Lecture 25

Laplace transforms of periodic functions

Example. $f(t) = \begin{cases} 3t-1 & \text{for } 0 \le t < 2 \\ 2 & \text{for } 2 \le t < 5 \\ 10-t & \text{for } t \ge 5 \end{cases}$

Example. $y'' - 6y' + 18y = 10e^{4t}$

$y(0) = 2 \quad y'(0) = 4$

$$s^2 Y(s) - 4 - 2s - 6(sY(s) - 2) + 18Y(s) = \frac{10}{s-4}$$

$$(s^2 - 6s + 18)Y(s) = 2s - 8 + \frac{10}{s-4}$$

$$\{(s-3)^2 + 9\} Y(s) = 2(s-3) - 2 + \frac{10}{s-4}$$

$$Y(s) = \frac{2(s-3)}{(s-3)^2+9} - \frac{2}{3}\frac{3}{(s-3)^2+9} + \frac{10}{((s-3)^2+9)(s-4)}$$

$$\frac{10}{((s-3)^2+9)(s-4)} = \frac{A}{s-4} + \frac{B(s-3)+C}{(s-3)^2+9} \qquad s=4: \quad A = \frac{10}{(4-3)^2+9} = 1$$

$$s = 3+3i: \quad B(3i) + C = \frac{10}{3+3i-4} = \frac{10}{-1+3i} = \frac{10}{10}(-1-3i)$$

$$= -3i - 1 \Rightarrow B = -1 \quad C = -1$$

$$Y(s) = \frac{2(s-3)}{(s-3)^2+9} - \frac{2}{3}\frac{3}{(s-3)^2+9} + \frac{1}{s-4} - \frac{(s-3)+1}{(s-3)^2+9}$$

$$y(t) = 2e^{3t}\cos(3t) - \frac{2}{3}e^{-3t}\sin(3t) + e^{4t}$$

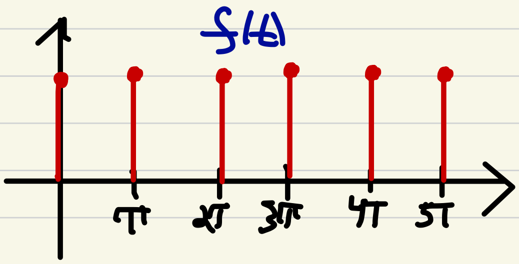
$$- e^{3t}\cos(3t) - \frac{1}{3}e^{-3t}\sin(3t)$$

$$= e^{3t}\cos(3t) - e^{-3t}\sin(3t) + e^{4t}$$

Example

$$y'' + y = f(t) = \sum_{k=0}^{\infty} \delta_{k\pi}(t)$$

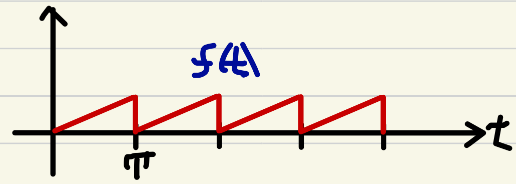
$$y(0) = 0, \quad y'(0) = 0$$



Example

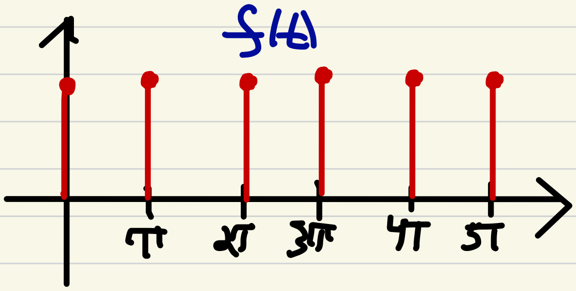
$$y'' + y = f(t) = \frac{t}{\pi} - \sum_{k=0}^{\infty} u_{k\pi}(t)$$

$$y(0) = 0, \quad y'(0) = 0$$

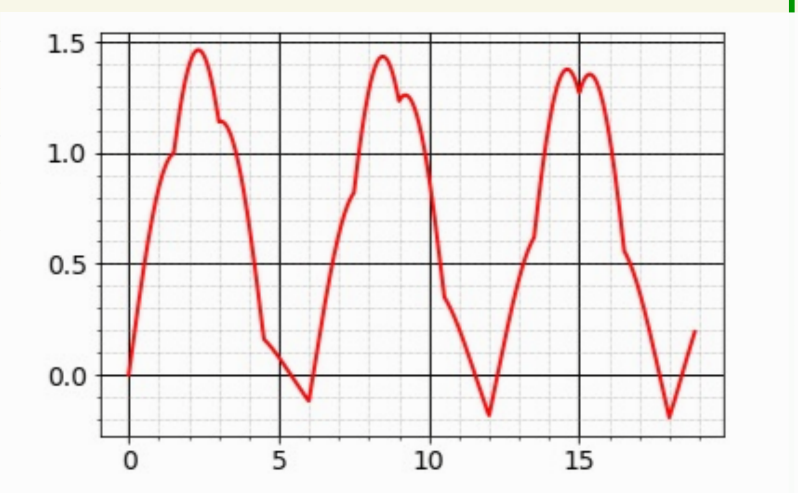


$$y'' + y = f(t) = \sum_{k=0}^{\infty} \delta_{kT}(t)$$

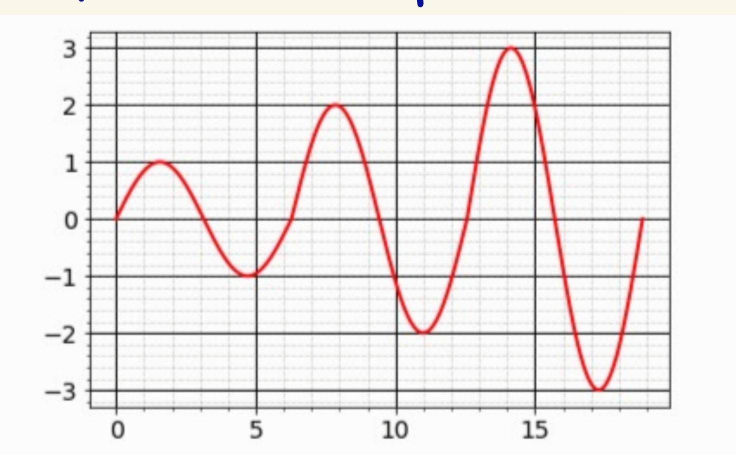
$y(0) = 0, \; y'(0) = 0$



T = 15

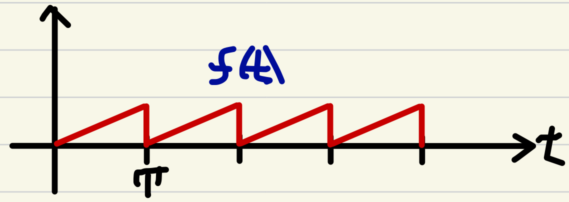


T = 2π (natural period)

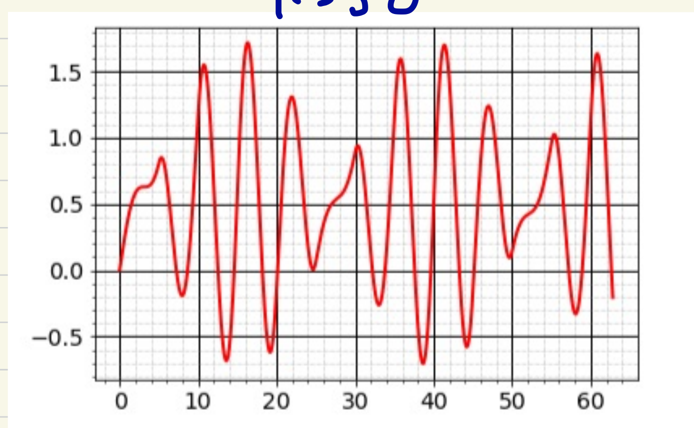


$$y'' + y = f(t) = \frac{t}{\pi} - \sum_{k=0}^{\infty} u_{kT}(t)$$

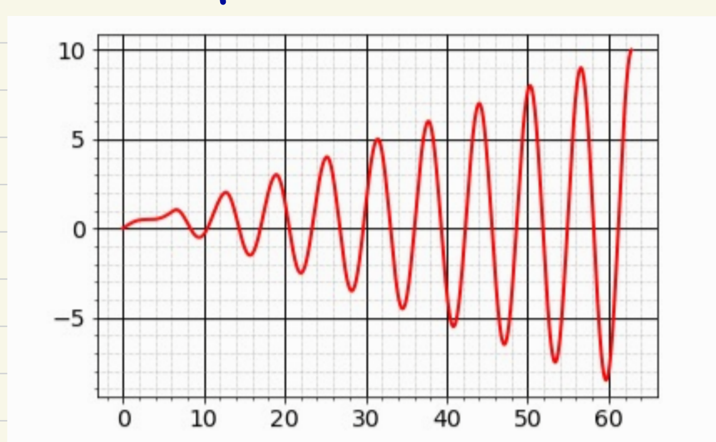
$y(0) = 0, \; y'(0) = 0$



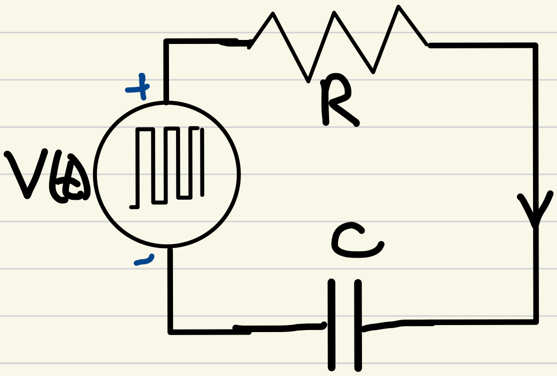
T = 50

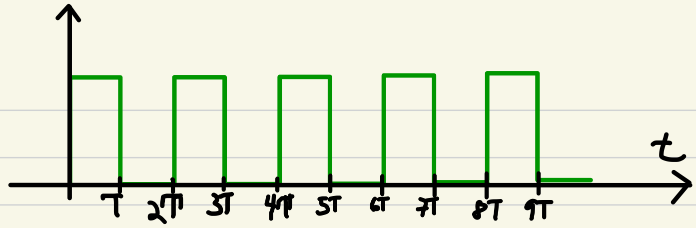


T = 2π



Example. (RC-circuit)

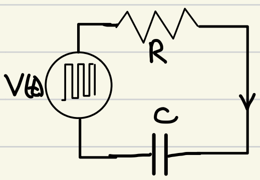




$$RC\,\frac{dV_c}{dt} + V_c = V(t)$$

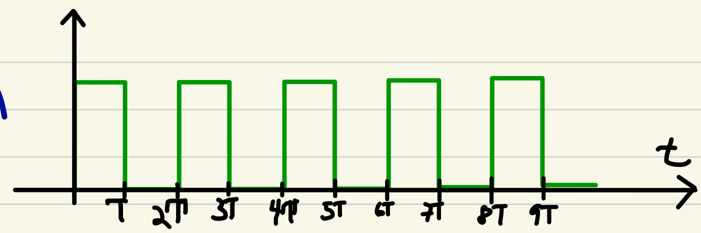
$$V_c(0) = 0$$

Example. (RC-circuit)



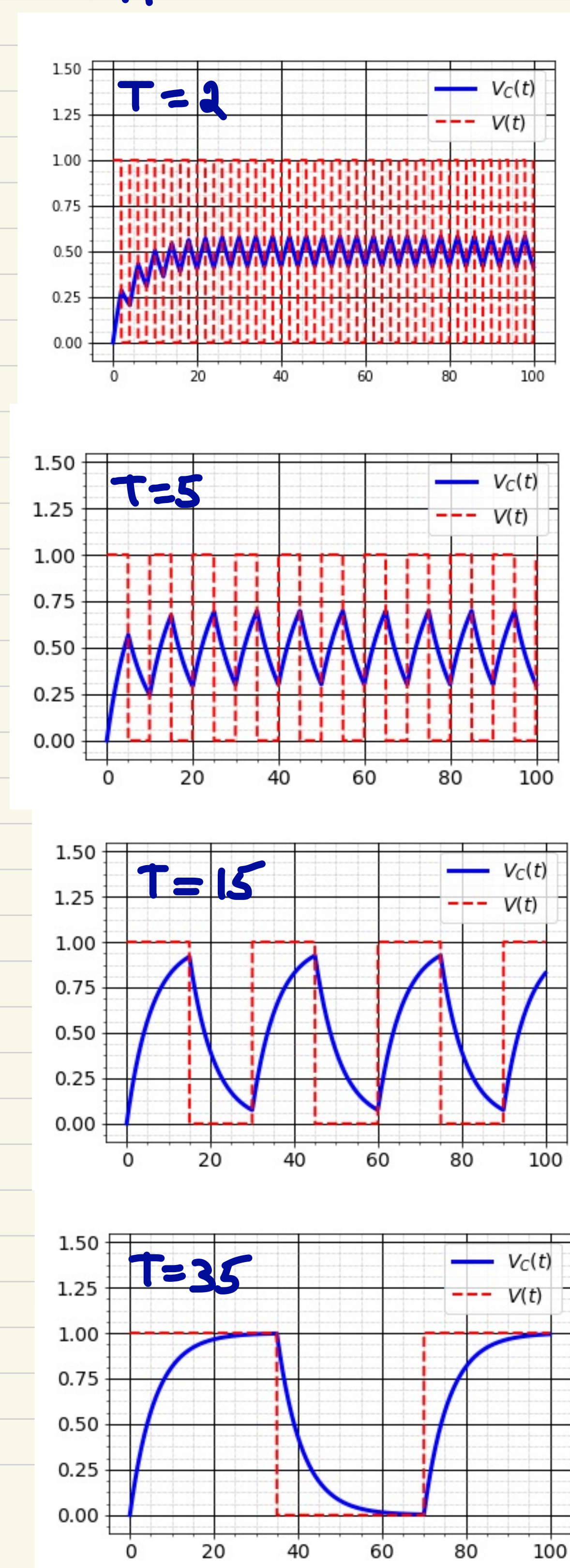
$$RC\frac{dV_c}{dt} + V_c = V(t)$$

$$V_c(0) = 0$$



$$V(t) = \sum_{k=0}^{\infty} (-1)^k u_{kT}(t)$$ (pulse wave of period $2T$)

RC = 6



Apply the Laplace transform

$$(RC\,s+1)Y(s) = \sum_{k=0}^{\infty} (-1)^k e^{-kTs} \cdot \frac{1}{s}$$

Solve for $Y(s)$

$$Y(s) = \sum_{k=0}^{\infty} (-1)^k e^{-kTs} H(s)$$

where

$$H(s) = \frac{1}{(RCs+1)s} = \frac{1}{s} - \frac{1}{s + 1/RC}$$

Let $h(t) = \mathcal{L}^{-1}\{H(s)\} = 1 - e^{-t/RC}$

Then
$$V_c(t) = \mathcal{L}^{-1}\left\{ \sum_{k=0}^{\infty} (-1)^k e^{-kTs} H(s) \right\}$$
$$= \sum_{k=0}^{\infty} (-1)^k u_{kT}(t)\, h(t-kT)$$
$$= \sum_{k=0}^{\infty} (-1)^k u_{kT}(t) \left\{ 1 - \exp\left(-\frac{(t-kT)}{RC}\right) \right\}$$