\,. \tag{8.6}$$

This mechanical system is called the *driven, damped harmonic oscillator.*

REMARK 8.1. Notice that under the change of symbols

$$x \leftrightarrow q\,, \quad m \leftrightarrow L\,, \quad \gamma \leftrightarrow R\,, \quad k \leftrightarrow \frac{1}{C}\,, \quad F(t) \leftrightarrow V(t)\,,$$

the differential equation modeling the driven, damped, harmonic oscillator is the same as the differential equation that models the RLC-circuit. This raises the possibility of modeling mechanical systems by electrical networks. The *analogue computers* used in the 1940's and 1950's to study complicated mechanical systems were based on this observation.

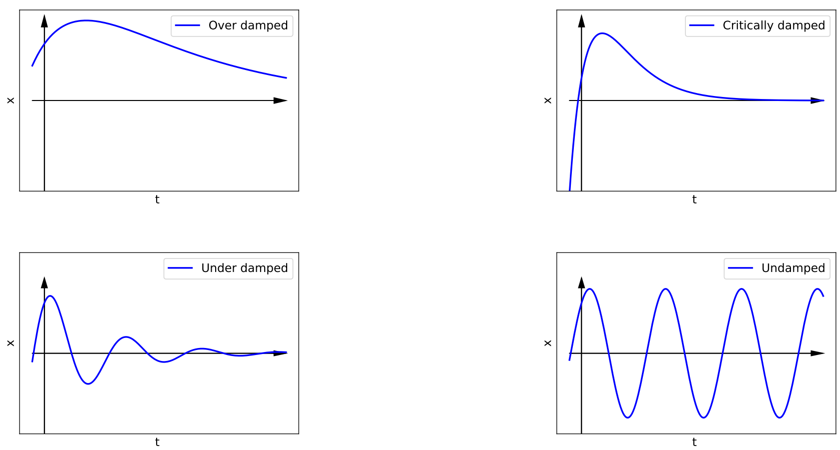
The observed behavior of the damped harmonic oscillator (or of an RLC circuit) depends on the numerical values of the parameters m, γ, and k (or L, R, and C). More precisely, it depends on the number and type of the roots

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

of the characteristic polynomial $mr^2 + \gamma r + k$. There are four cases:

(i) $\gamma^2 - 4mk > 0$ (Over-damped) Roots are real and negative.
(ii) $\gamma^2 - 4mk = 0$ (Critically-damped) A single double root.
(iii) $\gamma^2 - 4mk < 0$ (Under-damped) Roots are complex conjugates of one another.
(iv) $\gamma = 0$ (Un-damped) Roots are pure imaginary.

Figure 8.4 illustrates typical behaviors of the damped harmonic oscillator in each of these four cases.



Over-damped $x(t) = A\,e^{-\left(\frac{\gamma}{2m}+\frac{\sqrt{\gamma^2-4mk}}{2m}\right)t} + B\,e^{-\left(\frac{\gamma}{2m}-\frac{\sqrt{\gamma^2-4mk}}{2m}\right)t}$	Critically damped $x(t) = (A + Bt)\;e^{-\frac{\gamma}{2m}\,t}$
Under-damped $x(t) = A\,e^{-\frac{\gamma}{2m}\,t}\cos(\omega t - \phi)$	Undamped $x(t) = A\,\cos(\omega_0 t - \phi)$

FIGURE 8.4. The four behaviors of the damped harmonic oscillator.

Note on units: Because fundamental properties of the system, such as the criterion for critical damping, do not depend on units, it is worthwhile to express properties in terms of dimensionless quantities. Notice that since $\omega_0 t$ is dimensionless, the dimensions of ω_0 are $(time)^{-1}$. Because the dimensions of $(\gamma/m)x'$ are the dimensions of x'', the dimensions of γ/m must agree with the dimensions of x''/x', i.e. $(time)^{-1}$. It follows that the quantity $\dfrac{\gamma/m}{\omega_0}$ is dimensionless (i.e. independent of units). This suggests expressing the damping criterion terms of it. Since

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \omega_0^2} = -\frac{\gamma}{2m} \pm \omega_0 \sqrt{\left(\frac{\gamma/m}{2\omega_0}\right)^2 - 1}\,,$$

the four cases above can be expressed in the following dimensionless form:

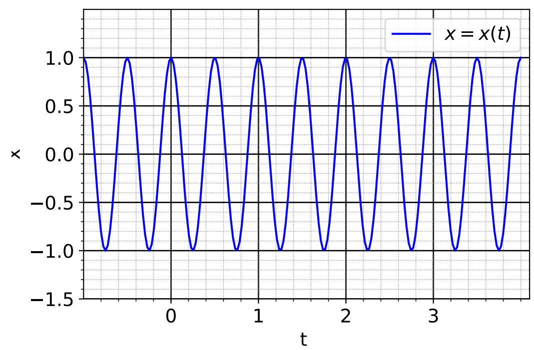
$$\boxed{\frac{\gamma/m}{2\omega_0} > 1 \text{ (over)},\ \frac{\gamma/m}{2\omega_0} = 1 \text{ (critical)},\ 0 < \frac{\gamma/m}{2\omega_0} < 1 \text{ (under)},\ \frac{\gamma/m}{2\omega_0} = 0 \text{ (undamped)}}$$

EXERCISES 4.

(1) A weight of mass $m = 5$ kg is suspended from a spring with unknown spring constant k. The weight is free to move up and down. Ignoring friction, its position relative to its equilibrium position satisfies a differential equation of the form

$$mu'' + ku = 0\,,$$

where t denotes time measured in seconds. To find the spring constant, the spring is set in motion and the graph of $u(t)$ plotted. The result is shown in the following figure. The horizontal axis is t (measured in seconds), vertical axis is u (measured in meters):



(a) The function $u = u(t)$ is the solution of an initial value problem. What are the initial conditions? That is, what are the values of $u(0)$ and $u'0$?
(b) What is the period T measured in seconds?
(c) What is the spring constant k? (*Your answer can most easily be expressed in terms of* π.)

(2) A cylindrical log of radius 1/10 meter, 5 meters in length, and with a mass of 50 kilograms is placed vertically in a lake so that it is free to bob up and down. Assume that there is no water resistance. A weight of 50 kilograms of negligible volume is attached to the bottom of the log so that it remains vertical (so the total mass of the log and weight together is 100 kilograms). The mass density of water is 1000 kilograms per cubic meter. (For convenience, assume that the acceleration due to gravity is $g = 10$ meters per sec^2 (*It is actually closer to* 9.81.)

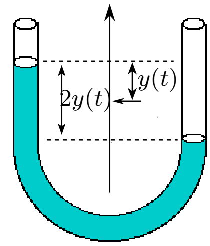
There are two forces acting on the log: gravity and the buoyant force of the water. The buoyant force can be computed from Archimedes' principle:

An object that is completely or partially submerged in a fluid is acted on by an upward (buoyant) force equal to the weight of the displaced fluid.

Let t be time in seconds and let $d(t)$ denote the depth (in meters) of the bottom of the log
.

(a) Compute the depth d_{eq} of the log in its equilibrium position, i.e. when the magnitude of the buoyant force is exactly equal the combined weight (in Newtons) of the log plus the mass. **Hint:** Draw a good picture!
(b) Write down a differential equation for $d(t)$.
(c) Now let $y(t) = d(t) - d_{eq}$, the displacement of the log from its equilibrium position. Assuming that $y(0) = 1$ meters and $y'(0) = 0$ meters/sec, write down an initial value problem for y.
(d) Solve the initial value problem you wrote down in part (c).

(3) Consider a "U"-shaped tube filled with liquid Mercury as shown in the figure below. The radius of the tube is 1 centimeter (so its diameter is 2 centimeters). There are 500 grams of Mercury in the tube. Liquid Mercury has a mass density of 13.5 grams per cubic centimeter. The mercury in the tube will oscillate with a certain period T, measured in seconds. Your task is to compute T by completing the following steps:



(a) Let $y(t)$ be the height above its equilibrium position of the liquid surface at the left vertical segment of the tube. (At equilibrium, both surfaces are at the same height above sea level and $y = 0$. When $y(t) < 0$ the right surface is higher than the left surface.) The only force acting on the mass of fluid in the tube is due to gravity. Compute the total force on the fluid (vertical component only) and use that formula to find a linear, homogeneous, constant coefficient, second order differential equation for $y = y(t)$.
(b) Compute T by solving the differential equation you found in part (a).
Hints: Treat the fluid as a single rigid body. The net force acting on the fluid is twice the weight of the fluid above the equilibrium level (all other forces cancel).

(4) A 1 kilogram mass is suspended from the end of a spring with a spring constant of 1 N/n (Newtons per meter). The mass is free to move up and down, y is the amount (measured in meters) that the spring is stretched and there is no gravity. In addition, there is a damping mechanism that exerts a force of $-\gamma\, y'$ Newtons, where γ is a constant.
  (a) What value of γ will make the system *critically damped*?
  (b) If at time $t = 0$ the spring is not stretched and the mass is moving at a rate of 0.5 m/sec (meters per second), what is the formula for $y(t)$. (Use the value of γ obtained in part a).
  (c) What is the maximum amount by which the spring will be stretched?

(5) Consider a mechanical system consisting of a spring with spring constant $k = 10$ lb/ft and a 100 pound weight that is free to move up and down. Assume that at time $t = 0$ sec the spring is unstretched but taut and the velocity of the weight is 0 ft/sec. Notice at time $t = 0$ sec the net force on the weight is the 100 pound downward force due to gravity. Describe the subsequent motion of the weight, particularly the period and frequency of the resulting periodic behavior and the amplitude of the oscillations.

(6) Suppose that a car weighing 2000 pounds is supported by four shock absorbers each with a spring constant of 520 lbs/inch.
  (a) Assume no damping and determine the period of oscillation of the vertical motion of the car. Hint: $g = 384\text{in/sec}^2$.
  (b) What were the initial conditions if after 10 seconds the car body is 4 inches above its equilibrium position and at the high point in its cycle?
  (c) Now assume that oil is added to the shock absorbers to produce a force of -83.2lb-sec/in times the vertical velocity of the car body (measured in/sec). Find the displacement $y(t)$ from equilibrium if $y(0) = 1$ in and $y'(0) = -12$ in/sec.

(7) Suppose that you are designing a new shock absorber for an automobile. The car has a mass of 1000 kg (kilograms) and the combined effect of the springs in the suspension system is that of a spring constant of 20000 N/m.
  (a) Before a damping mechanism is installed in the car, when the car hits a bump it will bounce up and down. How may bounces will a rider experience in the minute right after the car hits a bump?
  (b) Your job is to design a damping mechanism that eliminates oscillations when the car hits a bump. What is the minimum value of the effective damping constant that can be used?
  (c) Suppose that at time $t = 0$ the car hits a bump. Immediately before that time the car was not moving up and down and the effect of the bump is to add a vertical component to the speed of the car of 1.0 meter/sec. How high will the car rise above its equilibrium position if you design the system with the damping constant you found in part (b)?

CHAPTER 9

Solving Nonhomogeneous Differential Equations

Recall that the *general solution* of the homogeneous differential equation

$$L[y] = ay'' + by' + cy = 0\,.$$

is of the form

$$y_h(t) = C_1 y_1(t) + C_2 y_2(t)\,,$$

where $y_1(t)$ and $y_2(t)$ form a fundamental basis of solutions. The general solution of the nonhomogeneous differential equation

$$L[y] = ay'' + by' + cy = f(t), \tag{9.1}$$

can then be written in the form

$$y(t) = y_p(t) + y_h(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t)\,, \tag{9.2}$$

where $y_p(t)$ is a *particular solution* of the nonhomogeneous differential equation.

To show this, assume that $y(t)$ is any function with $L[y(t)] = f(t)$. Then the difference $y(t) - y_p(t)$ is a solution of the homogeneous differential equation as the following computation shows:

$$L[y(t) - y_p(t)] = L[y(t)] - L[y_p(t)] = f(t) - f(t) = 0\,.$$

But every solution of the homogeneous differential equation is of the form $y_h(t)$. Consequently,

$$y(t) - y_p(t) = C_1 y_1(t) + C_2 y_2(t)\,,$$

for some choice of C_1 and C_2. Hence, $y(t)$ satisfies (9.2).

Solving the initial value problem

$$L[y] = f(t)\,, \quad y(t_0) = y_0\,, \quad y'(t_0) = y_0'\,,$$

then reduces to solving the following pair of equations:

$$\begin{aligned} C_1 y_1(t_0) + C_2 y_2(t_0) + y_p(t_0) &= y_0 \\ C_1 y_1'(t_0) + C_2 y_2'(t_0) + y_p'(t_0) &= y_0'\,. \end{aligned}$$

The problem of finding $y_1(t)$ and $y_2(t)$ was addressed in the previous chapter. It remains to find techniques for finding a *particular solution* $y = y_p(t)$ of the nonhomogeneous equation

$$L[y] = ay'' + by' + cy = f(t)\,.$$

A *particular solution* denotes a solution that does not involve any arbitrary constants. This will become more clear through examples.

There are several approaches to finding a particular solution. Two will be addressed in these notes:

- *Undetermined Coefficients*
- *Laplace Transforms.*

Remark 9.1. There is a third approach, called *variation of parameters*. Because undetermined coefficients and Laplace transforms are sufficient in most cases, variation of parameters will not be included in these notes. The interested reader can find a number of explanations of this method on the web.

9.1. Undetermined Coefficients

When the forcing function $f(t)$ is of a special form, the method of *undetermined coefficients* reduces the problem of finding a particular solution to a problem algebra. Specifically, this method applies whenever $f(t)$ is of one of the forms

$$f(t) = p(t)e^{rt}$$

or

$$p(t)\sin(\omega t)e^{rt} + q(t)\cos(\omega t)e^{rt}\,,$$

where $p(t)$ and $q(t)$ are polynomials or when $f(t)$ is a sum of terms like these.

Here are some examples of differential equations where the method applies:

(1) $y'' + 2y' - y = (3t+1)e^{2t}$
(2) $y'' + 4y = (1-t^3)\cos(3t)$
(3) $y'' - 2y' + y = (1+t+t^2)e^{3t}\cos(3t) + te^{3t}\sin(3t)$
(4) $y'' - y = (1+2t)e^{t} + (t^2\sin(3t) + (2-t+t^2)\cos(3t))$

Rather than presenting the method in general and then giving examples, it is less confusing to work by example first followed by a description of the general method.

Example 9.1. Find a particular solution of the nonhomogeneous differential equation

$$L[y] = y'' + 3y' + 2y = (t-2)\,e^{2t}\,.$$

Solution. Notice that $f(t) = (t-2)\,e^{2t}$ is of the form $p(t)e^{rt}$, where $p(t) = t-2$, a polynomial of degree 1, and $r = 2$. Let

$$y_p(t) = (At+B)e^{2t}\,,$$

where A and B are to be determined. A direct computation gives:

$$L[y_p] = \{12A\,t + (7A+12B)\}\;e^{2t}\,.$$

Observe that y_p will satisfy the equation

$$L[y_p] = (t-2)\,e^{2t}\,,$$

provided that A and B satisfy the equation

$$\{12A\,t + (7A+12B)\}\;e^{2t} = (t-2)\,e^{2t}$$

for all t. Equating like terms results in two equations in two unknowns:

$$12A = 1 \text{ and } 7A + 12B = -2\,.$$

The first equation gives $A = 1/12$. Substituting this value into the second equation gives $B = -31/144$. We conclude that

$$y_p(t) = \left\{\frac{1}{12}t - \frac{31}{144}\right\}\;e^{2t}\,.$$

is a particular solution.

EXAMPLE 9.2. Find a particular solution of

$$L[y] = y'' + 3y' + 2y = 1 - 2t\,.$$

SOLUTION. Set

$$y_p(t) = (At + B)e^{0t} = (At + B)\,.$$

Comparing

$$L[y_p] = (3A + 2B) + 2A\,t\,.$$

with the function $f(t) = 1 - 2t$ yields the two equations

$$3A + 2B = 1 \text{ and } 2A = -2\,.$$

Solving the second equation gives $A = -1$ and substituting that value into the first equation gives $B = 2$. Hence

$$y_p = 2 - t\,.$$

EXAMPLE 9.3. Find a particular solution of

$$L[y] = y'' + 3y' + 2y = f(t) = (t^2 - 2)\,e^{2t}\,.$$

SOLUTION. In this case $p(t) = t^2 - 2$, a polynomial of degree 2, so set

$$y_p(t) = \left(At^2 + Bt + C\right)\,e^{2t}$$

$$L[y_p] = \left\{12A\,t^2 + (14A + 12B)\,t + (2A + 7B + 12C)\right\}\,e^{2t}$$

and

$$L[y_p] = (t^2 - 2)\,e^{2t}\,.$$

Equating coefficients gives the three equations

$$\begin{cases} 12A = 1 \\ 14A + 12B = 0 \\ 2A + 7B + 12C = -2\,. \end{cases}$$

The first equation shows $A = 1/12$.
The second (together with $A = 1/12$) forces $B = -7/72$.
And the third (together with $A = 1/12$ and $B = -7/12$) forces $C = -107/864$.

Hence,

$$y_p(t) = \left\{\frac{1}{12}t^2 - \frac{7}{72}t - \frac{107}{864}\right\}\,e^{2t}\,.$$

EXAMPLE 9.4. Find a particular solution of

$$L[y] = y'' + 3y' + 2y = (t - 2)\,e^{-2t}\,.$$

SOLUTION. In this case, the function $y_p(t) = (At + B)e^{-2t}$ cannot be a particular solution. Indeed, a simple computation gives

$$L[(At + B)\,e^{-2t}] = -Ae^{-2t}\,;$$

but

$$L[(At + B)\,e^{-2t}] = (t - 2)\,e^{-2t}\,.$$

Clearly, no choice of A and B can yield $(t - 2)e^{-2t}$. T The solution to this problem is to multiply by the original guess by t:

$$y_p(t) = t\;(At + B)\;e^{-2t}\,.$$

Then

$$L[t(At + B)\,e^{-2t}] = \{-2At + (2A - B)\}\;e^{-2t}\,.$$

The coefficients A and B can then be chosen to satisfy the equation

$$\{-2At + (2A - B)\}\;e^{-2t} = (t - 2)e^{-2t}\,.$$

Comparing terms as before yields two equations

$$-2A = 1 \text{ and } 2A - B = -2\,;$$

and solving for A and B gives $A = -1/2$ and $B = 1$. Hence,

$$y_p(t) = t(-t/2 + 1)e^{-2t} = (t - t^2/2)\,e^{-2t}\,.$$

EXAMPLE 9.5. Find a particular solution to the differential equation

$$L[y] = y'' - 6y' + 9y = (t-2)e^{3t}\,.$$

SOLUTION. Because $r^2 - 6r + 9 = (r-3)^2$,

$$L[e^{3t}] = 0 \text{ and } L[te^{3t}] = 0\,.$$

Consequently, the function $y_p(t) = (At + B)e^{3t}$ cannot be a solution. Neither can $y_p(t) = t(At+B)e^{3t}$ because

$$L[t(At+B)e^{3t}] = 2Ae^{3t}$$

This suggests multiplying by t^2, and setting $y_p(t) = t^2(At + B)e^{3t}$. A short computation shows that

$$L[y_p] = (6At + 2B)e^{3t} = (t-2)e^{3t}\,,$$

Comparing coefficients shows that

$$y_p(t) = t^2(t/6 - 1)e^{3t}$$

is a particular solution.

EXAMPLE 9.6. Find a particular solution for

$$L[y] = y'' - 6y' + 13y = 5\cos(2t)e^{4t}$$

SOLUTION.

The characteristic polynomial $r^2 - 6r + 13$ has roots $3 \pm 2\,i$. So the general solution of $L[y] = f(t)$ has the form

$$y(t) = y_p(t) + \{C_1\cos(2t) + C_2\sin(2t)\}\ e^{3t}\,.$$

Substituting

$$y_p(t) = (A\cos(2t) + B\sin(2t))\,e^{4t}$$

into the differential equation yields (after a lengthy computation)

$$\begin{aligned} L[y_p] &= \{(A + 4B)\cos(2t) + (-4A + B)\sin(2t)\}\ e^{4t} \\ &= 5\cos(2t)\,e^{4t}\,. \end{aligned}$$

Equating coefficients gives the system of equations

$$A + 4B = 5 \text{ and } -4A + B = 0$$

whose solution is $A = 5/17$ and $B = 20/17$. Hence, the function

$$y_p(t) = \left\{\frac{5}{17}\cos(2t) + \frac{20}{17}\sin(2t)\right\}\ e^{4t}$$

is a particular solution.

EXAMPLE 9.7. Find the general solution of the differential equation

$$L[y] = y'' - 6y' + 13y = 5\cos(2t)e^{4t} - 2t\sin(2t)e^{4t}\,.$$

SOLUTION. Let $y_p(t) = (A + Bt)e^{4t}\cos(2t) + (C + Dt)e^{4t}\sin(2t)$. A lengthy computation gives

$$\begin{aligned} L[y_p] = &\{(A + 2B + 4C + 4D) + (B + 4D)t\}\cos(2t)e^{4t} \\ &+ \{(-4A - 4B + C + 2D) + (-4B + D)t\}\sin(2t)e^{4t} \end{aligned}$$

Comparing coefficients leads to the following system of equations:

$$\begin{cases} A+2B+4C+4D &= 5 \\ B+4D &= 0 \\ -4A-4B+C+2D &= 0 \\ -4B+D &= -2\,. \end{cases}$$

One finds (after some computation) that

$$A = -\frac{67}{289}\,, \quad B = \frac{8}{17}\,, \quad C = \frac{344}{289}\,, \quad D = -\frac{2}{17}$$

and thus

$$y_p(t) = \left(-\frac{67}{289} + \frac{8}{17}t\right)\cos(2t)e^{4t} + \left(\frac{344}{289} - \frac{2}{17}t\right)\sin(2t)e^{4t}$$

The roots of the characteristic polynomial $r^2 - 6r + 13$ are $3 \pm 2i$. Therefore, the general solution is

$$\begin{aligned} y = &\left(-\frac{67}{289} + \frac{8}{17}t\right)\cos(2t)e^{4t} + \left(\frac{344}{289} - \frac{2}{17}t\right)\sin(2t)e^{4t} \\ &+ \{C_1\cos(2t) + C_2\sin(2t)\}\; e^{3t}\,. \end{aligned}$$

THE GENERAL CASE. Here is an outline of how to find a particular solution in the general case

$$ay'' + by' + cy = f(t)\,,$$

where $f(t)$ is one of the two forms $f(t) = \begin{cases} p(t)e^{r_0 t} \\ \{p(t)\cos(\omega t) + q(t)\sin(\omega t)\}\; e^{r_0 t} \end{cases},$
with

$$\begin{aligned} p(t) &= p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n \\ q(t) &= q_0 + q_1 t + q_2 t^2 + \cdots + q_n t^n\,. \end{aligned}$$

(1) Let r_1 and r_2 be the the roots of the characteristic polynomial $ar^2 + br + c$.
(2) If $f(t) = p(t)e^{r_0 t}$ then let

$$y_p(t) = \begin{cases} P(t)e^{r_0 t} & \text{if } r_0 \neq r_1, r_2 \\ tP(t)e^{r_0 t} & \text{if } r_0 = r_1 \text{ and } r_0 \neq r_2 \\ t^2 P(t)e^{r_0 t} & \text{if } r_0 = r_1 = r_2 \text{ (double root),} \end{cases}$$

where $P(t) = A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n$.

If $f(t) = \{p(t)\cos(\omega t) + q(t)\sin(\omega t)\}\; e^{r_0 t}$ then set

$$y_p(t) = \begin{cases} \{P(t)\cos(\omega t) + Q(t)\sin(\omega t)\}\, e^{r_0 t} & \text{if } r_0 + \omega\, i \neq r_1, r_2 \\ t\,\{P(t)\cos(\omega t) + Q(t)\sin(\omega t)\}\, e^{r_0 t} & \text{if } r_0 + \omega\, i = r_1 \text{ or } r_2 \end{cases}$$

$$\text{where } \begin{cases} P(t) &= A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n \\ Q(t) &= B_0 + B_1 t + B_2 t^2 + \cdots + B_n t^n\,. \end{cases}$$

(3) Equate coefficients of powers of t in the equation $L[y_p(t)] = f(t)$ to get a linear system of equations in the unknown coefficients A_i and B_i.
(4) Solve the system to get $P(t)$ (and $Q(t)$), and thus $y_p(t)$.
(5) If

$$L[y] = f(t) = f_1(t) + f_2(t)\,,$$

where $f_1(t)$ and $f_2(t)$ are both of the above form (but with different values of n, r_0 and/or ω), then set

$$y_p(t) = y_{p_1}(t) + y_{p_2}(t)\,,$$

where

$$L[y_{p_1}] = f_1(t) \text{ and } L[y_{p_2}] = f_2(t)\,.$$

9.2. Undetermined Coefficients Using Complex-valued Functions

When $f(t)$ involves trig functions, the algebra involved in applying the method of undetermined coefficients can be messy. In such cases, using complex-valued functions often simplifies the computations.

Recall from Section 5 that the complex exponential function is the function

$$e^{(\rho+i\omega)t} = (\cos(\omega t) + i\sin(\omega t))e^{\rho t}\,,$$

where ρ and ω are real numbers. Its derivative satisfies the identity

$$\frac{de^{(\rho+i\omega)t}}{dt} = (\rho+\omega i)e^{(\rho+i\omega)t}\,.$$

Consequently, its second derivative can be easily evaluated:

$$\frac{d^2e^{(\rho+i\omega)t}}{dt^2} = (\rho+\omega i)^2e^{(\rho+i\omega)t}\,.$$

As already mentioned in Section sec:complex-functions, this fact greatly simplifies computations of derivatives of functions of the form

$$x(t) = a\cos(\omega t)e^{\rho t} + b\sin(\omega t)e^{\rho t}\,.$$

Example 9.8. Find a particular solution of the differential equation

$$y'' + y' + y = (\cos(t) - \sin(t))e^{2t}$$

Solution. Since $(\cos(t) - \sin(t))e^{2t} = \mathrm{Re}\left((1+i)e^{(2+i)t}\right)$, there is a particular solution of the form $y_p(t) = \mathrm{Re}\,(z_p(t))$, where $z_p(t)$ is a particular solution solution of

$$z'' + z' + z = (1+i)e^{(2+i)t}\,.$$

Set $z_p(t) = Ae^{(2+i)t}$. Then

$$\begin{aligned} z_p''(t) + z_p'(t) + z_p(t) &= \left((2+i)^2 + (2+i) + 1\right)Ae^{(2+i)t} \\ &= (6+5i)Ae^{(2+i)t} = (1+i)e^{(2+i)t} \end{aligned}$$

Solving for A gives

$$A = \frac{1+i}{6+5i} = \frac{11}{61} + \frac{1}{61}i\,.$$

Hence,

$$z_p(t) = \left(\frac{11}{61} + \frac{1}{61}i\right)e^{(2+i)t}\,.$$

Taking the real part of $z_p(t)$ gives a particular solution of the original differential equation:

$$y_p(t) = \mathrm{Re}\,(z_p(t)) = \left(\frac{11}{61}\cos(t) - \frac{1}{61}\sin(t)\right)e^{2t}\,.$$

EXAMPLE 9.9. Solve the initial value problem

$$u'' + \omega_0^2 u = \sin(\omega t)\,, \quad u(0) = u'(0) = 0\,,$$

for $\omega \neq \omega_0$.

SOLUTION. The general solution of the corresponding homogeneous differential equation is

$$u_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)\,,$$

so the general solution of the original differential equation is of the form

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + u_p(t)\,.$$

Since $\sin(\omega t) = \mathrm{Re}\left(-ie^{i\omega t}\right)$, let $u_p(t) = \mathrm{Re}\left(z_p(t)\right)$, where $z_p(t) = Ae^{i\omega t}$ is a solution of the complex differential equation

$$z'' + \omega_0 z = -i\, e^{i\omega t}\,.$$

To find A substitute $z_p(t)$ into the complex differential equation:

$$z_p''(t) + \omega_0^2 z_p(t) = (-\omega^2 + \omega_0^2)Ae^{i\omega\, t} = -ie^{i\omega t}$$

If follows that $A = \dfrac{-i}{\omega_0^2 - \omega^2}$ and $u_p(t) = \mathrm{Re}\left(z_p(t)\right) = \dfrac{1}{\omega_0^2 - \omega^2}\sin(\omega t)$. The general solution is, therefore,

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{1}{\omega_0^2 - \omega^2}\sin(\omega t)\,.$$

The initial conditions

$$u(0) = C_1 = 0 \text{ and } u'(0) = \omega_0 C_2 + \frac{\omega}{\omega_0^2 - \omega^2} = 0$$

force $C_1 = 0$ and $C_2 = \dfrac{\omega/\omega_0}{(\omega^2 - \omega_0^2)}$. Consequently,

$$u(t) = \frac{\omega/\omega_0}{(\omega^2 - \omega_0^2)}\sin(\omega_0 t) + \frac{1}{\omega_0^2 - \omega^2}\sin(\omega t) = \frac{1}{(\omega^2 - \omega_0^2)}\left((\omega/\omega_0)\sin(\omega_0 t) - \sin(\omega t)\right)\,.$$

EXAMPLE 9.10. Find a particular solution of the differential equation

$$u'' + \omega_0^2 u = \sin(\omega_0 t)\,.$$

SOLUTION. Proceeding as before, the particular solution will be the real part of a particular solution of the complex differential equation

$$z'' + \omega_0^2 z = -ie^{\omega_0 it}\,.$$

The function $z_p(t) = Ae^{\omega_0 it}$ won't work since $z_p''(t) + \omega_0^2 z_p(t) = 0$. So try the next best thing: $z_p(t) = Ate^{\omega_0 it}$:

$$z_p'(t) = A(1 + \omega_0 it)e^{\omega_0 it} \implies z_p''(t) = A\left((\omega_0 i) + (1 + \omega_o it)\omega_0 i\right)e^{\omega_0 it} = A\left(2\omega_0 i - {}_0^2 t\right)e^{\omega_0 it}\,,$$

Then

$$z_p''(t) + \omega_0^2 z_P(t) = A\left(2\omega_0 i - \omega_0^2 t + \omega_0^2 t\right)e^{\omega_0 it} = (2\omega_0 i)Ae^{\omega_0 it} = -ie^{\omega_p it}\,.$$

Solving for A gives $A = \frac{-i}{2\omega_0 i} = -\frac{1}{2\omega_0}$. Hence, the function

$$u_p(t) = \mathrm{Re}\left(-\frac{1}{2\omega_0}te^{\omega_0 it}\right) = -\frac{t}{2\omega_0}\cos(\omega_0 t)$$

is a particular solution.

EXERCISES 5. Solve each of the following differential equations and initial value problems.

(1) $y'' + 3y = t^2 + 1$.
(2) $y'' + y = 2\sin(t)$, $y(0) = 0$, $y'(0) = 0$.
(3) $y'' + y' + y = e^t \sin(t)$.
(4) $y'' - y' = e^t$
(5) $y'' - 4y = e^{2t}$, $y(0) = 1$, $y'(0) = 1$.
(6) $y'' - 4y' + 4y = e^{2t}$
(7) $y'' + 4y = 3\cos(t) + 4\sin(t)$, $y(0) = 0$, $y'(0) = 0$.
(8) $y'' + 4y = 12\cos(4t) - 12\sin(4t)$, $y(0) = 0$, $y'(0) = 0$.
(9) $y'' - 2y' + y = \sin(2t)e^{-t}$.

CHAPTER 10

The Driven Harmonic Oscillator

In an earlier chapter, we studied the undriven harmonic oscillator. If there are additional time-dependent ("external") forces on the object, mechanical system is then modeled by the nonhomogeneous differential equation

$$m\,x'' + \gamma\,x' + k\,x = F(t)\,, \tag{10.1}$$

where $F(t)$ denotes the external "*driving force*." For simplicity, assume that $F(t)$ is of the form

$$F(t) = F_0\,\cos(\omega t)\,.$$

10.1. Resonance

First consider the special case where there is no damping. The system is then modeled on the differential equation

$$\boxed{x'' + \omega_0^2\,x = \frac{F_0}{m}\,\cos(\omega t)\,.} \tag{10.2}$$

Figure 10.1 indicates how an external force of the form $F(t) = F_0\cos(\omega\,t)$ might be applied.

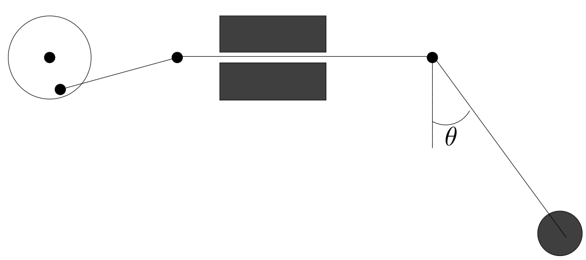


FIGURE 10.1. A simple pendulum with a driving force. For small angles, θ satisfies a differential equation of the form $\theta'' + \omega_0^2\theta = A\cos(\omega t)$.

When the frequency ω of the driving force equals ω_0 (the *natural frequency*), interesting things happen (see Figure 10.2). Recall that the solution of (10.2) is of the form

$$x(t) = x_p(t) + x_h(t) = x_p(t) + A\cos(\omega_0 t - \phi),$$

where $x_p(t)$ is a particular solution.

There are two cases consider: $\omega \neq \omega_0$ and $\omega = \omega_0$:

Case 1: $\omega \neq \omega_0$. Notice $(F_0/m)\cos(\omega t) = \mathrm{Re}\left((F_0/m)e^{i\omega t}\right)$, so try $x_p(t) = \mathrm{Re}\,(z_p(t))$ where $z_p(t)$ is a solution of

$$z_p'' + \omega_0^2 z_p = (F_0/m)e^{i\omega t}$$

Substitute $z_p(t) = Ce^{i\omega t}$ into the differential equation to get

$$z_p'' + \omega_0^2 z_p = (-\omega^2 + \omega_0^2)Ce^{i\omega t} = (F_0/m)e^{i\omega t}\,.$$

Hence, $C = \dfrac{F_0/m}{\omega_0^2 - \omega^2}$ and $x_p(t) = \operatorname{Re}\left(\dfrac{F_0/m}{\omega_0^2 - \omega^2}\, e^{i\omega t}\right) = \dfrac{F_0/m}{\omega_0^2 - \omega^2}\cos(\omega t)$.

Case 2: $\omega = \omega_0$. Try $z_p(t) = Cte^{i\omega_0 t}$ Then

$$z_p'' + \omega_0^2 z_p = \left((2\omega_0 i - \omega_0^2 t\right) Ce^{i\omega_0 t} + \omega_0^2 Cte^{i\omega_0 t} = 2\omega_0 iCe^{-\omega_0 t} = (F_0/m)e^{i\omega_0 t}$$

Therefore, $C = \dfrac{F_0/m}{2\omega_0 i}$ and $x_p(t) = \operatorname{Re}\left(\dfrac{F_0/m}{2\omega_0 i} te^{i\omega_0 t}\right) = \left(\dfrac{F_0/m}{2\omega_0}\right) t\,\sin(\omega_0 t)$.

Remark 10.1. The computation without using complex-valued functions is similar, but involves a little more algebra:

Case 1: $\omega \neq \omega_0$. Try $x_p = A\cos(\omega t) + B\sin(\omega t)$. Then

$$\begin{aligned} x_p'' + \omega_0^2 x_p &= -\,\omega^2(A\cos(\omega t) + B\sin(\omega t)) + \omega_0^2(A\cos(\omega t) + B\sin(\omega t)) \\ &= (\omega_0^2 - \omega^2)A\cos(\omega t) + (\omega_0^2 - \omega^2)\sin(\omega t) = \frac{F_0}{m}\cos(\omega t) \end{aligned}$$

Hence, $A = \dfrac{F_0/m}{\omega_0^2 - \omega^2}$, $B = 0$ and $x_p(t) = \dfrac{F_0/m}{\omega_0^2 - \omega^2}$.

Case 2: $\omega = \omega_0$. Try $x_p = t\,(A\cos(\omega_0 t) + B\sin(\omega_0 t))$. Then

$$\begin{aligned} x_p'' + \omega_0^2 x_p &= -\,2A\omega_0\sin(\omega_0 t) + 2B\omega_0\cos(\omega_0 t) \\ &\quad - \omega_0^2 t(A\cos(\omega_0 t) + B\sin(\omega_0 t)) \\ &\quad + \omega_0^2 t(A\cos(\omega_0 t) + B\sin(\omega_0 t)) \\ &= \frac{F_0}{m}\cos(\omega_0 t)\,. \end{aligned}$$

Therefore, $A = 0$ and $B = \dfrac{F_0/m}{2\omega_0}$ and $x_p(t) = \left(\dfrac{F_0/m}{2\omega_0}\right) t\,\sin(\omega_0 t)$.

Example 10.1. (Resonance) Consider the initial value problem:

$$x'' + \omega_0^2\, x = \frac{F_0}{m}\cos(\omega t)\,, \quad x(0) = 0\,, \quad x'(0) = 0$$

First suppose that $\omega = \omega_0$. The general solution of the differential equation is

$$x(t) = \frac{F_0/m}{2\omega_0}\, t\,\sin(\omega_0 t) + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)\,.$$

The initial condition $x(0) = 0$ implies $C_1 = 0$ and $x'(0) = 0$ implies $C_2 = 0$. Hence,

$$x(t) = \frac{F_0/m}{2\omega_0}\, t\,\sin(\omega_0 t)$$

Thus, *when the frequency of the external force is exactly equal to the natural frequency of the system ($\omega = \omega_0$), the amplitude of the oscillations of the system increases without bound—this is the phenomenon known as* resonance.

Now suppose that $\omega \neq \omega_0$. The general solution of the differential equation is then

$$x(t) = \left(\frac{F_0/m}{\omega_0^2 - \omega^2}\right)\cos(\omega t) + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)\,.$$

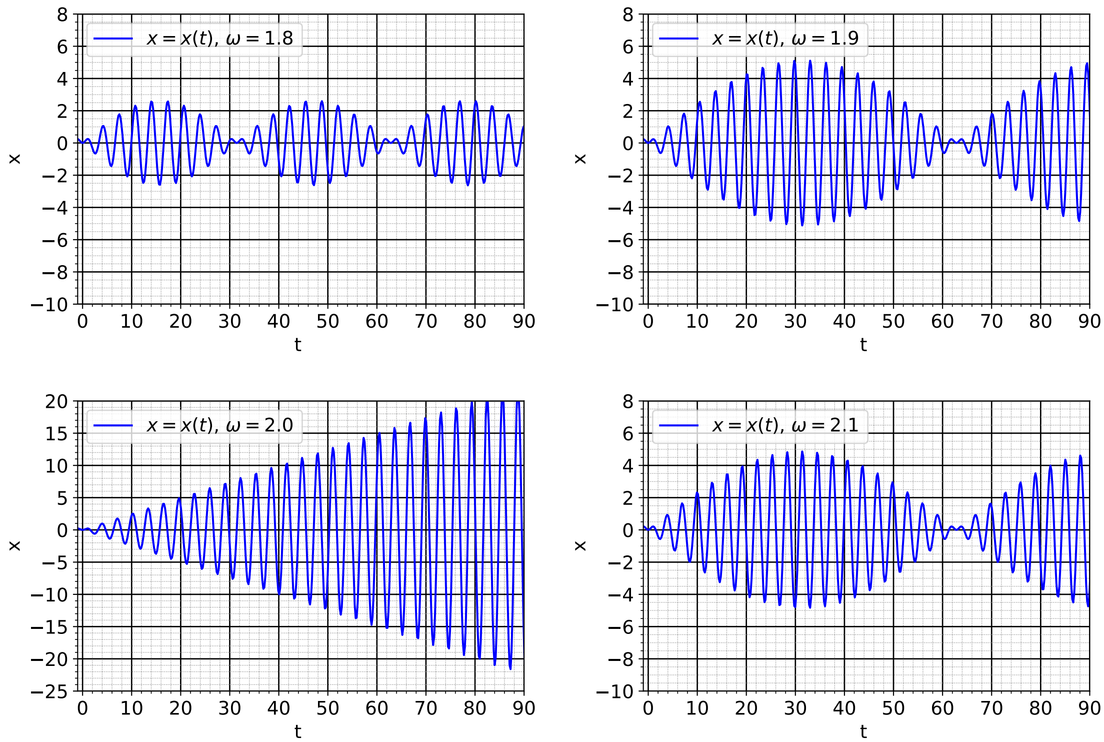


FIGURE 10.2. Graphs of solutions of $x'' + 4x = \cos(\omega t)$, $x(0) = x'(0) = 0$ for various values of ω. "Resonance" occurs when $\omega = 2.0$.

The initial conditions $x(0) = \dfrac{F_0/m}{\omega_0^2 - \omega^2} + C_1 = 0$ and $x'(0) = C_2 = 0$ together give

$$x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \quad (\cos(\omega t) - \cos(\omega_0 t))\ .$$

Applying a trig identity from Appendix A leads to the formula

$$\boxed{x(t) = \left\{ \left(\frac{2F_0/m}{\omega_0^2 - \omega^2} \right) \sin\left(\frac{\omega_0 - \omega}{2}\, t \right) \right\} \sin\left(\frac{\omega_0 + \omega}{2}\, t \right)}\,.$$

When ω is close to ω_0, the term in braces corresponds to a slowly varying amplitude, and the term $(\omega_0 + \omega)/2 \approx \omega_0$ corresponds to a high frequency. This leads to the phenomenon of *beats*, which is illustrated in Figure 10.2. As ω approaches ω_0 the frequency of the beats decreases, leading to the solution when $\omega = \omega_0$:

$$\lim_{\omega\to\omega_0} \left\{ \left(\frac{2F_0/m}{\omega_0^2 - \omega^2} \right) \sin\left(\frac{\omega_0 - \omega}{2}\, t \right) \right\} \sin\left(\frac{\omega_0 + \omega}{2}\, t \right) = \frac{F_0/m}{2\omega_0}\, t\, \sin(\omega_0 t)\,.$$

To see this, apply l'Hôpital's to the expression in braces:

$$\lim_{\omega\to\omega_0} \left\{ \frac{2F_0/m}{\omega_0^2 - \omega^2} \sin\left(\frac{\omega_0 - \omega}{2} t \right) \right\} = \lim_{\omega\to\omega_0} \frac{(2F_0/m) \sin\left(\frac{(\omega_0 - \omega)}{2} t \right)}{(\omega_0 + \omega)(\omega_0 - \omega)} = \lim_{\omega\to\omega_0} \frac{(F_0/m) t \cos\left(\frac{\omega_0 - \omega}{2} t \right)}{(\omega_0 + \omega)} = \frac{F_0/m}{2\omega_0} t$$

10.2. Forced Oscillations with Damping

The analysis of a forced, damped harmonic oscillator is similar to that of the forced (undamped) harmonic oscillator. In this case, the system is modeled by the differential equation

$$m\,x'' + \gamma\,x' + k\,x = F_0\,\cos(\omega\,t)$$

The general solution is of the form

$$x(t) = x_p(t) + C_1\,x_1(t) + C_2\,x_2(t)\,,$$

where $x_1(t)$ and $x_2(t)$ are solutions of the homogeneous differential equation and $x_p(t)$ is a particular solution of the nonhomogeneous equation.

For $\gamma > 0$ there are three cases, based on the number and type of the roots of the characteristic polynomial $mr^2 + \gamma r + k$:

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \omega_0^2}\,,$$

where $\omega_0 = \sqrt{k/m}$ (natural frequency). In all three cases, the term

$$x_h(t) = C_1 x_1(t) + C_2 x_2(t)$$

has the property that

$$\lim_{t\to\infty} x_h(t) = 0\,.$$

The function $x_h(t)$ is called a *transient* because when t is large it can be ignored. For t sufficiently large solution is approximately given by $x_p(t)$. That is

$$x(t) \approx x_p(t) \text{ for large } t$$

Using the method of undtermined coeffients alows us to write $x_p(t)$ in the form

$$x_p(t) = A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t) = R(\omega)\cos(\omega t - \phi)\,.$$

The function $x_p(t)$ is called the *steady state solution.* .

The computation of $x_p(t)$ using complex-valued functions proceeds as follows: Set $x_p(t) = \text{Re}\,(z_p(t)$, where $z_p(t) = Ce^{i\omega t}$. Substitute $z_p(t)$ into the equation $mz_p'' + \gamma z_p' + kz_p = F_0 e^{i\omega t}$ and note that $k = m\omega_0^2$ to get

$$(m(i\omega)^2 + \gamma(i\omega) + k)Ce^{i\omega t} = (m(\omega_0^2 - \omega^2) + i\gamma\omega)Ce^{i\omega t} = F_0 e^{i\omega t}\,.$$

Therefore,

$$(m(\omega_0^2 - \omega^2) + i\gamma\omega)C = F_0\,.$$

Putting the coefficient of C into polar form gives

$$\left(\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\,e^{i\delta}\right))\,C = F_0\,,$$

where $\boxed{\tan(\delta) = \dfrac{\gamma\omega}{m(\omega_0^2 - \omega^2)} \quad 0 < \delta < \pi}$ Hence,

$$C = \frac{F_0}{\sqrt{(m^2(\omega_0^2 - \omega^2)^2 + \gamma^2)}} e^{-i\delta}\,.$$

Consequently,

$$\boxed{x_p(t) = \text{Re}\left(\frac{F_0}{\sqrt{(m^2(\omega_0^2 - \omega^2)^2 + \gamma^2)}} e^{-i\delta} e^{i\omega t}\right) = \frac{F_0}{\sqrt{(m^2(\omega_0^2 - \omega^2)^2 + \gamma^2)}}\cos(\omega t - \delta)} \tag{10.3}$$

REMARK 10.2. In this case, the computation without using complex-valued functions is more involved. That computation proceeds as follows:

Substitute $x_p(t) = A\cos(\omega t) + B\sin(\omega t)$ into $m\,x'' + \gamma\,x' + k\,x = F_0\,\cos(\omega\,t)$ to get

$$\begin{aligned}&\big\{(-m\omega^2 + k)\,A + \gamma\,\omega\,B\big\}\cos(\omega t) + \\ &\big\{-\gamma\,\omega\,A + (-m\omega^2 + k)\,B\big\}\sin(\omega t) = F_0\,\cos(\omega\,t)\,.\end{aligned}$$

Thus,

$$(-m\omega^2 + k)\,A + \gamma\,\omega\,B = F_0 \text{ and } -\gamma\,\omega\,A + (-m\omega^2 + k)\,B = 0$$

Solving for A and B (requiring a bit of algebra), and simplifying gives

$$A = \frac{F_0\,m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \text{ and } B = \frac{F_0\,\gamma\omega}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\,.$$

Set

$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \qquad \cos(\delta) = \frac{m(\omega_0^2 - \omega^2)}{\Delta^2} \text{ and } \sin(\delta) = \frac{\gamma\omega}{\Delta^2}$$

Then $A = \dfrac{F_0}{\Delta}\cos(\delta)$ and $B = \dfrac{F_0}{\Delta}\sin(\delta)$

Hence,

$$x_p(t) = \frac{F_0}{\Delta}\{\cos(\omega t)\cos(\delta) + \sin(\omega t)\sin(\delta)\} = R(\omega)\,\cos(\omega\,t - \delta)\,,$$

where

$$R(\omega) = \frac{F_0}{\Delta} = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \text{ and } \tan(\delta) = \frac{\gamma\omega}{m(\omega_0^2 - \omega^2)}\,.$$

Resonant frequency. By analogy with the undamped case, it is useful to find the value of ω that results in the maximum amplitude of the oscillations in $x_p(t)$. In other words, we seek the value of ω that maximizes $R(\omega)$. This is again called the *resonant frequency.*

To get a feel for what happens, set

$$m = 1 \quad k = 1 \quad F_0 = 1$$

and graph $R(\omega)$ for a increasing values of the damping constant γ. Notice that (as one would expect) the maximum value of $R(\omega)$ decreases as the damping constant increases.

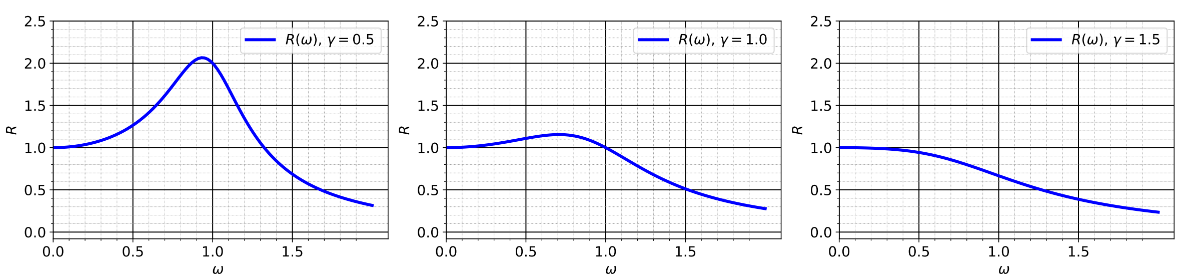


FIGURE 10.3. The amplitude $R(\omega)$ for three values of damping constant γ. Notice that as γ increases, the value of ω maximizing $R(\omega)$ decreases and eventually becomes 0, corresponding to a constant applied force.

The resonant frequency coincides with the value of ω at which the denominator

$$f(\omega) = m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\,.$$

of $R(\omega)$ achieves its minimum. Differentiating with respect to ω gives

$$f'(\omega) = 2\omega\,\left(\gamma^2 + 2m^2(\omega^2 - \omega_0^2)\right) = 4m^2\,\omega\,\left(\omega^2 - \left(\omega_0^2 - \frac{\gamma^2}{2m^2}\right)\right).$$

The critical values of $f(\omega)$ are, therefore, $\omega = 0$ and $\omega = \sqrt{\omega_0^2 - \dfrac{\gamma^2}{2m^2}}$.

Conclusion: If $\omega_0^2 > \dfrac{\gamma^2}{2m^2}$ or (equivalently) if $\left(\dfrac{\gamma/m}{\omega_0}\right)^2 < 2$, the resonant frequency is

$$\omega_{max} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2m^2}} = \omega_0\sqrt{1 - \frac{1}{2}\left(\frac{\gamma/m}{\omega_0}\right)^2}.$$

Otherwise, the maximum amplitude is achieved for $\omega = 0$, corresponding to a constant driving force.

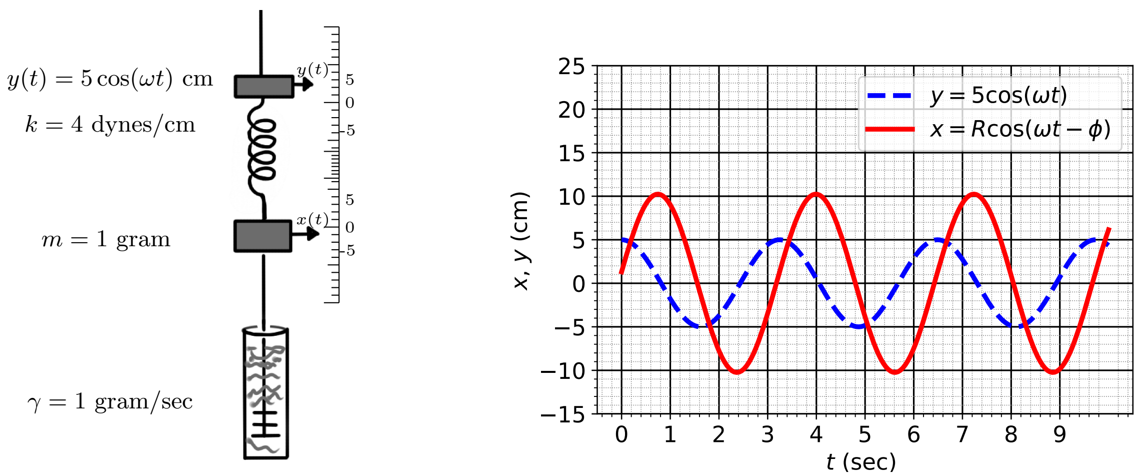


FIGURE 10.4. The oscillating plunger (dashed blue sine curve) causes the mass m to oscillate (red sine curve).

EXAMPLE 10.2. Consider the mechanical system pictured in Figure 10.4. Assume that in its rest configuration $x = 0$ and $y = 0$, where the forces exerted on the mass by gravity and the spring cancel. The net force exerted on the mass by gravity, the spring, and the damping mechanism is then

$$F = -k(x - y(t) - \gamma x'.$$

Applying Newton's second law of motion shows that the position of the mass $x = x(t)$ satisfies the differential equation

$$mx'' = -\gamma\,x' - k(x - y(t)) \text{ or } mx'' + \gamma\,x' + k\,x = ky(t).$$

Assume for simplicity that $m = 1$ gram, $k = 4.0$ dynes/cm, and $\gamma = 1.0$ grams/sec.

Finally assume that the plunger at the top of the figure moves up and down according to the rule $y(t) = 5\cos(\omega t)$ cm; causing the mass to also move up and down. Ignoring transients, $x(t)$ is of the form

$$x(t) = R(\omega)\cos(\omega t - \phi)$$

where both $R(\omega)$ and ϕ depend on ω. Find the value of ω that maximizes $R(\omega)$.

SOLUTION. Set $x(t) = \mathrm{Re}\,(z(t))$, where $z(t)$ satisfies the differential equation

$$mz'' + \gamma\,z' + k\,z = k\left(5e^{i\omega t}\right).$$

Using the values of m, γ, and k above this simplifies to

$$z'' + z' + 4z = 20\, e^{i\omega t}\,.$$

Substituting $z(t) = Ae^{i\omega t}$ into this equation gives

$$\{(4-\omega^2) + i\omega\}\}A = 20\,,$$

whose polar form is

$$\sqrt{(4-\omega^2)^2+\omega^2}\, e^{\phi i} A = 20\,, \qquad \text{where } \tan(\phi) = \frac{\omega}{4-\omega^2}\,, \text{ and } 0 < \phi < \pi\,.$$

It follows that

$$z(t) = \frac{20}{\sqrt{(4-\omega^2)^2+\omega^2}} e^{-i\phi} e^{i\omega t}$$

Consequently,

$$x(t) = R(\omega)\cos(\omega t - \phi) = \frac{20}{\sqrt{(4-\omega^2)^2+\omega^2}}\cos(\omega t - \phi)\,.$$

To maximize $R(\omega)$ is suffices to minimize $f(\omega) = (4-\omega^2)^2 + \omega^2$, which can be accomplished by solving

$$f'(\omega) = -4(4-\omega^2)\omega + 2\omega = 0$$

for ω. We can ignore the spurious solutions $\omega = 0$, and $\omega = -\sqrt{7/2}$. (Why?) Hence,

$$\omega = \sqrt{\frac{7}{2}} \approx 1.87\text{sec}^{-1}\,, R(\omega) = \frac{20}{\sqrt{(4-7/2)^2+(7/2)}} \approx 10.33 \text{ and } \phi = \arctan\left(\frac{\omega}{4-\omega^2}\right) \approx 1.31.$$

Thus, the amplitude of the oscillations in x is about twice the amplitude of the oscillations in y. (See Figure 10.4.)

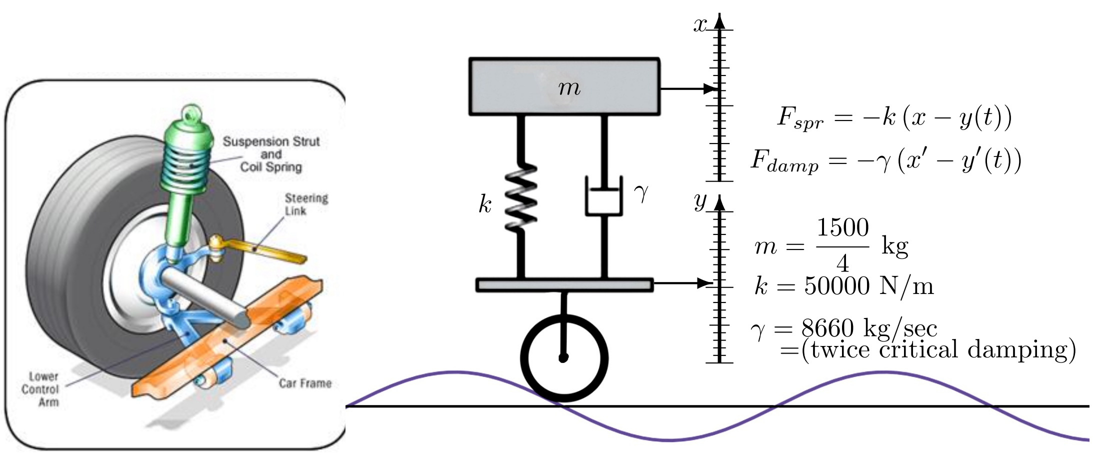


FIGURE 10.5. Left: sketch of strut on an automobile. Right: Simplified model of the system. The values of m, k, and γ in the figure are similar to those found in automobiles. The mass is divided by four because the weight of an automobile is distributed over four wheels.

EXAMPLE 10.3. (AUTOMOBILE STRUTS) A similar analysis can be done for the mechanical system modeling the struts on an automobile. Place the x-axis and the y-axis so that $x = y = 0$ at equilibrium, so the forces of gravity and the spring cancel—for this reason we make no mention of the force of gravity. Then, as shown in Figure 10.5, the spring force F_{spr} and the damping force F_{damp} both depend on the relative values of x and y. Newton's second law of motion then implies that the function $x = x(t)$ is a solution of the differential equation

$$mx'' = -\gamma(x' - y'(t)) - k(x - y(t)) \text{ or } m\,x'' + \gamma\,x' + k\,x = \gamma\,y'(y) + k\,y(t)\,.$$

Assume that the automobile is moving at a constant speed along a straight (but not flat!) road. For simplicity, also assume that the rise and fall of the road is given by the sine function

$$y(t) = a\cos(\omega t)\,,$$

where $a > 0$ is a constant and ω depends on the speed of the car. The steady-state solution is then

$$x(t) = aR(\omega)\cos(\omega t - \phi)\,,$$

where both $R(\omega)$ and ϕ have yet to be determined. Since both $R(\omega)$ and ϕ are independent of a, there is no loss of generality in setting $a = 1$.

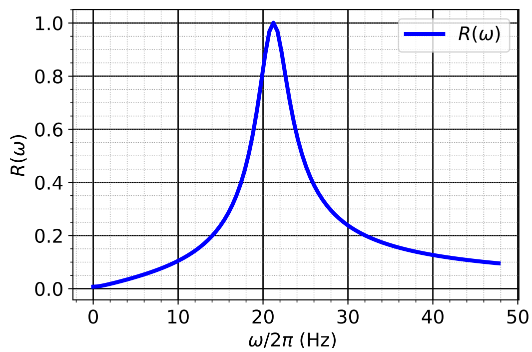


FIGURE 10.6. Notice that the maximum response is only slightly above 1. This implies that the amplitude of oscillations in the road is never amplified by the struts, and, in fact, it is reduced except at frequencies of around 21.5 Hz (cycles per second).

As in earlier examples, set $x(t) = \mathrm{Re}\,(z(t))$, where $z(t) = Ae^{i\omega t}$ is a solution of the complex differential equation

$$m\,z'' + \gamma\,z' + kz = k\,e^{i\omega t} + \gamma\,(e^{i\omega t})' = (k + i\gamma\omega)e^{i\omega t}$$

Proceeding as above we arrive at the equation $\left\{(-m\omega^2 + k) + \gamma\,\omega\,i\right\} Ae^{i\omega t} = (k + \gamma\,\omega i)e^{i\omega t}$. Hence,

$$z(t) = \frac{k + \gamma\,\omega\,i}{m(\omega_0^2 - \omega^2) + \gamma\omega i}e^{i\omega t} = \frac{\omega_0^2 + (\gamma/m)\,\omega\,i}{(\omega_0^2 - \omega^2) + (\gamma/m)\omega i}e^{i\omega t}$$

and

$$R(\omega) = \left|\frac{\omega_0^2 + (\gamma/m)\,\omega\,i}{(\omega_0^2 - \omega^2) + (\gamma/m)\omega i}\right| = \sqrt{\frac{\omega_0^4 + (\gamma/m)^2\omega^2}{(\omega_0^2 - \omega^2)^2 + (\gamma/m)^2\omega^2}}\,.$$

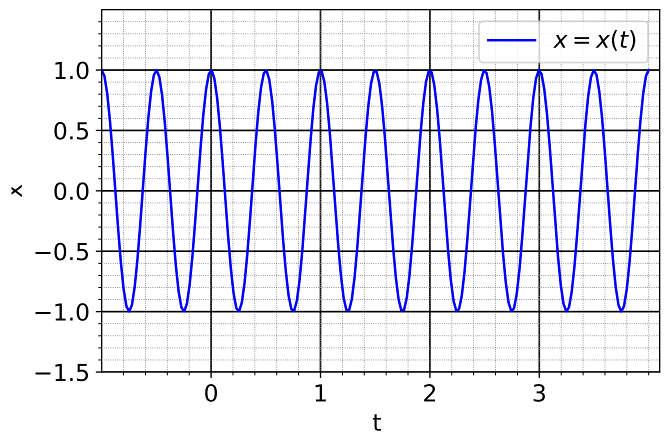
Figure 10.6 shows the graph of $R(\omega)$ for the numerical values of m, γ, and k given in Figure 10.5.

Exercises 6.

(1) A weight of mass $m = 5$ kg is suspended from a spring with unknown spring constant k. The weight is free to move up and down. Ignoring friction, its position relative to its equilibrium position satisfies a differential equation of the form

$$mx'' + kx = 0\,,$$

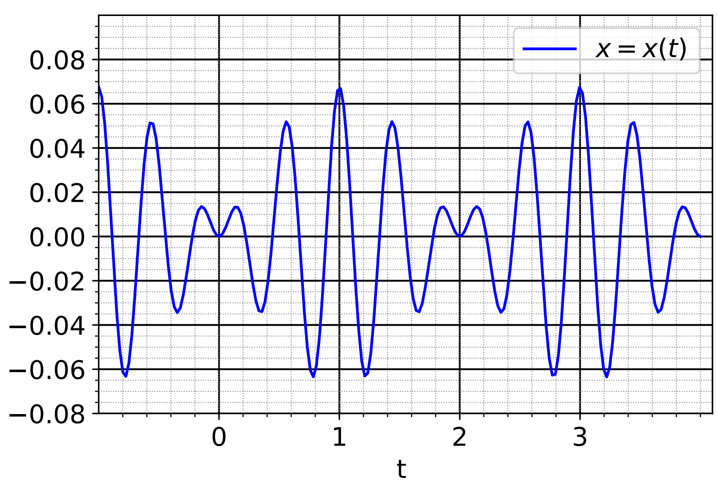
where t denotes time measured in seconds and x denotes its position measured in meters. To find the spring constant, the spring is set in motion and the graph of $u(t)$ plotted. The result is shown in the following figure (horizontal axis is t, vertical axis is x):



In a subsequent experiment, an external force of the form $F(t) = F_0\,\cos(\omega t)$ is applied to the mass so that the function $u(t)$ now obeys the differential equation

$$mx'' + kx = F_0\,\cos(\omega t)\,.$$

The graph of the position $x(t)$ in that experiment is shown in the graph below.



You estimated the the spring constant k in a previous exercise.) If you did not do it, do it now.) (*Your answer can most easily be expressed in terms of* π.)

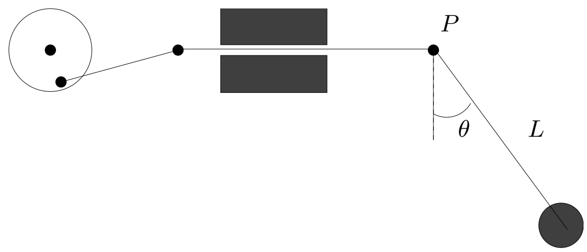
(a) Using your estimate of k, estimate as best you can the frequency ω of the applied force. (*Your answer can again most easily be expressed in terms of* π.)

(b) Estimate as best you can the amplitude F_0 of the applied force.

(2) A simple pendulum of mass m and length L is hinged at a point P (see figure). If the wheel at the left of the figure rotates at a rate of ω radians/second it forces the point P to move periodically back and forth. For small angle θ (where $\sin(\theta) \approx \theta$) the angle θ satisfies the differential equation

$$L\frac{d^2\theta}{dt^2} + g\theta = A\omega^2\cos(\omega t)\,.$$

Assume, for simplicity that $L = 1$ meter, $A = 1$ meter/sec^2, $\omega = 1$ rad/sec and $g = 9.8$ meter/sec^2. Find the solution that satisfies the initial conditions $\theta(0) = \theta'(0) = 0$.



(3) A ball of mass 1 kilogram moves in a viscous fluid. The viscous force on the ball is given by $-cv$, where v is the speed of the ball measured in meters per second, and $c = 2$ newton-sec/meter. An external force is applied to the ball along a fixed axis and with magnitude

$$F(t) = 2\cos(t)\ \text{N}$$

(t is time measured in seconds.) Let $y(t)$ be the displacement of the ball along the axis of the external force and assume that at time $t = 0$ the ball is at rest and $y = 0$. Find $y(t)$. Ignore gravity.

(4) In an experiment in a space station a charged metal sphere of mass 2 grams is placed in a graduated cylinder containing a viscous fluid. The sphere is free to move up and down and its vertical position is given by the variable y, measured in centimeters. The viscous force is given by the formula $-cy'$ with $c = 2$ dyne-sec/cm.

At time $t = 0$ seconds $y = 0$ cm and $y' = 0$ cm/sec and an external force

$$F(t) = 2\sin(10\pi t)\ \text{dynes}$$

is applied to the sphere via an electric field. After 1 second it is turned off. What are the position an velocity of the sphere when $t = 2$ seconds?

(5) A spring-mass system has spring constant 3 N/m (i.e. 3 Newtons per meter). A mass of 2 kg is attached to the spring and the motion takes place in a viscous fluid that offers a resistance (measured in Newtons) numerically equal to twice the magnitude of the instantaneous velocity (measured in meters per second).

Let u denote the displacement of the mass from its equilibrium position, If the system is driven by an external force of $3\cos(3t) - 2\sin(3t)$ N, determine the formula for $u(t)$ ignoring all "transients". Express your answer in the form $u(t) = A\cos(\omega t + \phi)$.

Part 3

Laplace Transforms

CHAPTER 11

Laplace Transforms

This chapter is an introduction to *Laplace transforms*, which provide an alternate way to solve initial value problems of the form

$$\begin{cases} L[y] = ay'' + by' + cy = f(t) & , a,\ b,\ c \text{ constant} \\ y(0) = y_0\,, y'(0) = y_0' \end{cases} \tag{11.1}$$

that is particularly useful when the forcing function $f(t)$ has discontinuities. The idea is to transform the initial value problem into a algebraic equation, solve the algebraic equation for the transformed of the solution, and then inverse transform to obtain the solution of the initial value problem.

More precisely, the Laplace transform turns the initial value problem (11.1) into the equation

$$(as^2 + bs + c)Y(s) = a\,y_0\,s + (b\,y_0 + a\,y_0') + F(s)\,, \tag{11.2}$$

where $Y(s)$ is the Laplace transform of $y(t)$ and $F(s)$ is the Laplace transform of $f(t)$. Solving Equation (11.2) for $Y(s)$ gives

$$\boxed{Y(s) = \frac{a\,y_0\,s + (b\,y_0 + a\,y_0')}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}\,,} \tag{11.3}$$

an explicit formula for the Laplace transform of the solution. Computing the *inverse Laplace transform* then solves the initial value problem.

Putting this idea into practice requires knowing how to compute $F(s)$ from $f(t)$ and how to compute $y(t)$ from $Y(s)$. In much the same way that derivatives and integrals are computed from a few basic properties (e.g. the product rule and integration by parts) together with a table of integral, so can Laplace transforms and inverse Laplace transforms be computed from a few basic rules, together with a table of Laplace transforms (see Appendix C).

11.1. Computing Laplace transforms

Suppose $f(t)$ is a function defined for all t with $0 \le t < \infty$. Its *Laplace transform* is the function

$$\mathcal{L}\{f\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)\,dt \tag{11.4}$$

provided this integral converges.[12]

Remark 11.1. Notice that values of $f(t)$ for $t < 0$ have no effect on its Laplace transform. When we use Laplace transforms, we are only interested in solving the initial value problem (11.1) for $t \ge 0$, that is, in the future. We get no information about the past. For all practical purposes, we might as well assume that $f(t) = 0$ for $t < 0$.

[12]Convergence is only briefly discussed in these notes. For virtually all functions encountered in practice, the integral converges when s is sufficiently large.

NOTATION. The notation $\mathcal{L}\{f\}$ is awkward. It is often more convenient to denote the Laplace transform of $f(t)$ by $F(s)$. Similarly, we write $Y(s) = \mathcal{L}\{f\}$, $G(s) = \mathcal{L}\{g\}(s)$, etc. For instance, $\mathcal{L}\{\cos(t)\}$ and $\mathcal{L}\{\cos\}$ both denote the Laplace transform of the function cos.

EXAMPLE 11.1. Here are three cases where the Laplace transform can be directly computed from the definition.

(a) If $f(t) = 1$, then

$$F(s) = \int_0^\infty e^{-st}\,dt = \lim_{A\to\infty}\int_0^A e^{-st}\,dt = \lim_{A\to\infty} \left.\frac{-e^{-st}}{s}\right|_0^A = \lim_{A\to\infty}\left(-e^{-st}+1\right)\frac{1}{s} = \frac{1}{s}\,.$$

Therefore,

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$$

Notice that the integral converges to $1/s$ only for $s > 0$ and diverges for $s < 0$.

(b) If $f(t) = e^{at}$, then

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{(-(s-a)t}\,dt \\ &= \lim_{A\to\infty}\int_0^A e^{-(s-a)t}\,dt = \lim_{A\to\infty}\left.\frac{-e^{-(s-a)t}}{(s-a)}\right|_0^A = \lim_{A\to\infty}\left(-e^{-(s-a)t}+1\right)\frac{1}{s-a} = \frac{1}{s-a} \end{aligned}$$

Therefore,

$$\boxed{\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}}$$

Notice that the integral converges to $1/(s-a)$ only for $s > a$ and diverges for $s < a$.

(c) Finally,

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}\,.}$$

Check this yourself. (Hint: use integration by parts.)

EXAMPLE 11.2. Sometimes, using complex valued functions simplifies the computation of the Laplace transform. Consider the following two cases:

$$\mathcal{L}\{\sin(at)\} = \int_0^\infty e^{-st}\sin(at)\,dt \text{ and } \mathcal{L}\{\cos(at)\} = \int_0^\infty e^{-st}\cos(at)\,dt\,.$$

Both integrals could be evaluated directly, but the computations are messy, involving integration by parts twice. It's easier to use complex-valued functions as follows:

Since $e^{iat} = \cos(at) + i\sin(at)$,

$$\mathcal{L}\left\{e^{iat}\right\} = \mathcal{L}\{\cos(at)\} + i\mathcal{L}\{\sin(at)\}\,.$$

Consequently,

$$\begin{aligned} \mathcal{L}\left\{e^{iat}\right\} &= \int_0^\infty e^{-st}e^{iat}\,dt = \int_0^\infty e^{-(s-ia)t}\,dt \\ &= \left.\frac{-e^{-(s-ia)t}}{s-ia}\right|_0^\infty \\ &= \frac{1}{s-ia} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2} \end{aligned}$$

Therefore,

$$\boxed{\mathcal{L}\left\{\cos(at)\right\} = \frac{s}{s^2+a^2} \text{ and } \mathcal{L}\left\{\sin(at)\right\} = \frac{a}{s^2+a^2}} \tag{11.5}$$

A similar computation shows that

$$\boxed{\mathcal{L}\left\{e^{(a+ib)t}\right\} = \frac{1}{s-(a+ib)}} \tag{11.6}$$

To see this, compute as before:

$$\begin{aligned}
\mathcal{L}\left\{e^{(a+bi)t}\right\} &= \int_0^\infty e^{-st}e^{(a+bi)t}\,dt \\
&= \int_0^\infty e^{-(s-(a+bi))t}\,dt \\
&= \lim_{A\to\infty} \left.\frac{-e^{-(s-(a+bi))t}}{s-(a+bi)}\right|_0^A \\
&= \frac{1}{s-(a+bi)}\,.
\end{aligned}$$

Note that the last step is only valid for $s > a$.

Because $e^{(a+ib)t} = e^{at}\cos(bt) + ie^{at}\sin(bt)$ and $\dfrac{1}{s-(a+ib)} = \dfrac{s-a}{(s-a)^2+b^2} + i\dfrac{b}{(s-a)^2+b^2}$, it follows that

$$\boxed{\mathcal{L}\left\{e^{at}\cos(bt)\right\} = \frac{s-a}{(s-a)^2+b^2}} \text{ and } \boxed{\mathcal{L}\left\{e^{at}\sin(bt)\right\} = \frac{b}{(s-a)^2+b^2}}\,.$$

11.2. Properties of the Laplace transform

Rather than continuing to derive Laplace transforms of specific functions, it is more efficient to find general properties of the Laplace transform.

The Laplace transform is a *linear operator*. This means that if $f(t)$ and $g(t)$ are functions and a and b are numbers, then

$$\boxed{\mathcal{L}\left\{af(t)+bg(t)\right\} = aF(s)+bG(s)\,,} \tag{11.7}$$

where $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$, respectively. Linearity follows immediately from linearity of the definite integral:

$$\mathcal{L}\left\{af(t)+bg(t)\right\} = \int_0^\infty e^{-st}(af(t)+bg(t))\,dt = a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt\,.$$

Because of linearity, we can decompose the Laplace transform of a sum of functions as a sum of the Laplace transform of each of the summands.

Example 11.3. By linearity and the table of Laplace transforms,

$$\mathcal{L}\left\{5e^{-2t}-3\sin(4t)\right\} = 5\mathcal{L}\left\{e^{-2t}\right\} - 3\mathcal{L}\left\{\sin(4t)\right\} = 5\frac{1}{s-(-2)} - 3\frac{4}{s^2+16} = \frac{5}{s+2} - \frac{12}{s^2+16}\,.$$

The next theorem shows that the Laplace transform of the derivatives of a function can be expressed in terms of the Laplace transform of the function, itself.

THEOREM 4. *Suppose $g(t)$ is a continuously differentiable with Laplace transform $G(s)$, then*

$$\boxed{\mathcal{L}\left\{g'\right\} = sG(s) - g(0).}$$

If $g(t)$ is has continuous second derivatives, then

$$\boxed{\mathcal{L}\left\{g''\right\} = s^2G(s) - sg(0) - g'(0)\,.}$$

PROOF OF THEOREM. In order for $\mathcal{L}\left\{g\right\}$ to exist, $g(t)$ must be *piecewise continuous* (required for the integral to exist) and it must be of *exponential order*: there are constants M and c so that $y(t) \leq Me^{ct}$. This implies that when $s > c$,

$$\lim_{a\to\infty} g(a)e^{-sa} = 0.$$

To compute

$$\mathcal{L}\left\{g'\right\} = \int_0^\infty e^{-st}g'(t)\,dt,$$

we use integration by parts: $u = e^{-st}$, $dv = g'(t)dt$, so $du = -se^{-st}\,dt$ and $v = g(t)$, so

$$\begin{aligned}
\mathcal{L}\left\{g'\right\} &= \int_0^\infty e^{-st}g'(t)\,dt \\
&= e^{-st}g(t)\big|_0^\infty + \int_0^\infty se^{-st}g(t)\,dt \\
&= -g(0) + s\int_0^\infty e^{-st}g(t)\,dt \\
&= sG(s) - g(0),
\end{aligned}$$

as desired.

If, in addition, $g(t)$ has continuous second derivatives, apply the first part of the theorem, twice as follows:

$$\mathcal{L}\left\{g''(t)\right\} = s\mathcal{L}\left\{g'(t)\right\} - g'(0) = s(sG(s) - g(0)) - g'(0) = s^2G(s) - sg(0) - g'(0)\,.$$

$\square$

Linearity and Theorem 4 are key ingredients for solving initial value problems. For suppose we have a linear constant coefficient differential equation

$$ay'' + by' + cy = f(t)\,,$$

together with the initial conditions $y(0) = y_0$ and $y'(0) = y_0'$. By linearity, applying $\mathcal{L}\left\{-\right\}$ to the differential equation gives

$$a\mathcal{L}\left\{y''\right\} + b\mathcal{L}\left\{y'\right\} + c\mathcal{L}\left\{y\right\} = \mathcal{L}\left\{f\right\} = F(s).$$

By Theorem 4 applying $\mathcal{L}\left\{-\right\}$ to the terms $\mathcal{L}\left\{y''\right\}$ and $\mathcal{L}\left\{y'\right\}$ gives

$$a(s^2Y(s) - sy_0 - y_0') + b(sY(s) - y_0) + cY(s) = F(s)\,,$$

which simplifies to

$$(as^2 + bs + c)Y(s) - (ay_0s + ay_0' + by_0) = F(s)\,. \tag{11.8}$$

This equation can be solved for $Y(s)$:

$$Y(s) = \frac{F(s)}{as^2 + bs + c} + \frac{ay_0s + ay_0' + by_0}{as^2 + bs + c}\,. \tag{11.9}$$

EXAMPLE 11.4. Consider the initial value problem $y'' - 3y' + 2y = 0$, $y(0) = 2$, $y'(0) = 1$. Applying the Laplace operator $\mathcal{L}\{-\}$, the equation becomes

$$(s^2Y(s) - 2s - 1) - 3(sY(s) - 2) + 2Y(s) = 0,$$

or

$$(s^2 - 3s + 2)Y(s) = 2s - 5.$$

Therefore,

$$Y(s) = \frac{2s-5}{s^2-3s+2} = \frac{2s-5}{(s-1)(s-2)} = \frac{3}{s-1} + \frac{-1}{s-2}.$$

We write $\mathcal{L}^{-1}\{-\}$ for the operator that undoes the Laplace transform: if $Y(s) = \mathcal{L}\{y\}$, then $\mathcal{L}^{-1}\{Y\}(t) = y(t)$. Using this notation,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{3}{s-1} + \frac{-1}{s-2}\right\}(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t) = 3e^t - e^{2t},$$

where we have used linearity of $\mathcal{L}\{-\}$ and (from Appendix C) the formula $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$.

In the above example, we implicitly assumed that $Y(s)$ determines $y(t)$. In fact, this is the case, as the next theorem shows.

THEOREM 5. *Suppose that f and g are continuous. Let $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. If for some $c > 0$, $F(s) = G(s)$ for all $s > c$, then $f(t) = g(t)$ for all $t > 0$.*

REMARK 11.2. This theorem is *not* obvious, and in fact the proof is difficult in beyond the scope of this course.

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\boxed{\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a) \text{ (the \textit{exponential shift formula})}.} \tag{11.10}$$

PROOF.

$$\mathcal{L}\left\{e^{at}f(t)\right\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a).$$

□

EXAMPLE 11.5. Since $\mathcal{L}\{\cos(bt)\} = s/(s^2 + b^2)$,

$$\mathcal{L}\left\{e^{at}\cos(bt)\right\} = \frac{s-a}{(s-a)^2 + b^2}.$$

Similarly,

$$\mathcal{L}\left\{e^{at}\sin(bt)\right\} = \frac{b}{(s-a)^2 + b^2}.$$

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\boxed{\mathcal{L}\{tf(t)\}) = -F'(s).} \tag{11.11}$$

PROOF. It's easiest to work backwards as follows:

$$F'(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{d}{ds}\left(e^{-st}f(t)\right)dt = -\int_0^\infty e^{-st}tf(t)dt = -\mathcal{L}\{tf(t)\}.$$

□

EXAMPLE 11.6. Since $\mathcal{L}\{1\} = \frac{1}{s}$, $\mathcal{L}\{t\} = -\left(\frac{1}{s}\right)' = \frac{1}{s^2}$.

Since $\mathcal{L}\{t\} = \frac{1}{s^2}$, $\mathcal{L}\left\{t^2\right\} = \mathcal{L}\{t \cdot t\} - \left(\frac{1}{s^2}\right)' = \frac{2}{s^3}$.

More generally, suppose $\mathcal{L}\left\{t^{n-1}\right\} = \frac{(n-1)!}{s^n}$. Then, $\mathcal{L}\{t^n\} = \mathcal{L}\left\{t \cdot t^{n-1}\right\} - \left(\frac{(n-1)!}{s^n}\right)' = \frac{(n)!}{s^{n+1}}$. Therefore, by mathematical induction,

$$\boxed{\mathcal{L}\{t^n\} = \frac{(n)!}{s^{n+1}}\,.}$$

for all positive integers.

EXAMPLE 11.7. Since, $\mathcal{L}\left\{t^3\right\} = \frac{6}{s^4}$,

$$\mathcal{L}\left\{t^3 e^{5t}\right\} = \frac{6}{(s-5)^4}$$

EXAMPLE 11.8. Since, $\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$,

$$\mathcal{L}\{t\sin(at)\} = -\left(\frac{a}{s^2+a^2}\right)' = \frac{(2as}{(s^2+a^2)^2}$$

Similarly, since $\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$,

$$\mathcal{L}\{t\cos(at)\} = -\left(\frac{s}{s^2+a^2}\right)' = \frac{(s^2-a^2}{(s^2+a^2)^2}$$

Suppose $\mathcal{L}\{f(t)\} = F(s)$, then

$$\boxed{\mathcal{L}\{f(at)\} = \frac{1}{a}F(s/a)\,.}$$

PROOF. Compute as follows, using the "u-substitution" $u = at$, $du = adt$:

$$\mathcal{L}\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt = \int_0^\infty e^{-s(u/a)} f(u)\frac{du}{a} = \frac{1}{a}\int_0^\infty e^{-(s/a)u} f(u)du = \frac{1}{a}F(s/a)\,.$$

□

EXAMPLE 11.9. Because $\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}$, it follows that

$$\mathcal{L}\{\cos(at)\} = \frac{1}{a}\frac{(s/a)}{(s/a)^2+1} = \frac{s}{s^2+a^2}\,.$$

11.3. Computing the Inverse Laplace Transform

In Section 11.2, we found a general formula for the Laplace transform of the solution of an initial value problem. To find the solution, itself, we have to compute the inverse Laplace transform.

Computing the inverse Laplace transform often involves the *partial fraction expansion*[13] of the Laplace transform.

EXAMPLE 11.10. Find the inverse Laplace transform of $F(s) = \dfrac{3s}{s^2 - s - 6}$.

SOLUTION. Compute the partial fractions expansion of $F(s)$ as follows:

$$\frac{3s}{s^2 - s - 6} = \frac{3s}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} = \frac{A(s+2) + B(s-3)}{(s-3)(s+2)}$$

Comparing numerators gives $3s = A(s+2) + B(s-3)$. Set $s = 3$ to conclude that $9 = A(5)$ or $A = 5/9$. Set $s = -2$ to conclude that $-6 = B(-5)$ or $B = 6/5$. Hence

$$\frac{3s}{s^2 - s - 6} = \frac{5/9}{s-3} + \frac{6/5}{s+2}\,.$$

We can now use the table of Laplace transforms to compute as follows:

$$\mathcal{L}^{-1}\left\{\frac{3s}{s^2 - s - 6}\right\} = \frac{5}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = \frac{5}{9}e^{3t} + \frac{6}{5}e^{-2t}\,.$$

EXAMPLE 11.11. Find the inverse Laplace transform of $F(s) = \dfrac{8s^2 - 4s + 12}{s(s^2+4)}$.

SOLUTION. First compute the partial fractions expansion of $F(s)$:

$$\frac{8s^2 - 4s + 12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{A(s^2+4) + s(Bs + C)}{s(s^2+4)} = \frac{(A+B)s^2 + Cs + 4A}{s(s^2+4)}$$

Comparing coefficients of powers of s in the numerator, we find that

$$A = 3\,, \quad C = -4\,, \text{ and } B = 9 - 3 = 5\,.$$

Therefore,

$$F(s) = \frac{3}{s} + \frac{5s-4}{s^2+4} = 3\left(\frac{1}{s}\right) + 5\left(\frac{s}{s^2+4}\right) - \frac{4}{2}\left(\frac{2}{s^2+4}\right)\,.$$

From the table of Laplace transforms, it now follows that

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\{F(s)\} &= 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{4}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= 3 + 5\cos(2t) - 2\sin(2t)\,. \end{aligned}$$

EXAMPLE 11.12. Find the inverse Laplace transform of $F(s) = \dfrac{2s-3}{s^2 + 2s + 10}$.

SOLUTION. The denominator has complex roots, so complete the square and rewrite $F(s)$ as follows:

$$\frac{2s-3}{s^2+2s+10} = \frac{2s-3}{(s+1)^2 + 9} = \frac{2(s+1) - 5}{(s+1)^2 + 9}$$

[13] This is a good time to read Appendix B, which presents a quick review of partial fractions.

Therefore,

$$F(s) = 2\left(\frac{(s+1)}{(s+1)^2+9}\right) - \frac{5}{3}\left(\frac{3}{(s+1)^2+9}\right)$$

From the table of Laplace transforms, it now follows that the inverse Laplace transform of $F(s)$ is

$$f(t) = 2e^{-t}\cos(3t) - \frac{5}{3}e^{-t}\sin(3t)$$

EXAMPLE 11.13. Find the inverse Laplace transform of $Y(s) = \dfrac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)}$.

SOLUTION. First compute the partial fractions expansion of $Y(s)$:

$$\begin{aligned} Y(s) &= \frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\ &= \frac{(As+B)(s^2+4)+(Cs+D)(s^2+1)}{(s^2+1)(s^2+4)} = \frac{(A+C)s^3+(B+D)s^2+(4A+D)s+(4B+D)}{(s^2+1)(s^2+4)} \end{aligned}$$

Comparing the coefficients of powers of s in numerators results in the system of four equations in four unknowns

$$A+C=2\,, \quad B+D=1\,, \quad 4A+C=8\,, \quad 4B+D=6\,,$$

which we can solve to obtain $A=2$, $B=5/3$, $C=0$, and $D=-2/3$. Therefore,

$$Y(s) = 2\left(\frac{s}{s^2+1}\right) + \frac{5}{3}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\left(\frac{2}{s^2+4}\right).$$

Consequently, the inverse Laplace transform of $Y(s)$ is

$$y(t) = 2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t)\,.$$

11.4. Initial Value Problems with Continuous Forcing Function

Below are some examples illustrating the use of Laplace transforms for solving initial value problems. All of these examples could (sometimes more easily) be done using the method of undetermined coefficients. The purpose of these examples is mainly to illustrate the method. In later sections, more interesting examples are presented where the forcing function is not continuous and the method of undetermined coefficients does not apply.

EXAMPLE 11.14. Solve the initial value problem $y''+4y=\cos(3t)$, $y(0)=0$, $y'(0)=0$.

SOLUTION. Applying $\mathcal{L}\{-\}$ gives

$$(s^2+4)Y(s) = \frac{s}{s^2+9}, \text{ or } Y(s) = \frac{s}{(s^2+4)(s^2+9)} = \frac{s/5}{s^2+4} - \frac{s/5}{s^2+9}.$$

From the table of Laplace transforms,

$$y(t) = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \frac{1}{5}(\cos(2t)-\cos(3t))\,.$$

EXAMPLE 11.15. Solve the initial value problem $y'' + 4y = \cos(2t)$, $y(0) = 0$, $y'(0) = 0$.

SOLUTION. Applying $\mathcal{L}\{-\}$ gives

$$(s^2+4)Y(s) = \frac{s}{s^2+4}, \text{ or } Y(s) = \frac{s}{(s^2+4)^2}.$$

Using the table of Laplace transforms:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{2\cdot 2\cdot s}{(s^2+2^2)^2}\right\} = \frac{1}{4}t\sin(2t).$$

EXAMPLE 11.16. Laplace transforms can also be used to solve linear constant coefficient first order initial value problems. For instance, consider the initial value problem

$$y' + 2y = \cos(t)\,, \quad y(0) = 1\,.$$

Computing the Laplace transform of both sides gives

$$sY(s) - 1 + 2Y(s) = \frac{s}{s^2+1}\,,$$

which can be solved for $Y(s)$:

$$\begin{aligned} Y(s) &= \frac{s}{(s+2)(s^2+1)} + \frac{1}{s+2} = \frac{(2/5)s + (1/5)}{s^2+1} - \frac{2/5}{s+2} + \frac{1}{s+2} \\ &= \frac{(2/5)s}{s^2+1} + \frac{1/5}{s^2+1} + \frac{3/5}{s+2}\,. \end{aligned}$$

Hence,

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\ &= \frac{2}{5}\cos(t) + \frac{1}{5}\sin(t) + \frac{3}{5}e^{-2t} \end{aligned}$$

EXAMPLE 11.17. Solve the initial value problem

$$y'' - 3y' + 2y = 2e^{-3t}\,, \quad y(0) = 1\,, \quad y'(0) = 0\,.$$

SOLUTION. Applying $\mathcal{L}\{-\}$ and setting $Y(s) = \mathcal{L}\{y\}$ yields the equations

$$(s^2Y(s) - s) - 3(sY(s) - 1) + 2Y(s) = \frac{2}{s+3}\,,$$

which simplifies to

$$(s^2 - 3s + 2)Y(s) - s + 3 = \frac{2}{s+3}\,.$$

Solving for $Y(s)$ yields

$$\begin{aligned} Y(s) &= \frac{2}{(s+3)(s^2-3s+2)} + \frac{s-3}{s^2-3s+2} \\ &= \frac{s^2-7}{(s-1)(s-2)(s+3)} = \frac{3/2}{s-1} + \frac{-3/5}{s-2} + \frac{1/10}{s+3}. \end{aligned}$$

Consequently,

$$y(t) = \frac{3}{2}e^{t} - \frac{3}{5}e^{2t} + \frac{1}{10}e^{-3t}\,.$$

EXAMPLE 11.18. Solve the initial value problem $y'' + 2y' + 2y = \cos(2t)$, $y(0) = 1$, $y'(0) = 0$.

SOLUTION. Proceeding as in the previous example, apply $\mathcal{L}\{-\}$, solve for $Y(s)$, compute the partial fractions expansion for $Y(s)$, and finally, compute the inverse Laplace transform.

Here's the (somewhat messy!) computation omitting some algebra:

$$(s^2Y(s) - s) + 2(sY(s) - 1) + 2Y(s) = \frac{s}{s^2+4}$$
$$(s^2 + 2s + 2)Y(s) = \frac{s}{s^2+4} + s + 2\,.$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{s}{(s^2+4)(s^2+2s+2)} + \frac{s+2}{s^2+2s+2} \\ &= \frac{As+B}{s^2+4} + \frac{C(s+1)+D}{(s+1)^2+1} + \frac{(s+1)+1}{(s+1)^2+1}\,. \\ &= \frac{-\frac{1}{10}s + \frac{4}{10}}{s^2+4} + \frac{\frac{1}{10}(s+1) - \frac{3}{10}}{(s+1)^2+1} + \frac{(s+1)+1}{(s+1)^2+1} \\ &= -\frac{1}{10}\,\frac{s}{s^2+4} + \frac{2}{10}\,\frac{2}{s^2+4} + \frac{11}{10}\,\frac{s+1}{(s+1)^2+1} + \frac{7}{10}\frac{1}{(s+1)^2+1}\,. \end{aligned}$$

Using the table of Laplace transforms gives

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{10}\cos(2t) + \frac{1}{5}\sin(2t) + \frac{11}{10}e^{-t}\cos(t) + \frac{7}{10}e^{-t}\sin(t).$$

11.5. The Laplace Transform of Piecewise Continuous Functions

The Laplace transform is a useful tool when the forcing function $f(t)$ is piecewise continuous. Piecewise continuous forcing functions, such as those pictured in Figure 11.1, routinely occur in engineering applications, particularly in engineering applications involving signal processing.

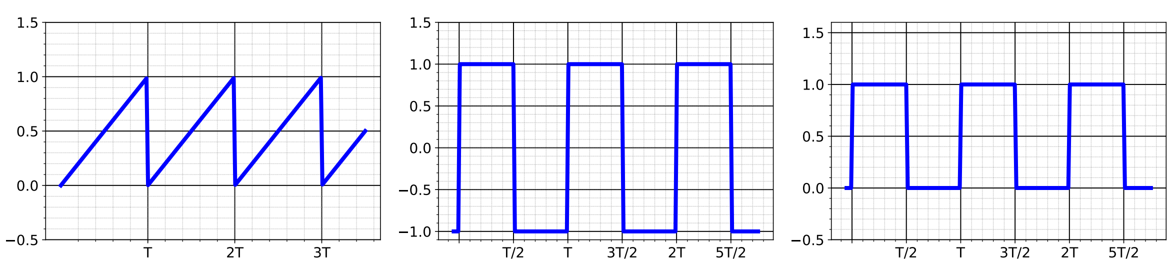


FIGURE 11.1. From left to right: a sawtooth wave, a square wave, and a pulse wave.

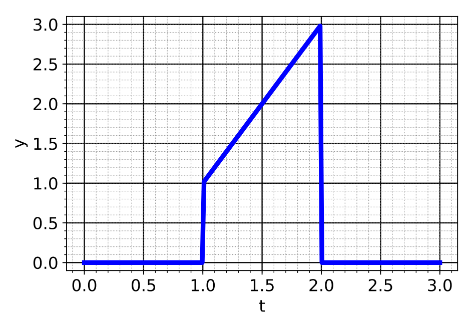
The *Heaviside step function*, denoted by $u_a(t)$ is the basic building block for constructing piecewise continuous function. It is defined as follows:

$$u_a(t) = \begin{cases} 0 & \text{if } t < a, \\ 1 & \text{if } t \geq a\,. \end{cases}$$

The difference $u_a(t) - u_b(t)$, $b > a$, of two Heaviside step functions forms a pulse.

FIGURE 11.2. The Heaviside step function $u_a(t)$ and its difference $u_a(t) - u_b(t)$ are the basic building blocks for constructing piecewise continuous functions.

EXAMPLE 11.19. Let $f(t)$ be the function defined by $f(t) = \begin{cases} 0 & \text{if } t < 1, \\ 2t-1 & \text{if } 1 \le t < 2, \\ 0 & \text{if } t \ge 2, \end{cases}$ then $f(t) = (2t-1)(u_1(t) - u_2(t))$.



EXAMPLE 11.20. The Heaviside step function is particularly useful in representing waves commonly found in engineering applications, such as *sawtooth waves*, *square waves*, and *pulse waves.*, illustrated in Figure 11.1. A sawtooth wave of period T and amplitude 1 can be represented as follows

$$f_{saw}(t) = \frac{t}{T} - \sum_{k=1}^{\infty} u_{kT}(t)\,; \tag{11.12a}$$

while a square wave of period T and amplitude 1 can be represented by

$$f_{sqr}(t) = u_0(t) + 2\sum_{k=1}^{\infty} (-1)^k u_{kT/2}(t)\,; \tag{11.12b}$$

and a pulse wave of period T and amplitude 1 can we represented by

$$f_{pulse}(t) = u_0(t) + \sum_{k=1}^{\infty} (-1)^k u_{kT/2}(t)\,. \tag{11.12c}$$

PROPOSITION 6. *The Laplace transform of $u_a(t)$ is e^{-as}/s. If $f(t)$ is a function with Laplace transform $F(s)$, then*

$$\boxed{\mathcal{L}\left\{u_a(t) f(t-a)\right\} = e^{-as} F(s).} \tag{11.13}$$

PROOF. The integral defining the Laplace transform is

$$\mathcal{L}\{u_a(t)f(t-a)\} = \int_0^\infty e^{-st}u_a(t)f(t-a)\,dt = \int_a^\infty e^{-st}f(t-a)\,dt.$$

Now make a change of variables: let $w = t - a$. When $t = a$, $w = 0$, and when $t = \infty$, $w = \infty$, so the integral becomes

$$\int_0^\infty e^{-s(w+a)}f(w)\,dw = \int_0^\infty e^{-sw}e^{-sa}f(w)\,dw = e^{-sa}\int_0^\infty e^{-sw}f(w)\,dw = e^{-sa}\mathcal{L}\{f\}\,.$$

The formula for the Laplace transform of $u_a(t)$ is a special case: set $f(t) = 1$ and recall that $1/s$ is the Laplace transform of 1, □

EXAMPLE 11.21. If $f(t) = u_1(t)(t-1)$, then $\mathcal{L}\{f\} = e^{-s}/s^2$.

EXAMPLE 11.22. Suppose $f(t) = (u_1(t) - u_2(t))(2t-1)$. To make use of Proposition 6, rewrite $f(t)$ as follows

$$f(t) = u_1(t)(2t-1) - u_2(t))(2t-1) = u_1(t)(2(t-1)+1) - u_2(t)(2(t-2)+3)\,.$$

Proposition 6 then yields the formula $\mathcal{L}\{f\} = e^{-s}\left(\frac{2}{s^2} + \frac{1}{s}\right) - e^{-2s}\left(\frac{2}{s^2} + \frac{3}{s}\right)$.

EXAMPLE 11.23. The Laplace transforms of the sawtooth, square, and pulse waves are, respectively,

$$F_{saw}(s) = \frac{1}{Ts^2} - \left(\sum_{k=1}^{\infty} e^{-kTs}\right)\frac{1}{s}\,,$$
$$F_{sqr}(s) = \frac{1}{s} + \left(\sum_{k=1}^{\infty}(-1)e^{-k(T/2)s}\right)\frac{2}{s}\,,$$

and

$$F_{saw}(s) = \frac{1}{s} + \left(\sum_{k=1}^{\infty}(-1)^k e^{-k(T/2)s}\right)\frac{1}{s}\,.$$

REMARK 11.3. The following variant of the formula (11.13) is occasionally useful:

$$\boxed{\mathcal{L}\{u_a(t)f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}} \tag{11.14}$$

To show this, let $g(t) = f(t+a)$. Then $f(t) = g(t-a)$. Applying (11.13) to $g(t)$ shows

$$\mathcal{L}\{u_a(t)f(t)\} = \mathcal{L}\{u_a(t)g(t-a)\} = e^{-as}\mathcal{L}\{g(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}\,.$$

For instance,

$$\begin{aligned}\mathcal{L}\{u_3(t)(2t-1)\} &= e^{-2s}\mathcal{L}\{2(t+3)-1\} = e^{-2s}\mathcal{L}\{2t+5\}\\ &= e^{-2s}\left(2\mathcal{L}\{t\} + 5\mathcal{L}\{1\}\right) = e^{-2s}\left(\frac{2}{s^2} + \frac{5}{s}\right)\,.\end{aligned}$$

11.6. Initial Value Problems with Piecewise Continuous Forcing Functions

Consider the differential equation

$$ay'' + by' + cy = f(t)\,,$$

where $f(t)$ is piecewise-continuous. What does it mean for $y(t)$ to be a solution of an equation like this? If $f(t)$ is discontinuous at some point $t = t_0$, will $y''(t)$ even be defined there? If not, how can the equation be satisfied? To avoid these issues, declare a function $y(t)$ to be a solution to an equation like this if

- $y(t)$ is continuous everywhere,
- $y'(t)$ is continuous everywhere, and
- $y(t)$ satisfies the differential equation at every point where the right-hand side $f(t)$ is continuous.

Thus $y''(t)$ need not be defined (and in practice usually won't be defined) at points of discontinuity of the right side; b it will, however, be defined at all other points.

EXAMPLE 11.24. Solve the initial value problem

$$y'' + 5y' + 4y = f(t), \quad y(0) = 0,\ y'(0) = 0\,,$$

where $f(t) = u_1(t) - u_{10}(t)$.

SOLUTION. Applying the Laplace transform yields the equation

$$(s^2 + 5s + 4)Y(s) = \frac{e^{-s}}{s} - \frac{e^{-10s}}{s}\,.$$

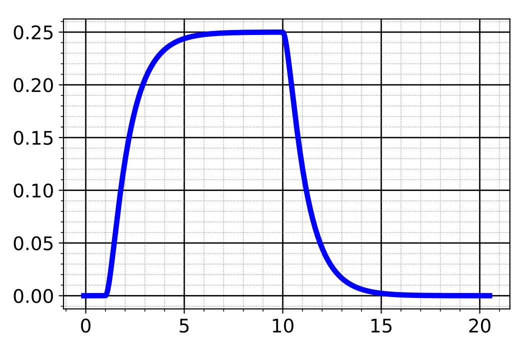
Hence,

$$Y(s) = \left(e^{-s} - e^{-10s}\right)\frac{1}{s(s+1)(s+4)} = \left(e^{-s} - e^{-10s}\right)\left(\frac{1/4}{s} + \frac{-1/3}{s+1} + \frac{1/12}{s+4}\right).$$

Let $p(t) = 1/4 - 1/3e^{-t} + 1/12e^{-4t}$, so that $p(t)$ is the inverse Laplace transform of the last term on the right. Then

$$\begin{aligned} y(t) &= u_1(t)p(t-1) - u_{10}(t)p(t-10) \\ &= u_1(t)\left(\frac{1}{4} - \frac{1}{3}e^{-t+1} + \frac{1}{12}e^{-4t+4}\right) + u_{10}(t)\left(\frac{1}{4} - \frac{1}{3}e^{-t+10} + \frac{1}{12}e^{-4t+40}\right)\,. \end{aligned}$$

The solution is graphed below.



EXAMPLE 11.25. Solve the initial value problem

$$y'' + 3y' + 2y = f(t), \quad y(0) = 2,\ y'(0) = 0\,,\ \text{where } f(t) = \begin{cases} 0 & \text{if } t < 1, \\ t-1 & \text{if } 1 \le t < 2, \\ 0 & \text{if } t \ge 2. \end{cases}$$

SOLUTION. In this case,

$$f(t) = u_1(t)(t-1) - u_2(t)(t-1) = u_1(t)(t-1) - u_2(t)((t-2)+1).$$

Apply $\mathcal{L}\{-\}$ to the differential equation:

$$\begin{aligned}(s^2Y - 2s) - (3sY - 6) + 2Y &= (e^{-s} - e^{-2s})\frac{1}{s^2} - e^{-2s}\frac{1}{s} \\ (s^2 - 3s + 2)Y - 2s + 6 &= (e^{-s} - e^{-2s})\frac{1}{s^2} - e^{-2s}\frac{1}{s},.\end{aligned}$$

Solving for $Y(s)$ gives

$$\begin{aligned}Y(s) &= \frac{2s-6}{s^2-3s+2} + \left(e^{-s} - e^{-2s}\right)\frac{1}{s^2(s^2-3s+2)} - e^{-2s}\frac{1}{s(s^2-3s+2)} \\ &= \frac{4}{s-1} + \frac{-2}{s-2} + (e^{-s} - e^{-2s})\left(\frac{1/2}{s^2} + \frac{3/4}{s} + \frac{-1}{s-1} + \frac{1/4}{s-2}\right) - e^{-2s}\left(\frac{1/2}{s} + \frac{-1}{s-1} + \frac{1/2}{s-2}\right).\end{aligned}$$

If we let $p(t)$ denote the inverse Laplace transform of the sum of fractions in the left-hand parentheses, and $q(t)$ the the inverse Laplace transform of the terms in the right-hand set, then

$$p(t) = \frac{1}{2}t + \frac{3}{4} - e^t + \frac{1}{4}e^{2t}, \quad q(t) = \frac{1}{2} - e^t + \frac{1}{2}e^{2t},$$

and the solution $y(t)$ can be written as follows:

$$y(t) = 4e^t - 2e^{2t} + u_1(t)p(t-1) - u_2(t)p(t-2) - u_2(t)q(t-2).$$

After lots of algebra this reduces to

$$y(t) = \begin{cases} 4e^t - 2e^{2t} & \text{if } 0 \le t < 1, \\ 4e^t - 2e^{2t} + \frac{1}{2}(t-1) + \frac{3}{4} - e^{t-1} + \frac{1}{4}e^{2t-2} & \text{if } 1 \le t < 2, \\ (4 - e - 2e^{-2})e^t + \left(-2 + \frac{1}{4}e^{-2} - \frac{3}{4}e^{-4}\right)e^{2t} & \text{if } t \ge 2. \end{cases}$$

EXAMPLE 11.26. Find the solution to the initial value problem $y''+y = f(t)$, $\quad y(0) = 0, \quad y'(0) = 0,$ where $f(t)$ is the sawtooth wave with period $T = 2\pi$:

$$f(t) = \frac{t}{2\pi} - \sum_{k=1}^{\infty} u_{2k\pi}(t).$$

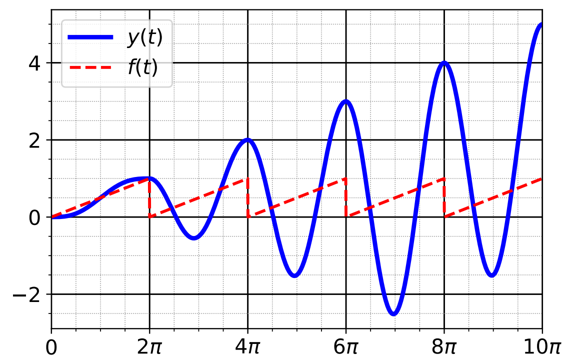
SOLUTION. Applying the Laplace transform to this initial value problem and solving for $Y(s)$, we find that

$$\begin{aligned}Y(s) &= \frac{1}{2\pi s^2(s^2+1)} - \frac{1}{s^2+1}\left(\sum_{k=1}^{\infty} e^{-2k\pi s}\right)\frac{1}{s} \\ &= \frac{1}{2\pi}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) - \left(\sum_{k=1}^{\infty} e^{-2k\pi s}\right)\left(\frac{1}{s} - \frac{s}{(s^2+1)}\right).\end{aligned}$$

The inverse transform of $Y(s)$ is then

$$y(t) = \frac{1}{2\pi}(t - \sin(t)) - \sum_{k=1}^{\infty} u_{2k\pi}(t)h(t - 2k\pi)$$

where $h(t) = 1 - \cos(t)$, the inverse Laplace transform of $\dfrac{1}{s} - \dfrac{s}{s^2+1}$. The solution $y(t)$ together with the forcing function $f(t)$ are graphed above.



EXAMPLE 11.27. Suppose that $f(t)$ is defined by $f(t) = \begin{cases} 100\sin(40t) & \text{when } 0 \le t < 7, \\ 0 & \text{when } t \ge 7. \end{cases}$

Solve the initial value problem $y'' + 3y' + 2y = f(t), \quad y(0) = 0,\ y'(0) = 0.$

SOLUTION. In this case, it's easier to work with complex-valued functions. Notice that since $f(t) = (1 - u_7(t))100\sin(40t) = 100(1 - u_7(t))\mathrm{Re}\left(-ie^{i40t}\right)$, the solution of the original initial value problem is the real part of the solution of the initial value problem

$$z'' + 3z' + 2z = g(t)\,, \quad z(0) = 0, z'(0) = 0\,,$$

where $g(t) = -100i(1 - u_7(t))e^{i40t}$. Applying the Laplace transform gives

$$Z(s) = \frac{G(s)}{s^2 + 3s + 2} = \frac{G(s)}{(s+2)(s+1)}\,.$$

The Laplace transform of $g(t)$ is

$$\begin{aligned} G(s) &= -100i\left(\mathcal{L}\left\{e^{i40t}\right\} - e^{-7s}\mathcal{L}\left\{e^{i40(t+7)}\right\}\right) = -100i\left(1 - e^{280i}e^{-7s}\right)\mathcal{L}\left\{e^{i40t}\right\} \\ &= -\left(1 - e^{280i}e^{-7s}\right)\frac{100i}{s - 40i}\,. \end{aligned}$$

Therefore, (by a messy partial fractions computation, that can be skipped[14])

$$\begin{aligned} Z(s) &= \left(1 - e^{280i}e^{-7s}\right)\frac{(-100i)}{(s-40i)(s+2)(s+1)} \\ &= \left(1 - e^{280i}e^{-7s}\right)\left(\frac{-0.00467 + 0.0622i}{s - 40i} - \frac{2.4938 + 0.1247i}{s+2} + \frac{2.498 + 0.06246i}{s+1}\right) \\ &= \left(1 - e^{280i}e^{-7s}\right)\mathcal{L}\left\{h(t)\right\} \end{aligned}$$

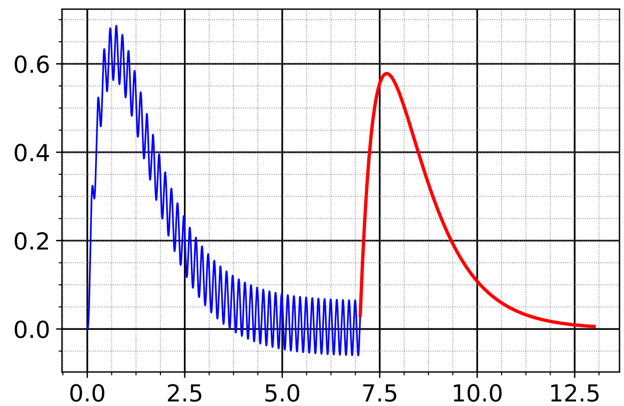
where $\quad h(t) = (-0.00467 + 0.0622i)e^{i40t} - (2.4938 + 0.1247i)e^{-2t} + (2.498 + 0.06246i)e^{-t}$

Hence, $z(t) = h(t) - e^{280i}u_7(t)h(t-7)$

Finally,

$$y(t) = \mathrm{Re}\left(z(t)\right) = \begin{cases} -2.49e^{-2t} + 2.50e^{-t} - 0.0622\sin(40t) - 0.00467\cos(40t), & \text{if } 0 \le t < 7, \\ -2.25e^{-2(t-7)} + 2.28e^{-(t-7)}, & \text{if } t \ge 7. \end{cases}$$

[14]The computation without using complex-valued functions is worse!



11.7. The Dirac Delta Function/Impulse Response

Laplace transform techniques are useful in cases where the forcing function $f(t)$ represents "impulses" of short duration.

As a motivating example, consider an object of mass m (in kilograms) free to move in a straight line. Let $x(t)$ be the position (in meters) of the object at time t seconds and let $v(t)$ be its velocity. Suppose also that $x(0) = 0$ and $v(0) = 0$.

Suppose that (as shown in Figure 11.3) at time $t = a$ a positive force is exerted on the object for ε seconds and vanishes for $t > a + \varepsilon$, where $\varepsilon > 0$ is assumed to be a small number. For instance, the object could be a football or baseball suddenly struck by a foot or a bat.

Label this force $f_\varepsilon(t)$, and assume that $f_\varepsilon(t)$ satisfies the following condition:

$$\int_a^{a+\varepsilon} f_\varepsilon(t)\, dt = J\,,$$

where J is a fixed constant. This integral is called an *impulse* and has the dimensions of momentum (Newton-seconds or kilogram-meters/second).

In this situation, Newton's second law of motion assumes the simple form

$$m\frac{dv}{dt} = f(t)\,, \quad v(0) = 0\,,$$

which we can integrate to find

$$mv(t) = \int_0^t f(\tau)\, d\tau\,.$$

Notice what happens: $v(t) = 0$ until $t = a$, at which time $v(t)$ increases until time $t = a + \varepsilon$. After that time, $v(t) = J/m$ because no force is being exerted on the object after that time.

Imagine now what happens if the impulse J stays constant, but ε approaches zero. To keep J constant, the values of $f_\varepsilon(t)$ have to become large on the interval $a \le t < a + \varepsilon$. For very small values of ε, the graph of $v(t)$ will become almost indistinguishable from the graph of the step function $(J/m)u_a(t)$.

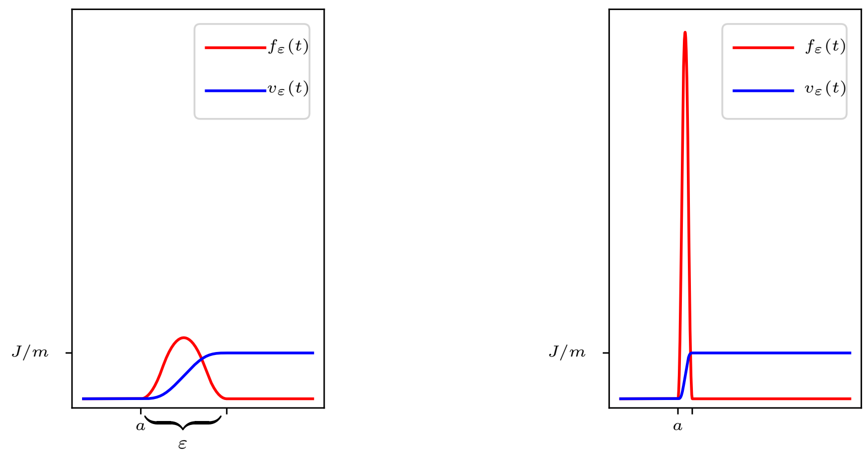


FIGURE 11.3. As ε approaches zero, $y_\varepsilon(t)$ approaches $(J/m)u_a(t)$, a multiple of the Heaviside step function.

The specific choice of $f_\varepsilon(t)$ is unimportant: we only need to insist that it vanishes outside the interval $a \le t < a + \varepsilon$ and that its integral remains equal to J.

To understand the behavior of the Laplace transform of $f_\varepsilon(t)$ as ε approaches 0, assume for simplicity, assume that $m = 1$, $J = 1$, and that $f_\varepsilon(t)$ has the special form:

$$f_\varepsilon(t) = \frac{1}{\varepsilon}\left(u_a(t) - u_{a+\varepsilon}(t)\right) = \begin{cases} 0 & t < a, \\ 1/\varepsilon & a \le t < a + \varepsilon, \\ 0 & t > a + \varepsilon. \end{cases}$$

The Laplace transform of $f_\varepsilon(t)$ can then be computed as follows:

$$\mathcal{L}\left\{f_\varepsilon(t)\right\} = \frac{1}{\varepsilon}\left(\mathcal{L}\left\{u_a(t)\right\} - \mathcal{L}\left\{u_{a+\varepsilon}(t)\right\}\right) = \frac{1}{\varepsilon}\left(\frac{e^{-as}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right) = e^{-as}\left(\frac{1 - e^{-\varepsilon s}}{\varepsilon s}\right).$$

Using l'Hôpital's rule, the limit as ε approaches zero of the Laplace transforms is easily found:

$$\lim_{\varepsilon \to 0} \mathcal{L}\left\{f_\varepsilon(t)\right\} = e^{-as} \lim_{\varepsilon \to 0}\left(\frac{1 - e^{-\varepsilon s}}{\varepsilon s}\right) = e^{-as} \lim_{\varepsilon \to 0}\left(\frac{s e^{-\varepsilon s}}{s}\right) = e^{-as}.$$

Roughly speaking, the *Dirac Delta Function* $\delta_a(t)$ is defined by

$$\delta_a(t) = \lim_{\varepsilon \to 0^+} f_\varepsilon(t).$$

Although this is not a well-defined function[15], it does have a well-defined Laplace transform:

$$\boxed{\mathcal{L}\left\{\delta_a(t)\right\} = e^{-as}.} \tag{11.15}$$

REMARK 11.4. When $a = 0$, the subscript is dropped and the notation $\delta(t)$ is used. The identity (11.15) then reduces to

$$\mathcal{L}\left\{\delta(t)\right\} = 1.$$

An alternate notation for $\delta_a(t)$ is $\delta(t - a)$. Then the delta function satisfies the identity

$$\mathcal{L}\left\{\delta(t - a)\right\} = e^{-as}\mathcal{L}\left\{\delta(t)\right\} = e^{-as}\, 1 = e^{-as}$$

[15] It is something called a "generalized function" or a "distribution," not an actual function.

which is consistent with the general formula $\mathcal{L}\left\{u_a(t)f(t-a)\right\} = e^{-as}\mathcal{L}\left\{f(t)\right\}$.

EXAMPLE 11.28. Laplace transforms give a way to model the dynamics of a force that acts instantaneously on an object of mass m:

$$m\frac{dv}{dt} = \lim_{\varepsilon\to 0} f_\varepsilon(t) = J\delta_a(t)\,, \quad v(0) = 0\,.$$

Taking Laplace transforms gives

$$msV(s) = Je^{-as} \implies V(s) = (J/m)\frac{e^{-as}}{s}\,.$$

Therefore,

$$v(t) = (J/m)\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = (J/m)u_a(t)\,.$$

Rather than solving for $v(t)$, one can apply Newton's second law of motion:

$$mx''(t) = J\delta_a(t)\,, \quad x(0) = 0\,, \quad x'(0) = 0\,.$$

Taking Laplace transforms gives $X(s) = (J/m)e^{-as}/s^2$. Consequently,

$$x(t) = (J/m)\mathcal{L}^{-1}\left\{e^{-as}/s^2\right\} = (J/m)(t-a)u_a(t)\,,$$

as expected.

EXAMPLE 11.29. Solve the initial value problem

$$y'' + 2y' + 2y = \delta_1(t), \quad y(0) = 0,\ y'(0) = 0.$$

SOLUTION. Apply the Laplace transform and solve for $Y(s)$:

$$(s^2 + 2s + 2)Y(s) = e^{-s} \implies Y(s) = e^{-s}\frac{1}{s^2 + 2s + 2}.$$

Complete the square and write $Y(s)$ in the form

$$Y(s) = e^{-s}\frac{1}{s^2 + 2s + 2} = e^{-s}\frac{1}{(s+1)^2 + 1}.$$

Therefore

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = u_1(t)h(t-1)\,,$$

where

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2 + 1}\right\} = e^{-t}\sin(t).$$

So

$$y(t) = u_1(t)e^{-(t-1)}\sin(t-1) = \begin{cases} 0 & \text{if } t < 1, \\ e^{-(t-1)}\sin(t-1) & \text{if } t \geq 1. \end{cases}$$

Note that this function is continuous everywhere, but it is not differentiable at $t = 1$. This is not surprising, because $t = 1$ is when the delta function is applied—this example models what happens in a damped spring system when you hit the mass with a hammer.

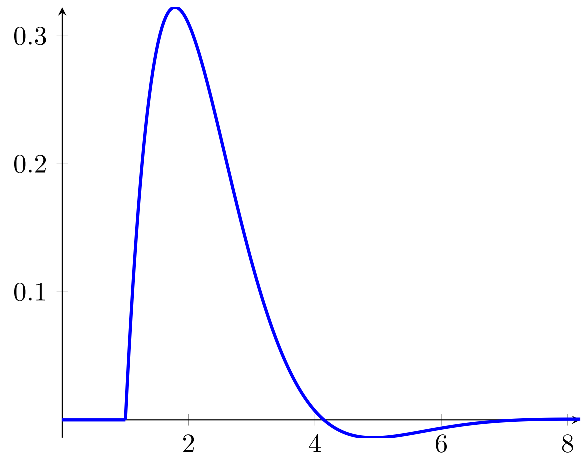


FIGURE 11.4. The graph of $y(t) = u_1(t)e^{-(t-1)}\sin(t-1)$.

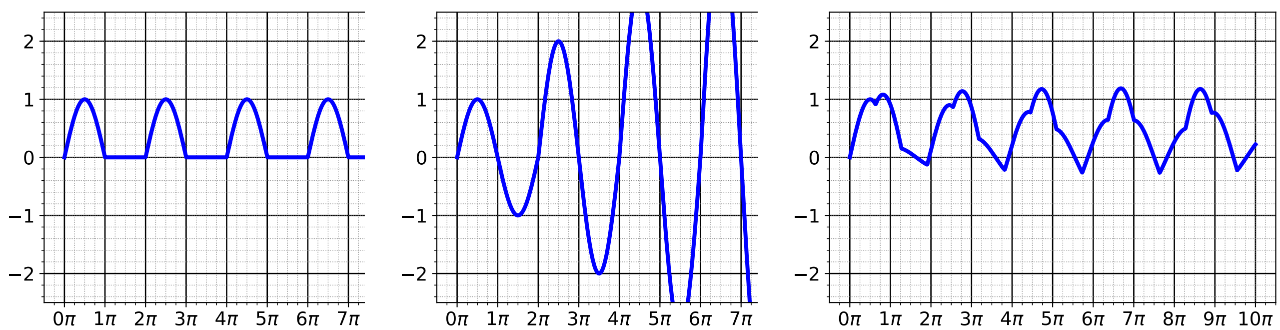


FIGURE 11.5. From left to right the solution of the initial value problem (11.16) for $T = \pi$, $T = 2\pi$, and $T = 2.0$. Resonance occurs when $T = 2\pi$, the natural frequency of the oscillator.

11.8. Modeling Examples

EXAMPLE 11.30. Consider a mass-spring system, with mass $m = 1$ kilogram and spring constant $k = 1$ Newton/meter. Suppose, in addition, the mass is repeatedly struck with a unit impulse every T seconds. The following initial value problem models this situation:

$$y'' + y = f(t), \quad y(0) = y'(0) = 0, \tag{11.16}$$

where $f(t) = \sum_{j=0}^{\infty} \delta_{jT}(t)$, and $T > 0$. Notice that the natural frequency of this harmonic oscillator is 2π. Therefore, we expect to observe some sort of resonance when $T = 2\pi$ (see Figure 11.5).

Taking Laplace transforms gives

$$Y(s) = \sum_{j=0}^{\infty} \frac{e^{-jTs}}{s^2 + 1}.$$

Taking inverse Laplace transforms yields the solution

$$y(t) = \sum_{j=0}^{\infty} u_{jT}(t) \sin(t - jT) .$$

EXAMPLE 11.31. (A MIXING PROBLEM) Suppose a large tank contains algae that grows exponentially with a doubling time of 24 hours. The tank initially contains 100 kilograms of algae. Every 12 hours, h kilograms are instantaneously removed. How large can h be so that this process can be repeated indefinitely?

SOLUTION. Let t denote time in hours and let $y(t)$ denote the total mass of algae in the tank at time t. Then $y(t)$ is a solution of the initial value problem

$$y' = ky - \sum_{j=1}^{\infty} h\, \delta_{12j}(t) , \quad y(0) = 100 , \tag{11.17}$$

where k and h are to be determined. Observe that we modeled instantaneously removing h kilograms of algae at time $t = 12j$ by the impulse $-h\, \delta_{12j}(t)$.

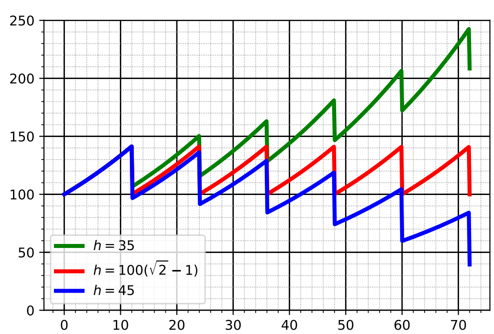


FIGURE 11.6. The solution of the initial value problem (11.17) for values of h below, at, and above the critical value $h = 100(\sqrt{2} - 1) \approx 41.4$ kilograms.

Taking the Laplace transform of the initial value problem gives

$$(s - k)Y(s) - 100 = -\sum_{j=1}^{\infty} he^{-12js} \implies Y(s) = \frac{100}{s-k} - h \sum_{j=1}^{\infty} \frac{e^{-12js}}{s-k} .$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y(t) &= 100e^{kt} - h \sum_{j=1}^{\infty} \mathcal{L}^{-1} \left\{ e^{-12jt}/(s-k) \right\} \\ &= 100e^{kt} - h \sum_{j=1}^{\infty} u_{12j}(t) e^{k(t-12j)} = \left(100 - h \sum_{j=1}^{\infty} u_{12j}(t) e^{-12jk} \right) e^{kt} . \end{aligned} \tag{11.18}$$

If the term in parentheses ever became negative, then the tank would be empty, so the condition on h is that the term in parentheses be positive for all t, no matter how large. Since $u_{12j}(t) = 1$ for t large, this amounts to the condition

$$100 - h \sum_{j=1}^{\infty} e^{-12jk} > 0 \text{ or } h < \frac{100}{\sum_{j=1}^{\infty} e^{-12jk}} .$$

Using the sum formula for the geometric series $\sum_{j=1}^{\infty} r^j = \frac{r}{1-r}$, with $r = e^{-12k}$, this can be rewritten as

$$h < \frac{100(1-e^{-12k})}{e^{-12k}} = 100(e^{12k} - 1)$$

Since the doubling time is 24 hours, $e^{24k} = 2$, so $k = \ln(2)/24$. Hence $e^{12k} = e^{\ln(2)/2} = \sqrt{2}$. We conclude that

$$h < 100(\sqrt{2} - 1) = 100(0.4121) \approx 41.4 \text{ kilograms.}$$

11.9. Convolutions

Equation (3.7) of Section 3.2.2, gave the formula

$$y(t) = e^{-kt} \int_0^t e^{ku} f(u)\, du + y_0\, e^{-kt}$$

for the solution of the first order initial value problem

$$y' + ky = f(t)\,, \quad y(0) = y_0,.$$

Notice that if $y(0) = 0$, then the formula simplifies to

$$y(t) = e^{-kt} \int_0^t e^{ku} f(u)\, du = \int_0^t f(u) e^{-k(t-u)}\, du\,.$$

Notice also that e^{-kt} is the solution of the initial value problem

$$y' + ky = 0\,, \quad y(0) = 1\,.$$

There is a similar formula for the solution of the initial value problem

$$ay'' + by' + cy = f(t)\,, \quad y(0) = y'(0) = 0\ :$$

$$\boxed{y(t) = \int_0^t f(u) g(t-u)\, du\,,} \tag{11.19}$$

where $g(t)$ is the solution of the initial value problem

$$ay'' + by' + cy = 0\,, \quad y(0) = 0\,, \quad y'(0) = 1/a\,.$$

The right-hand side of Equation (11.19) is called the *convolution* of the functions $f(t)$ and $g(t)$.

More generally, if $f(t)$ and $g(t)$ are any two functions defined for $t \geq 0$, then their convolution is defined to be

$$\boxed{(f * g)(t) = \int_0^t f(u) g(t-u)\, du\,.} \tag{11.20}$$

The asterisk $*$ does not mean ordinary multiplication: it is a new operation, *convolution*, defined by the integral on the right side.)

As it applies to differential equations, the most important property of convolution is given by the following theorem:

THEOREM 7 (The Convolution Theorem). *If $\mathcal{L}\{f\} = F(s)$ and $\mathcal{L}\{g\} = G(s)$, then*

$$\boxed{\mathcal{L}\{f * g\} = F(s)G(s).}$$

PROOF. (*Skip this proof if you haven't taken Math 126.*) Compute as follows, using the definition of the Laplace transform, followed by the formula for convolution:

$$\begin{aligned}\mathcal{L}\{f*g\} = \int_0^\infty e^{-st}(f*g)\,dt &= \int_0^\infty e^{-st}\left(\int_0^t f(u)g(t-u)du\right)dt \\ &= \int_0^\infty \left(\int_0^t f(u)g(t-u)e^{-st}du\right)dt = \iint_R f(u)g(t-u)e^{-st}dudt\,.\end{aligned}$$

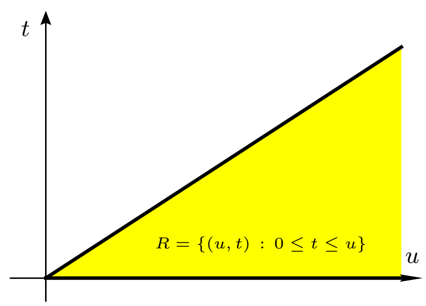


FIGURE 11.7.

This is a double integral over the infinite region R in Figure 11.7. Now change variables, letting $v = t - u$, so $t = u + v$ and $dv = dt$:

$$\begin{aligned}\mathcal{L}\{f*g\} &= \int_0^\infty \int_0^\infty f(u)g(v)e^{-s(u+v)}\,dv\,du \\ &= \left(\int_0^\infty e^{-su}f(u)\,du\right)\left(\int_0^\infty e^{-sv}g(v)\,dv\right) = \mathcal{L}\{f\}\,\mathcal{L}\{g\}\,.\end{aligned}$$

□

A number of properties of convolution follow immediately from The Convolution Theorem:

COROLLARY 8. *Let $f(t)$, $g(t)$, and $h(t)$ be continuous functions. Then the following identities hold:*

$$f*g = g*f \tag{11.21a}$$

$$(f*g)*h = f*(g*h) \tag{11.21b}$$

PROOF. By Theorem 5, to prove each identity, we need only show that the left-hand side and the right-hand side have the same Laplace transform:

$$\begin{aligned}&\text{(i)}\ \ \mathcal{L}\{f*g\} = F(s)G(s) = G(s)F(s) = \mathcal{L}\{g*f\} \\ &\text{(ii)}\ \ \mathcal{L}\{(f*g)*h\} = \mathcal{L}\{f*g\}\,\mathcal{L}\{h\} = F(s)G(s)H(s) \\ &\qquad \mathcal{L}\{f*(g*h)\} = \mathcal{L}\{f\}\,\mathcal{L}\{g*h\} = F(s)G(s)H(s)\,.\end{aligned}$$

□

Equation (11.19) follows immediately from the Convolution Theorem. For, consider the initial value problem

$$ay'' + by' + cy = f(t)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,.$$

Apply the Laplace operator to get

$$(as^2 + bs + c)Y(s) = F(s)\,.$$

Therefore,

$$Y(s) = F(s)G(s)\,, \quad \text{where } G(s) = \frac{1}{as^2+bs+c}.$$

Let $g(t) = \mathcal{L}^{-1}\{G(s)\}$. It then follows from The Convolution Theorem that

$$y(t) = (f * g)(t)\,.$$

That $g(t)$ is the solution of the initial value problem

$$ay'' + by' + cy = 0\,, \quad y(0) = 0\,, \quad y'(0) = 1/a$$

follows by taking Laplace transforms of the initial value problem

$$a(s^2Y(s) - y'(0) - sy(0)) + b(sY(s) - y(0)) + cY(s) = 0\,.$$

Solving for $Y(s)$ and recalling that $y(0) = 0$ and $y'(0) = 1/a$, shows that $Y(s) = G(s)$. Hence, the solution is $g(t) = \mathcal{L}^{-1}\{G(s)\}$.

Remark 11.5. The function $g(t)$ is perhaps best viewed as the solution of the initial value problem

$$ay'' + by' + cy = \delta(t)\,, y(0) = y'(0) = 0\,.$$

For taking the Laplace transform of this initial value problem also yields $G(s)$:

$$(as^2 + bs + c)Y(s) = 1 \text{ or } Y(s) = \frac{1}{as^2+bs+c} = G(s)\,.$$

Example 11.32. Consider $y'' + 3y' + 2y = \sin(t)$, $y(0) = 0$, $y'(0) = 0$. Then

$$G(s) = \frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2},$$

So

$$g(t) = \mathcal{L}^{-1}\{G\}(t) = e^{-t} - e^{-2t}\,,$$

and the solution is, therefore, given by the convolution

$$y(t) = (e^{-t} - e^{-2t}) * \sin(t).$$

Remark 11.6. The solution to the initial value problem

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0,\; y'(0) = 0. \tag{11.22a}$$

is called the *(unit) impulse response function*; it is often denoted by $g(t)$. Taking the Laplace transform of (11.22a) shows that

$$G(s) = \mathcal{L}\{g(t)\} = \frac{1}{as^2+bs+c}\,. \tag{11.22b}$$

$G(s)$ is called the *transfer function.*

There is a rather nice formula for $g(t)$ in terms of the roots $r_1, r_2 = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2-4ac}}{2a}$ of the characteristic polynomial $as^2 + bs + c$. Let $\rho = \dfrac{b}{2a}$.

$$g(t) = \begin{cases} \dfrac{e^{-\rho t}}{a}\dfrac{\sinh(wt)}{w}, & \text{if } b^2 - 4ac > 0,\; w = \frac{\sqrt{b^2-4ac}}{2a}, \\[2ex] \dfrac{e^{-\rho t}}{a}t, & \text{if } b^2 - 4ac = 0, \\[2ex] \dfrac{e^{-\rho t}}{a}\dfrac{\sin(\omega t)}{\omega}, & \text{if } b^2 - 4ac < 0,\; \omega = \frac{\sqrt{4ac-b^2}}{2a}. \end{cases} \tag{11.22c}$$

Proof. (i) If $b^2-4ac>0$, let $w=\dfrac{\sqrt{b^2-4ac}}{2a}$. Then

$$G(s)=\frac{1}{as^2+bs+c}=\frac{1}{a(s+\rho+w)(s+\rho-w)}=\frac{1}{2aw}\left(\frac{1}{s+\rho-w}-\frac{1}{s+\rho+w}\right).$$

Hence,

$$g(t)=\mathcal{L}^{-1}\left\{G(s)\right\}=\frac{1}{2aw}\left(e^{-(\rho-w)\,t}-e^{-(\rho+w)\,t}\right)=\frac{e^{-\rho t}}{aw}\sinh(w\,t)$$

(ii) If $b^2-4ac=0$, then $G(s)=\dfrac{1}{a(s+\rho)^2}$. Therefore

$$g(t)=\frac{1}{a}\mathcal{L}^{-1}\left\{\frac{1}{s-(-\rho)}\right\}=\frac{1}{a}te^{-\rho t}.$$

(iii) If $b^2-4ac<0$, let $\omega=\dfrac{\sqrt{4ac-b^2}}{2a}$. Then

$$G(s)=\frac{1}{as^2+bs+c}=\frac{1}{a(s+(\rho+\omega i)(s+\rho-\omega i)}=\frac{1}{2a\omega i}\left(\frac{1}{s+\rho-\omega i}-\frac{1}{s+\rho+\omega i}\right).$$

Hence, by Equation (11.6),

$$g(t)=\mathcal{L}^{-1}\left\{G(s)\right\}=\frac{1}{2a\omega i}\left(e^{-(\rho-\omega i)\,t}-e^{-(\rho+\omega i)\,t}\right)=\frac{e^{-\rho t}}{a\omega}\left(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right)=\frac{e^{-\rho t}}{a\omega}\sin(\omega t).$$

□

The *state-free solution* is the solution to the initial value problem

$$ay''+by'+cy=f(t),\quad y(0)=0,\ y'(0)=0. \tag{11.23}$$

Taking Laplace transforms gives

$$Y(s)=G(s)F(s). \tag{11.24}$$

By The Convolution Theorem, the state-free solution is the function $(f*g)(t)$. The *input-free solution* is the solution to

$$ay''+by'+cy=0,\quad y(0)=y_0,\ y'(0)=y_0'. \tag{11.25}$$

Proposition 9. *The solution of the initial value problem*

$$ay''+by'+cy=f(t),\quad y(0)=y_0,\quad y'(0)=y_0'$$

is the sum of the state-free and input-free solutions:

$$y(t)=(f*g)(t)+ay_0\,g'(t)+(ay_0'+by_0)\,g(t). \tag{11.26}$$

Proof. Taking the Laplace transform gives of the initial value problem gives the formula

$$Y(s)=F(s)G(s)+(ay_0\,s+(ay_0'+by_0)G(s).$$

for the Laplace transform of the solution. On the other hand, taking the Laplace transform of (11.26) using the convolution theorem gives the same thing. Consequently, the two functions agree and (11.26) is the solution of the initial value problem. □

Example 11.33. Consider $y''+4y=f(t)$, $y(0)=2$, $y'(0)=3$. Then $g(t)=\mathcal{L}^{-1}\left\{1/(s^2+4)\right\}=1/2\sin(2t)$. The state-free solution is

$$\frac{1}{2}\sin(2t)*f(t)=\frac{1}{2}\int_0^t\sin(2u)f(t-u)\,du=\frac{1}{2}\int_0^t\sin(2(t-u))f(u)\,du.$$

The input-free solution is

$$ay_0\,g'(t)+(ay_0'+by_0)g(t)=2\cos(2t)+\frac{3}{2}\sin(2t).$$

Therefore,

$$y(t) = 2\cos(2t) + \frac{3}{2}\sin(2t) + \frac{1}{2}\int_0^t \sin(2(t-u))f(u)\,du.$$

EXERCISES 7.

(1) Using Laplace Transforms, find the solution of each of the following initial value problems, (*Notice that the left hand sides are the same.*)
 (a) $y'' - 3y' + 2y = 0$, $y(0) = 0$, $y'(0) = 0$.
 (b) $y'' - 3y' + 2y = t$, $y(0) = 0$, $y'(0) = 0$.
 (c) $y'' - 3y' + 2y = e^{2t}$, $y(0) = 0$, $y'(0) = 0$.
(2) Evaluate each of the following Laplace transforms or inverse Laplace transforms
 (a) $\mathcal{L}\left\{2t + u_2 \cos(t-2)\right\}$
 (b) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s-2)(s+1)}\right\}$
 (c) $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s^2+2s+5}\right\}$
 (d) $\mathcal{L}\left\{t \sin(2t)\right\}$
 (e) $\mathcal{L}^{-1}\left\{\dfrac{2s+1}{4s^2+4s+5}\right\}$
 (f) $\mathcal{L}^{-1}\left\{\dfrac{2s+1}{s(s^2+4)}\right\}$
 (g) $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+4s+5)}\right\}$
(3) Solve the initial value problem $y'' - y = u_4(t) - u_5(t)$, $y(0) = 1$, $y'(0) = 0$.
(4) Consider the following initial value problem:

$$y'' + 2y' + 5y = u_2(t)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,.$$

 (a) Let $Y(s)$ denote the Laplace transform for the solution. Find $Y(s)$.
 (b) Find the solution $y(t)$ by computing the inverse Laplace transform of $Y(s)$.
 (c) Give the numerical value of $y(3)$. (*Use a calculator for this part.*)
(5) Compute $y(7)$, where $y(t)$ is the solution of the the initial value problem

$$y'' - y = u_4(t) - u_5(t) + u_6(t)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,.$$

(6) Compute $y(10\pi)$, where $y(t)$ is the solution of the the initial value problem

$$y'' + y = \delta(t-\pi)) - \delta(t-2\pi) + \delta(t-3\pi)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,.$$

(7) Consider the initial value problem

$$y'' + y = f(t)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,,$$

where

$$f(t) = 1 + 2\sum_{k=1}^{\infty}(-1)^k u_{k\pi}(t)$$

 (a) Draw the graph of $f(t)$ for $0 \le t \le 6\pi$.
 (b) Find a formula for $F(s)$, the Laplace transform of $f(t)$.
 (c) Find $Y(s)$, the Laplace transform of the solution $y(t)$. Express your answer in the form

$$Y(s) = H(s)\sum_{k=0}^{\infty} e^{-kTs}\,.$$

 (d) Find $h(t)$, the inverse Laplace transform of $H(s)$ (from part (c)) and use this to find a formula for the solution $y(t)$.
 (e) Graph $y(t)$ for $0 \le t < 8\pi$. (Notice that although $y(t)$ is expressed as an infinite series, most terms in the series vanish for $t < 8\pi$.)

(8) Consider the initial value problem

$$y'' + 0.2y' + y = f(t)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,,$$

where $f(t)$ is the same as in the previous problem. Repeat steps (b)–(e) of that problem.

(9) When making prescriptions for drugs that will be taken over a prolonged period of time it is necessary to take into account the fact that the concentration of a drug in the bloodstream grows after each subsequent dose. In this problem you derive a formula in standard use by physicians.

Let c_0 be the concentration of a drug immediately after the first dose (this is proportional to the size of the dose and the weight of the patient and is information known for all commonly used drugs). After t units of time the concentration will be given by the formula $c = c_0 e^{-rt}$ where r is a constant that depends on the drug (this is just the law of exponential decay and again the value of r is known for all commonly used drugs).

Now suppose that the same dose is taken every T units of time (e.g. every 4 hours). Let $y(t)$ denote the concentration of the drug in the bloodstream t hours after the first dose. Then $y(t)$ is a solution of the following initial value problem

$$y' + ry = f(t) = \sum_{k=0}^{\infty} c_0 \delta(t - kT)\,, y(0) = 0$$

(a) Compute $Y(s)$, the Laplace Transform of $y(t)$. (Note: it is an infinite series.)

(b) Now compute the inverse Laplace Transform to obtain a formula for $y(t)$ as another infinite series.

(c) Use part (b) to find a formula for $c_k = y(kT)$, the concentration of the drug right after a dose is administered at time $t = kT$.

(d) Initially, c_k will grow pretty rapidly, but it will eventually level off and approach

$$c_\infty = \lim_{k\to\infty} c_k\,.$$

Use the formula for the sum of a geometric series to find a formula for c_∞.

(e) Find the value of r if the half-life of the drug in the bloodstream is 3 hours.

(f) Use the result of the previous parts of the problem to obtain a graph of the ratio c_∞/c_0 as a function of T for a drug with a half-life of 3 hours. What is the time between doses if the stable concentration is twice the initial concentration?

Appendices

A. Basic Formulas from Algebra, Trigonometry, and Calculus

Algebra:

Completing the square: $X^2 + bX + c = (X + \frac{b}{2})^2 - \frac{b^2}{4} + c.$

Quadratic formula: roots of $aX^2 + bX + c$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Exponents: $a^b \cdot a^c = a^{b+c}$; $\dfrac{a^b}{a^c} = a^{b-c}$; $(a^b)^c = a^{bc}$; $a^{1/b} = \sqrt[b]{a}$

Logarithms: $\ln(1) = 0$; $\ln(e) = 1$; $\ln(ab) = \ln(a) + \ln(b)$; $\ln\left(\dfrac{a}{b}\right) = \ln(a) - \ln(b)$

Geometry:

Circle: circumference $= 2\pi r$; area $= \pi r^2$;

Sphere: vol $= \frac{4}{3}\pi r^3$; surface area $= 4\pi r^2$

Cylinder: vol $= \pi r^2 h$; lateral area $= 2\pi rh$; surface area $= 2\pi rh + 2\pi r^2$.

Cone: vol $= \frac{1}{3}\pi r^2 h$; lateral area $= \pi r\sqrt{r^2 + h^2}$; surface area $= \pi r\sqrt{r^2 + h^2} + \pi r^2$

Analytic geometry

Point-slope formula for straight line: $y = y_0 + m(x - x_0)$

Equation for circle centered at (h, k): $(x - h)^2 + (y - k)^2 = r^2$

Equation for ellipse centered at (h, k): $\dfrac{(x - a)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$

Trigonometry

$\sin = \dfrac{\text{opposite}}{\text{hypotenuse}}$; $\cos = \dfrac{\text{adjacent}}{\text{hypotenuse}}$; $\tan = \dfrac{\text{opposite}}{\text{adjacent}}$;

$\sec = \dfrac{1}{\cos}$; $\csc = \dfrac{1}{\sin}$; $\cot = \dfrac{1}{\tan}$; $\tan = \dfrac{\sin}{\cos}$; $\cot = \dfrac{\cos}{\sin}$;

$\sin(x) = \cos\left(\dfrac{\pi}{2} - x\right)$; $\cos(x) = \sin\left(\dfrac{\pi}{2} - x\right)$

$\sin(x + \pi) = -\sin(x)$; $\cos(x + \pi) = -\cos(x)$

$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$; $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

;

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$; $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$;

$\sin(x) + \sin(y) = 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{y-x}{2}\right)$ $\quad \sin(x) - \sin(y) = 2\sin\left(\dfrac{x-y}{2}\right)\cos\left(\dfrac{x+y}{2}\right)$

$\cos(x) + \cos(y) = 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{y-x}{2}\right)$ $\quad \cos(x) - \cos(y) = 2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{y-x}{2}\right)$

$\sin^2(x) + \cos^2(x) = 1$; $\tan^2(x) + 1 = \sec^2(x)$; $1 + \cot^2(x) = \csc^2(x)$.

$\sin^2 x = \dfrac{1 - \cos(2x)}{2}$; $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$

Values at common angles:

$\theta =$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin(\theta) =$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos(\theta) =$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan(\theta) =$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	—

The phase-shift formula:
$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t) = A\cos(\omega t - \phi) = A\cos\left(\omega(t - t_0)\right)$,
where $t_0 = \dfrac{\phi}{\omega}$; $A = \sqrt{C_1^2 + C_2^2}$,
$\cos(\phi) = \dfrac{C_1}{\sqrt{C_1^2 + C_2^2}}$. $\sin(\phi) = \dfrac{C_2}{\sqrt{C_1^2 + C_2^2}}$, $\tan(\phi) = \dfrac{C_2}{C_1}$

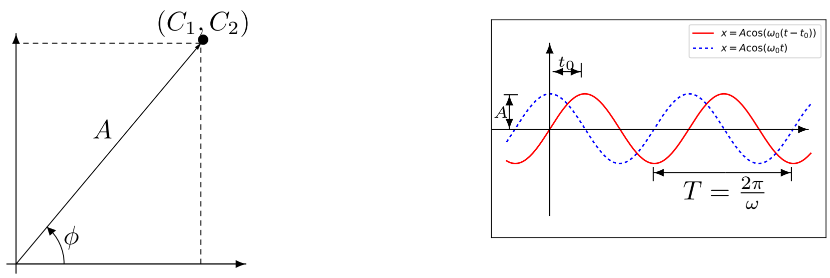
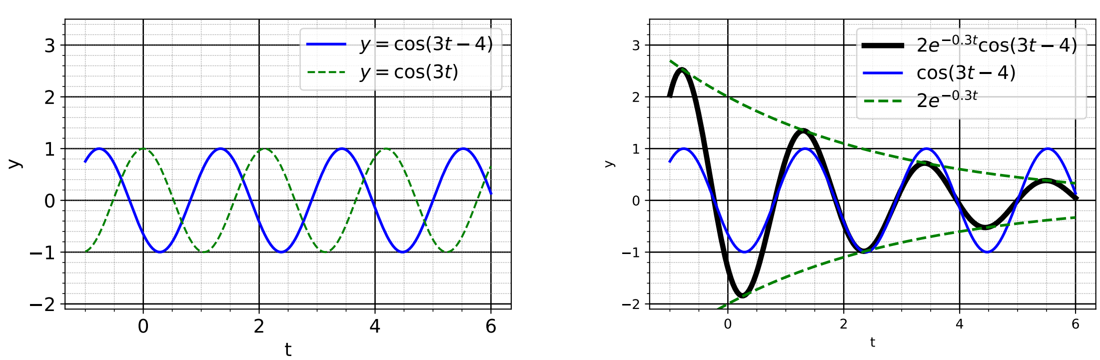


FIGURE A.1. The figure above illustrates how to graph the function $y(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$. It is a "cosine curve" of amplitude $A = \sqrt{C_1^2 + C_2^2}$, period $T = \frac{2\pi}{\omega}$, shifted by $t_0 = \frac{\phi}{\omega}$ units. The angle ϕ is called the phase angle or phase shift. Notice, however, that the actual shift is the quantity $t_0 = \phi/\omega$ rather than ϕ.

EXAMPLE A.1. *Sketch the graph of the function* $y(t) = 2\,e^{-0.3t}\cos(3\,t - 4)$.

Step 1: Graph the function $f(t) = \cos(3t)$, a cosine function with period $2\pi/3 \approx 2$.
Step 2: Graph the function $f(t - 4/3) = \cos(3\,t - 4) = \cos\left(3(t - 4/3)\right)$. This is the graph of $\cos(3t)$ shifted to the right by $4/3$ units.
Step 3: Next graph the functions $g(t) = 2e^{-0.3\,t}$ and $-g(t) = -2e^{-0{,}3\,t}$.
Step 4: Finally graph $y(t) = 2e^{-0.3t}\cos(3\,t - 4)$, which is the product $g(t)\cdot f(t - 4/3)$ —a shifted cosine function $\cos(3t - 4)$ with varying amplitude $g(t) = 2e^{-0.3\,t}$. Notice that in the right-hand figure below, the graph of $y(t)$ touches the graphs of $g(t)$ and $-g(t)$ (the dotted curves) when $\cos(3t - 4) = \pm 1$, which is where $3t - 4$ is an integer multiple of π:

$$3t - 4 = n\pi \qquad \Longleftrightarrow \qquad t = \frac{4}{3} + n\frac{\pi}{3}\,.$$



Calculus

Basic differentiation formulas:

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}, \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right), \text{ for } v \neq 0.$$

Chain rule:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}.$$

Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_{f(x)}^{g(x)} h(u)\,du = h\left(g(x)\right)g'(x) - h\left(f(x)\right)f'(x)$$

Derivatives of specific functions:

$$\frac{dx^n}{dx} = nx^{n-1};\ \frac{de^x}{dx} = e^x;\ \frac{dln|x|}{dx} = \frac{1}{x};$$

$$\frac{d\sin(x)}{dx} = \cos(x);\ \frac{d\cos(x)}{dx} = -\sin(x);\ \frac{d\tan(x)}{dx} = \sec^2(x);$$

$$\frac{d\arcsin(x)}{dx} = \frac{1}{\sqrt{1-x^2}};\ \frac{d\arctan(x)}{dx} = \frac{1}{1+x^2}.$$

Basic integration formulas:

$$\int (u+v)\,dx = \int u\,dx + \int v\,dx;\ \int au\,dx = a\int u\,dx;$$

Substitution:

$$\int f\left(u(x)\right)u'(x)\,dx = F\left(u(x)\right), \text{ where } \int f(u)du = F(u);$$

Integration by parts:

$$\int u\,dv = uv - \int v\,du;$$

Standard integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C\ (n \neq -1);\ \int \frac{dx}{x} = \ln|x| + C;\ \int e^x\,dx = e^x + C;$$

$$\int \sin(x)\,dx = -\cos(x) + C;\ \int \cos(x)\,dx = \sin(x) + C;\ \int \tan(x)\,dx = -\ln|\cos(x)| + C;$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C;\ \int \frac{xdx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + C;$$

$$\int \frac{dx}{1+x^2} = \arctan(x) + C;\ \int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C;$$

$$\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\ln(1+x^2) + C$$

B. Review of Partial Fractions

When computing integrals and inverse Laplace transforms, *rational functions*, i.e. ratios of polynomials, arise:

$$R(s) = \frac{P(s)}{Q(s)} = \frac{p_n s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0}{q_m s^m + q_{m-1}s^{m-1} + \cdots + q_1 s + q_0}$$

It is useful to express $R(s)$ as a sum of simple fractions, this is called the *partial fractions expansion* of $R(s)$. Here's how to do that:

Step 0: If degree $P(s) \geq$ degree $Q(s)$, first perform a long division.

$$\textit{Example:} \quad \frac{s^5+1}{s^4+2s^2+1} = s - \frac{2s^3+s-1}{s^4+2s^2+1}\,.$$

Step 1: If the denominator hasn't already been factored, factor it completely.

$$\textit{Example:} \quad -\frac{2s^3+s-1}{s^4+2s^2+1} = -\frac{2s^3+s-1}{(s^2+1)^2}\,.$$

Step 2: For each factor in the denominator of the form $(s+a)^p$ include terms of the form

$$\frac{A_1}{(s+a)} + \frac{A_2}{(s+a)^2} + \cdots + \frac{A_p}{(s+a)^p}\,,$$

and for each term in the denominator of the form $(s^2+bs+c)^q$, include terms of the form

$$\frac{A_1 s + B_1}{(s^2+bs+c)} + \frac{A_2 s + B_2}{(s^2+bs+c)^2} + \cdots \frac{A_p s + B_p}{(s^2+bs+c)^q}$$

in the partial fractions expansion.

Examples:

$$-\frac{2s^3+s-1}{(s^2+1)^2} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+1)^2}\,,$$

$$\frac{2s^3-s^2+2s}{(s-1)^2(s^2+s+1)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{Cs+D}{(s^2+s+1)}\,,$$

$$\frac{3s^4+3s^3-3s^2-2s+4}{s^2(s-1)(s^2+2s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{Ds+E}{(s^2+2s+2)}\,.$$

$$\frac{3s^4+3s^3-3s^2-2s+4}{s^2(s-1)(s^2+2s+2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{Ds+E}{(s^2+2s+2)} + \frac{Fs+G}{(s^2+2s+2)^2}\,.$$

Step 3: Determine a system of equations for the unknown constants by collecting terms in the partial fractions expansion and equating the numerator of the result with the numerator of the original fraction.

$$\textit{Example:} \quad \frac{-2s^3-s+1}{(s^2+1)^2} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+1)^2} = \frac{As^3+Bs^2+(A+C)s+(B+D)}{(s^2+1)^2}\,.$$

So $A=-2 \quad B=0\,, \quad A+C=-1\,, \quad B+D=1.$

Step 4: Solve the system to determine the unknown constants.

Example (From Step 3): $A=-2\,, \quad B=0\,, \quad C=1\,, \quad D=1.$

Hence, $\dfrac{-2s^3-s+1}{(s^2+1)^2} = \dfrac{-2s}{(s^2+1)} + \dfrac{s+1}{(s^2+1)^2}.$

C. Table of Laplace Transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\dfrac{1}{s}$	e^{bt}	$\dfrac{1}{s-b}$
t^n, $n = 1, 2, 3 \ldots$	$\dfrac{n!}{s^{n+1}}$	$t^n e^{bt}$	$\dfrac{n!}{(s-b)^{n+1}}$
$\sin(at)$	$\dfrac{a}{s^2+a^2}$	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
$e^{bt}\sin(at)$	$\dfrac{a}{(s-b)^2+a^2}$	$e^{bt}\cos(at)$	$\dfrac{(s-b)}{(s-b)^2+a^2}$
$t\sin(at)$	$\dfrac{2as}{(s^2+a^2)^2}$	$t\cos(at)$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$
$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
$e^{bt}\sinh(at)$	$\dfrac{a}{(s-b)^2-a^2}$	$e^{bt}\cosh(at)$	$\dfrac{(s-b)}{(s-b)^2-a^2}$
$u_c(t)$	$\dfrac{e^{-cs}}{s}$	$\delta(t-c)$	e^{-cs}

General Formulas			
$a\,f(t) + b\,g(t)$	$a\,F(s) + b\,G(s)$	$f(at)$	$\dfrac{1}{a}F(s/a)$
$e^{bt}f(t)$	$F(s-b)$	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	$u_c(t)f(t)$	$e^{-cs}\mathcal{L}\{f(t+c)\}$
$\displaystyle\int_0^t f(u)\,du$	$\dfrac{F(s)}{s}$	$\dfrac{1}{t}f(t)$	$\displaystyle\int_s^\infty F(u)\,du$
$f'(t)$	$sF(s) - f(0)$	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f * g(t) = \displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	$f(t+T) = f(t)$	$\dfrac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$

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