Lecture 26

Convolutions.
The impulse-response function

Example.

$f(t)$ = rate (in kg/yr) that chemical is added to lake.

t = time in years.

$y(t)$ = quantity (in kg) of chemical in the lake.

V = volume of lake in meters^3

b = rate (in m^3/yr) that water both enters & leaves the lake.

$a = b/V$

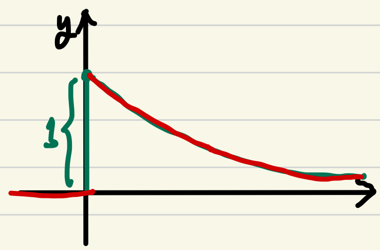
$$\frac{dy}{dt} = f(t) - b\left(\frac{y}{V}\right)$$

$$\begin{cases} \frac{dy}{dt} + ay = f(t) \\ y(0) = 0 \end{cases}$$

Take the Laplace transform:

$$(s+a)Y(s) = F(s) \Rightarrow Y(s) = \frac{1}{s+a}\cdot F(s) \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+a}\cdot F(s)\right\}$$

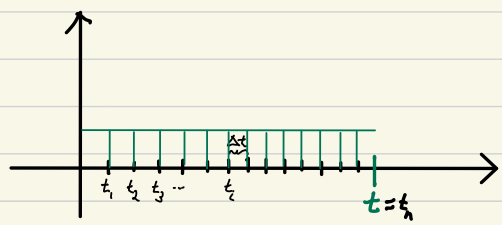
- Suppose at time $t=0$, 1 kg of the chemical is dumped into the lake.



$$f(t) = \delta(t) \Rightarrow F(s) = 1$$

Then
$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+a}\cdot 1\right\} = e^{-at}$$

- Now suppose the chemical is added at a constant rate: $f(t) = r$



Idea: Replace $f(t)$ by $\sum_{i=1}^{n} r\Delta t\,\delta_{t_i}(t)$

Then
$$Y(s) \approx \sum_{i=1}^{n} r\Delta t \frac{e^{-t_i s}}{s+a} \Rightarrow y(t) \approx \sum_{i=1}^{n} r\Delta t\, u_{t_i}(t)\, e^{-a(t-t_i)}$$
$$= \sum_{i=1}^{n} r e^{-a(t-t_i)}\Delta t$$

Let $\Delta t \to 0$ to get
$$y(t) = \int_0^t r e^{-a(t-\tau)}\,d\tau = \frac{r}{a}\left(1 - e^{-at}\right)$$

$$\int_0^t f(\tau) e^{-a(t-\tau)}\,d\tau = f * e^{-at}$$

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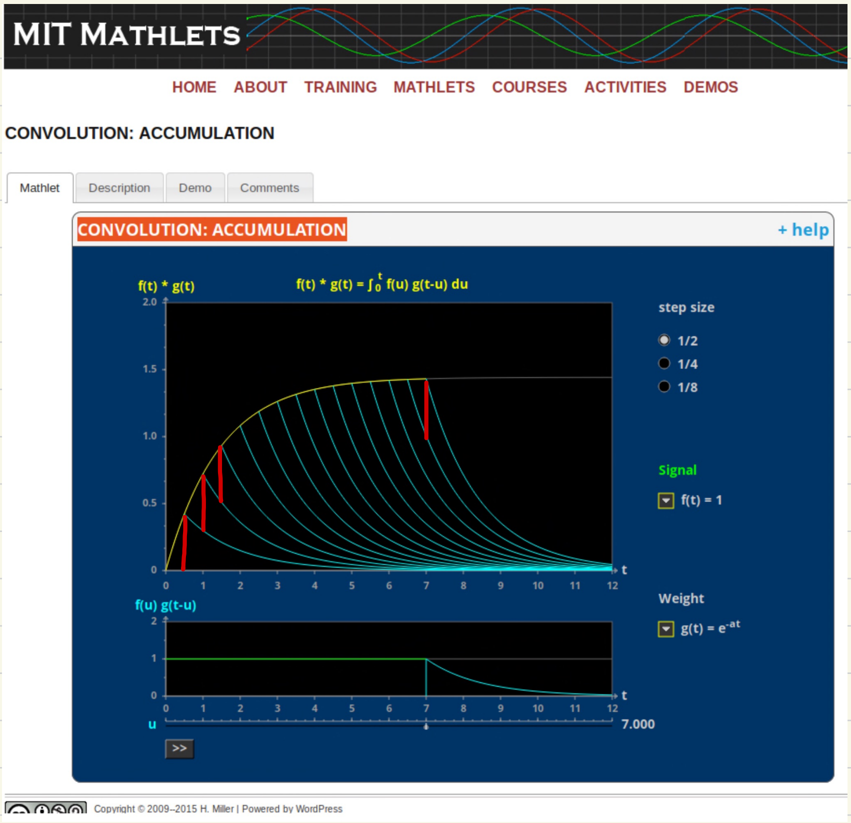
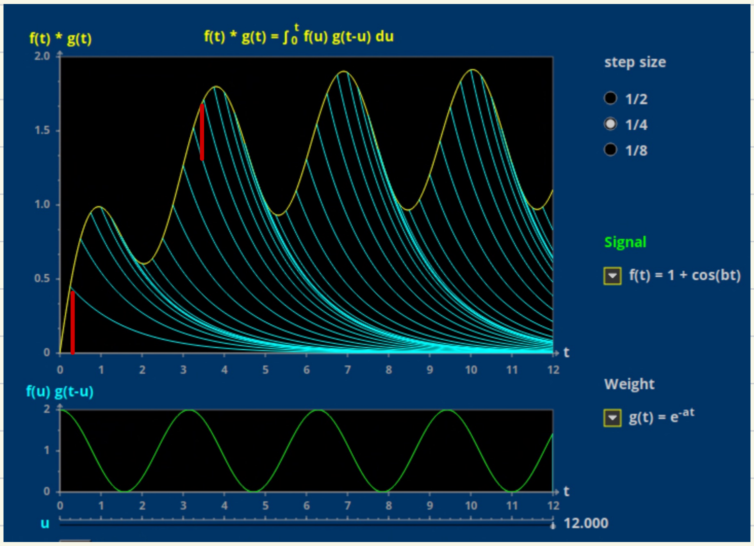




Goal: Obtain a better understanding of solutions of the initial value problem

$$ay'' + by' + cy = f(t)$$
$$y(0) = y_0, \quad y'(0) = y_0'$$

Idea: Break problem into 2 parts:

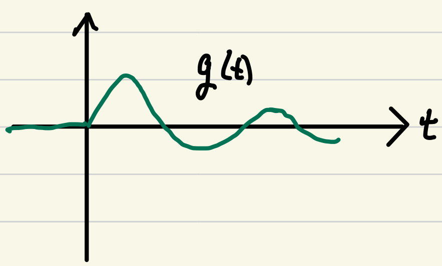
$$(1) \quad \begin{cases} ay'' + by' + cy = 0 \\ y(0) = y_0, \quad y'(0) = y_0' \end{cases}$$ the <u>input-free solution</u>

$$(2) \quad \begin{cases} ay'' + by' + cy = f(t) \\ y(0) = 0, \quad y'(0) = 0 \end{cases}$$ the <u>state-free solution</u>

Both of these will be expressed in terms of the (unit) <u>impulse-response function</u> $g(t)$, which is the solution to the following initial value problem:

$$(*) \quad \begin{cases} ay'' + by' + cy = \delta(t) \\ y(0) = 0, \quad y'(0) = 0 \end{cases}$$

The Laplace transform of $(*)$ is $(as^2 + bs + c)\,G(s) = 1$
so
$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{as^2 + bs + c}\right\}$$



<u>Note.</u> $g(t)$ is also the solution of

$$(**) \quad \begin{cases} ay'' + by' + cy = 0 \\ y(0) = 0, \quad y'(0) = 1/a \end{cases}$$

To see this, take the Laplace transform of $(**)$:

$$a\left(s^2 Y(s) - y'(0)\right) + b\left(sY(s)\right) + cY(s) = 0$$

$$\Rightarrow \quad a\left(s^2 Y(s) - 1/a\right) + b\left(sY(s)\right) + cY(s) = 0$$

$$\Rightarrow \quad Y(s) = \frac{1}{as^2 + bs + c} = G(s).$$ Hence, $g(t) = \mathcal{L}^{-1}\{Y(s)\}$

(I) <u>The input-free solution:</u>

<u>Goal:</u> Find a formula in terms of $g(t)$ for the solution of

$$\begin{cases} a y'' + b y' + c y = 0 \\ y(0) = y_0 \ , \ y'(0) = y_0' \end{cases}$$

Take the Laplace transform:

$$a\left(s^2 Y(s) - y_0' - y_0 s\right) + b\left(s Y(s) - y_0\right) + c Y(s) = 0$$

$$\Rightarrow (a s^2 + bs + c)\, Y(s) = (a y_0)\, s + (a y_0' + b y_0)$$

$$\Rightarrow Y(s) = (a y_0) \frac{s}{a s^2 + bs + c} + (a y_0' + b y_0) \frac{1}{a s^2 + bs + c}$$

$$= (a y_0) \cdot s \cdot G(s) + (a y_0' + b y_0)\, G(s)$$

So $y(t) = (a y_0)\, \mathcal{L}^{-1}\{s\, G(s)\} + (a y_0' + b y_0)\, \mathcal{L}^{-1}\{G(s)\}$

$$= \boxed{(a y_0)\, g'(t) + (a y_0' + b y_0)\, g(t)}$$ <u>input-free solution</u>

(2) The state-free solution.

Goal: Find a formula in terms of $g(t)$ for the solution of

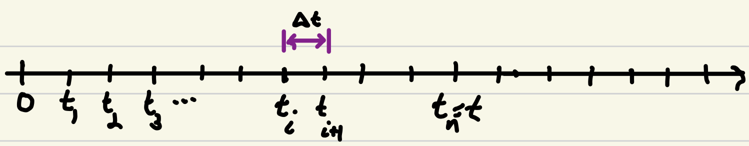
$$\begin{cases} a y'' + b y' + c y = f(t) \\ y(0) = 0 \ , \ y'(0) = 0 \end{cases}$$

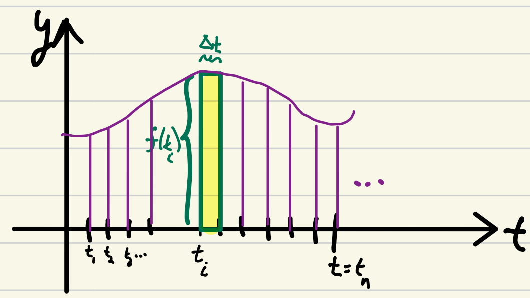
Take the Laplace transform:

$$(a s^2 + bs + c) Y(s) = F(s) = \int_0^\infty f(\tau) e^{-\tau s} d\tau$$

Then $\quad y(t) = \mathcal{L}^{-1}\left\{ F(s) \cdot G(s) \right\}$ where $\quad G(s) = \dfrac{1}{as^2+bs+c}$

Fix t and divide the real line into small subintervals of width Δt with $t = t_n$ as shown:





Notice

$$F(s) = \int_0^\infty f(\tau) e^{-\tau s} d\tau \approx \sum_{i=1}^\infty f(t_i) e^{-t_i s} \Delta t = \mathcal{L}\left\{ \sum_{i=1}^\infty f(t_i) \Delta t \, \delta(t - t_i) \right\}$$

(Riemann sum)

So

$$y(t) \approx \mathcal{L}^{-1}\left\{ \sum_{i=1}^\infty f(t_i) \Delta t \cdot \frac{e^{-t_i s}}{as^2+bs+c} \right\}$$

$$= \sum_{i=1}^\infty f(t_i) \Delta t \cdot \mathcal{L}^{-1}\left\{ \frac{e^{-t_i s}}{as^2+bs+c} \right\}$$

$$= \sum_{i=1}^\infty f(t_i) \Delta t \cdot u_{t_i}(t) \cdot g(t - t_i)$$

$$u_{t_i}(t) = \begin{cases} 1 & \text{for } t_i \le t \\ 0 & \text{for } t_i > t \end{cases}$$

$$= \sum_{i=1}^n f(t_i) \, g(t - t_i) \, \Delta t$$

Let $\Delta t \to 0$:

$$\boxed{y(t) = \int_0^t f(\tau) \, g(t - \tau) \, d\tau}$$

Convolution of f and g

$f * g \, (t)$

Convolutions

Def^{n} The convolution of $f(t)$ and $g(t)$

$$f*g\,(t) = \int_0^t f(u)g(t-u)\,du$$

Theorem (The Convolution Theorem).

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\cdot\mathcal{L}\{g\}$$

Additional Properties

$$\begin{cases} f*g = g*f \\ (f*g)*h = f*(g*h) \\ f*(g+h) = f*g + f*h \\ f*(cg) = c\, f*g \quad (c = \text{const}) \end{cases}$$

Summary

Solution of $\begin{cases} ay'' + by' + cy = f(t) \\ y(0) = y_0 \quad y'(0) = y_0' \end{cases}$

$$y(t) = \underbrace{f*g\,(t)}_{\text{"state free"}} + \underbrace{a y_0\, g'(t) + (a y_0' + b y_0)\, g(t)}_{\text{"input-free"}}$$

where $g(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{as^2+bs+c}\right\}$ (unit) impulse-response function

(the solution to $ay''+by'+cy = \delta(t)$ $y(0)=0$, $y'(0)=0$

<u>Example.</u> $L[y] = y'' + 2y' + 10y$

$$G(s) = \frac{1}{s^2+2s+10} = \frac{1}{(s+1)^2+9} = \frac{1}{3}\,\frac{3}{(s-1)^2+3^2}$$

$$g(t) = \frac{e^{-t}}{3}\sin(3t)$$

$$g'(t) = e^{-t}\left(\cos(3t) - \frac{1}{3}\sin(3t)\right)$$

$$\begin{cases} y'' + 2y' + 10y = t \\ y(0)=0, \quad y'(0)=0 \end{cases} \Rightarrow$$

$$y(t) = \int_0^t (t-u)\,\frac{e^{-u}}{3}\sin(3u)\,du$$

Aargh!

$$\begin{cases} y'' + 2y + 10y = 0 \\ y(0) = 5, \quad y'(0) = 7 \end{cases} \Rightarrow$$

$$\begin{aligned} y(t) &= 5g'(t) + (7 + 2\cdot 5)\,g(t) \\ &= 5g'(t) + 17g(t) \\ &= 5\,e^{-t}\left(\cos(3t) - \frac{1}{3}\sin(3t)\right) \\ &\quad + 17\,\frac{e^{-t}}{3}\sin(3t) \\ &= e^{-t}\left(5\cos(3t) + 4\sin(3t)\right) \end{aligned}$$

$$Y(s) = \frac{1}{s^2((s+1)^2+9)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C(s+1)+D}{(s+1)^2+9}$$

$$= \frac{1/10}{s^2} - \frac{1/50}{s} + \frac{\frac{1}{50}(s+1) - \frac{4}{50}}{(s+1)^2+9}$$

$$y(t) = \frac{1}{10}t - \frac{1}{50} + \frac{1}{50}e^{-t}\cos(3t) - \frac{4}{150}e^{-t}\sin(3t)$$