Lecture 24

Laplace Transforms

Impulses

Review

The Heaviside step function.

Main identities:

$$\mathcal{L}\{u_a(t) f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u_a(t) f(t-a)$$

Example $\mathcal{L}\{u_3(t)\, t^2\} = e^{-3s}\mathcal{L}\{(t+3)^2\}$

$$= e^{-3s}\mathcal{L}\{t^2+6t+9\}$$

$$= e^{-3s}\left(\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right)$$

Example Find $\mathcal{L}\{u_3(t)\cos(8t)\}$

Soln

$\mathcal{L}\{u_3(t)\cos(8t)\}$

$$= e^{-3s}\mathcal{L}\{\cos(8(t+3))\}$$

$$= e^{-3s}\mathcal{L}\{\cos(8t+24)\}$$

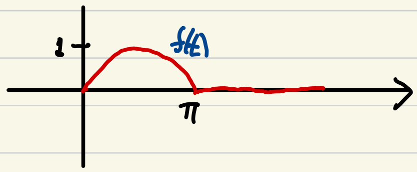
$$= e^{-3s}\mathcal{L}\{\cos(8t)\cos(24) - \sin(8t)\sin(24)\}$$

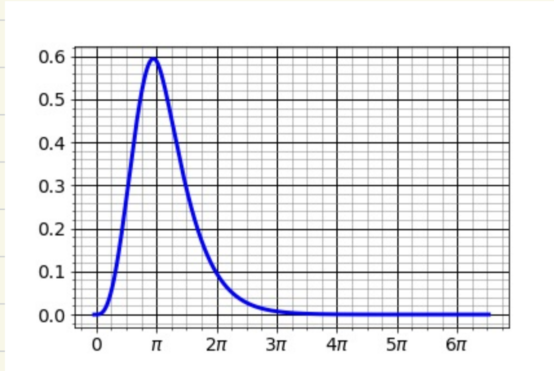
$$= e^{-3s}\left(\cos(24)\frac{s}{s^2+64} - \sin(24)\frac{8}{s^2+64}\right)$$

Example. Solve the initial value problem

$$y'' + 2y' + y = f(t) = \begin{cases} \sin(t) & \text{for } 0 \le t \le \pi \\ 0 & \text{for } t \ge \pi \end{cases}$$

$$y(0) = y'(0) = 0$$





Solution. $f(t) = \sin(t) - u_\pi(t)\sin(t)$

Take the Laplace transform

$$F(s) = \frac{1}{s^2+1} - e^{-\pi s}\mathcal{L}\{\underbrace{\sin(t+\pi)}_{-\sin(t)}\} = (1 - e^{-\pi s})\frac{1}{s^2+1}$$

$$(s^2+2s+1)Y(s) = F(s) \Rightarrow (s+1)^2 Y(s) = (1 - e^{-\pi s})\frac{1}{s^2+1}$$

$$\Rightarrow Y(s) = (1 - e^{-\pi s})\frac{1}{(s+1)^2(s^2+1)}$$

Let $H(s) = \frac{1}{(s+1)^2(s^2+1)}$ $\quad h(t) = \mathcal{L}^{-1}\{H(s)\}$

Then $y(t) = h(t) - u_\pi(t)h(t-\pi)$

What is $h(t)$? Use cover-up method

$$H(s) = \frac{1}{(s+1)^2(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$$

$$B = \frac{1}{((-1)^2+1)} = \frac{1}{2} \qquad C(i) + D = \frac{1}{(i+1)^2} = \frac{1}{2i} = -\frac{1}{2}i$$

$$\Rightarrow D = 0,\ C = -\frac{1}{2}$$

$$H(s) = \frac{1/2}{(s+1)^2} - \frac{1/2}{s^2+1} + \frac{A}{s+1} = \frac{1}{(s+1)^2(s^2+1)}$$

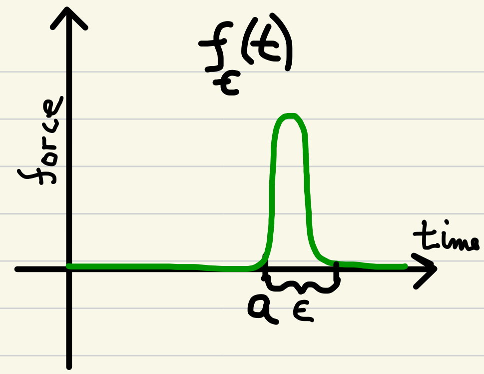
Set $s = 0$: $A = 1$ $\quad$ So $H(s) = \frac{1}{s+1} + \frac{1}{2}\frac{1}{(s+1)^2} - \frac{1}{2}\frac{1}{s^2+1}$

Let $h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{2}\sin(t)$

Impulses.

Model case Force applied to mass over very short time interval

$p = mv$: momentum

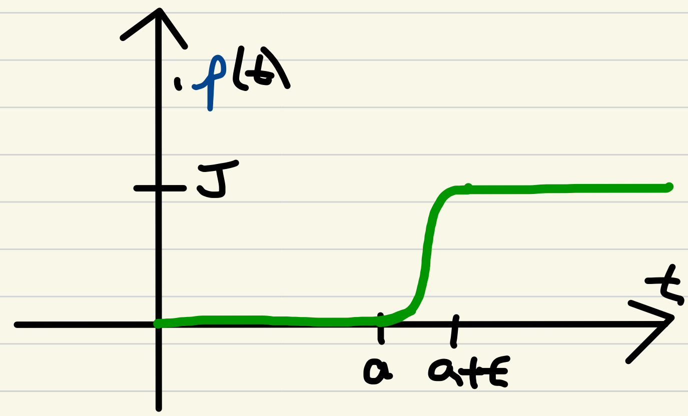


$$\frac{dp}{dt} = f_\epsilon(t) \qquad v(0)=0$$

$$J = \int_a^{a+\epsilon} f_\epsilon(t)\,dt \quad \text{(impulse)}$$

Integrate to find change in velocity

$$p(t) = \int_0^t f_\epsilon(u)\,du = \begin{cases} 0 & \text{for } t<a \\ J & \text{for } t>a+\epsilon \end{cases}$$



$\boxed{\text{Key point: If } \epsilon \text{ is small, values of } p(t) \text{ for } a \le t \le a+\epsilon \text{ are not important}}$

In the limit as $\epsilon \to 0$, with $J = 1$ (unit impulse)

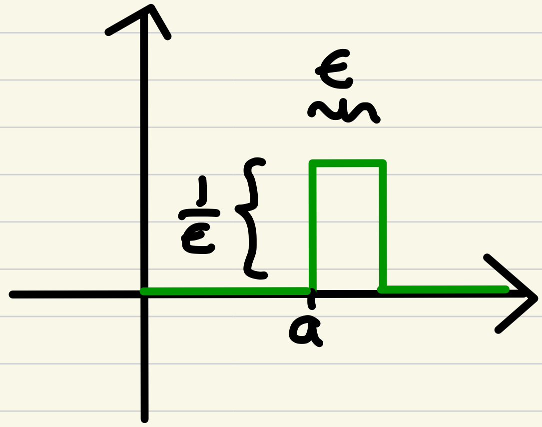
$$\int_a^{a+\epsilon} f_\epsilon(t)\,dt = 1$$

we get the $\boxed{\text{Dirac delta function } \delta_a(t)}$

Goal To Compute $\mathcal{L}\{\delta_a(t)\}$

Let

$$f_\epsilon(t) = \frac{1}{\epsilon}\left(u_a(t) - u_{a+\epsilon}(t)\right)$$



Note $\displaystyle\int_a^{a+\epsilon} f_\epsilon(t)\,dt = 1$

$$\mathcal{L}\{f_\epsilon(t)\} = \frac{e^{-as} - e^{-(a+\epsilon)s}}{\epsilon s} = e^{-as}\left(\frac{1-e^{-\epsilon s}}{\epsilon s}\right)$$

$$\lim_{\epsilon\to 0}\mathcal{L}\{f_\epsilon(t)\} = e^{-as}\lim_{\epsilon\to 0}\left(\frac{1-e^{-\epsilon s}}{\epsilon s}\right)$$

$$= e^{-as}\lim_{\epsilon\to 0}\frac{se^{-\epsilon s}}{s} = e^{-as}$$

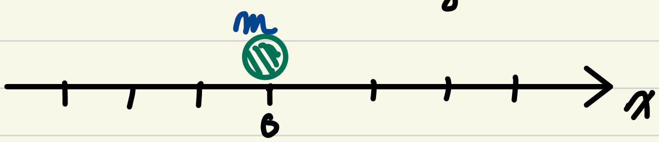
(l'Hôpital)

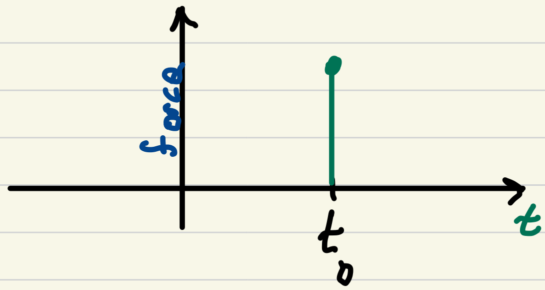
Conclusion

$$\boxed{\mathcal{L}\{\delta_a(t)\} = e^{-as}}$$

Example.

$$F = F_0\,\delta_{t_0}(t)$$





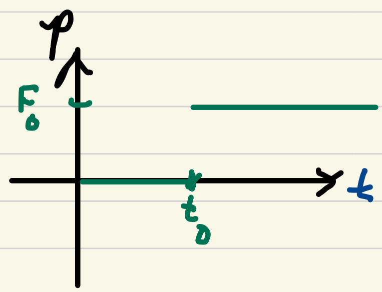
$$\begin{cases} \dfrac{dp}{dt} = F_0\,\delta_{t_0}(t) \\ p(0) = 0. \end{cases} \qquad m\,\frac{d^2x}{dt^2} = F_0\,\delta_{t_0}(t), \quad \begin{cases} x(0) = 0 \\ x'(0) = 0 \end{cases}$$

Take the Laplace transform:

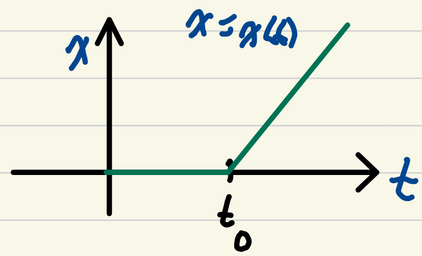
$$\begin{cases} s\,P(s) = F_0\,e^{-t_0 s} \\ m\,s^2\,X(s) = F_0\,e^{-t_0 s} \end{cases} \qquad \begin{cases} P(s) = F_0\,e^{-t_0 s}\,\dfrac{1}{s} \\ X(s) = (F_0/m)\,e^{-t_0 s}\,\dfrac{1}{s^2} \end{cases}$$

$\Downarrow \mathcal{L}^{-1}$

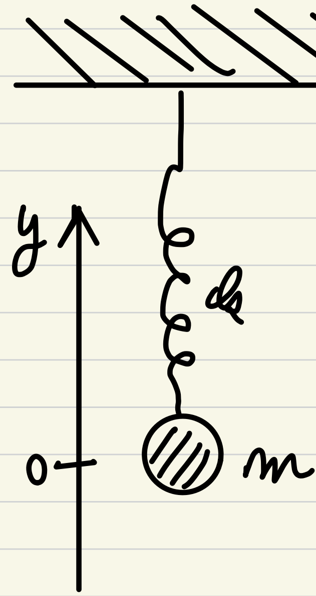
$$p(t) = F_0\,u_{t_0}(t)$$



$$x(t) = \frac{F_0}{m}\,u_{t_0}(t)\,(t - t_0)$$



Example.



$$\omega_0 = \sqrt{k/m}$$

$$\begin{cases} y'' + \omega_0^2 y = F_0/m \, \delta_{t_0}(t) \\ y(0) = 0 \quad y'(0) = 0 \end{cases}$$

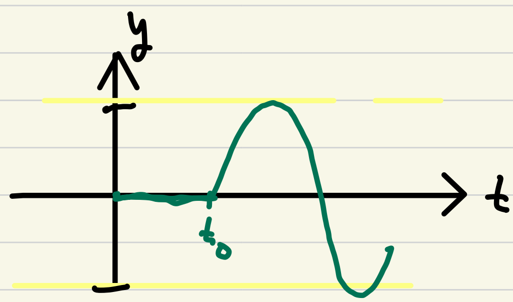
$$\Downarrow \mathcal{L}$$

$$(s^2 + \omega_0^2)\, Y(s) = \left(F_0/m\right) e^{-t_0 s}$$

$$Y(s) = \frac{F_0}{m} e^{-t_0 s} \frac{1}{s^2 + \omega_0^2}$$

$$\mathcal{L}^{-1} \Downarrow$$

$$y(t) = \frac{F_0}{m\omega_0} u_{t_0}(t) \sin\left(\omega_0 (t - t_0)\right)$$



$$m y'(t_0^+) = F_0$$

__Example__ Solve the IVP $\begin{cases} y'' + 2y' + 10y = \delta_1(t) + \delta_2(t) \\ y(0) = 2 \quad y'(0) = 1 \end{cases}$

__Solution.__

Apply the Laplace transform

$$(s^2 Y(s) - s\,y(0) - y'(0)) + 2(sY(s) - y(0)) + 10Y(s) = e^{-s} + e^{-2s}$$

Find $Y(s)$

$$(s^2+2s+10)Y(s) = e^{-s} + e^{-2s} + y(0)s + y'(0) + 2y(0)$$

$$((s+1)^2+9)Y(s) = e^{-s} + e^{-2s} + \underbrace{2s+5}_{=2(s+1)+5}$$

So

$$Y(s) = (e^{-s}+e^{-2s})\frac{1}{(s+1)^2+9} + 2\frac{(s+1)}{(s+1)^2+9} + \frac{3}{(s+1)^2+9}$$

$$= (e^{-s}+e^{-2s})\,\mathcal{L}\left\{\frac{e^{-t}\sin(3t)}{3}\right\} + 2\,\mathcal{L}\{e^{-t}\cos(3t)\} + \mathcal{L}\{e^{-t}\sin(3t)\}$$

$$= \mathcal{L}\Big\{2e^{-t}\cos(3t) + e^{-t}\sin(3t) + \frac{1}{3}u_1(t)\,e^{-(t-1)}\sin(3(t-1)) + \frac{1}{3}u_2(t)\,e^{-(t-2)}\sin(3(t-2))\Big\}$$

Therefore,

$$\boxed{y(t) = e^{-t}(2\cos(3t) + \sin(3t)) + \frac{1}{3}u_1(t)\,e^{-(t-1)}\sin(3(t-1)) + \frac{1}{3}u_2(t)\,e^{-(t-2)}\sin(3(t-2))}$$