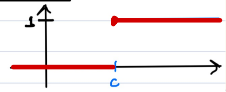
Lecture 23

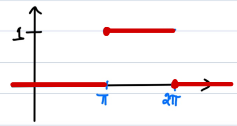
Laplace transform of discontinuous functions

The Heaviside Step Function

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$



$$u_\pi(t) - u_{2\pi}(t)$$



$$(t-2)\,u_2(t)$$

The basic building block:

* $$\boxed{\mathcal{L}\{u_a(t) f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}}$$

$$\boxed{\mathcal{L}\{u_a(t) f(t-a)\} = e^{-as}F(s)}$$

* $$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\} = u_a(t) f(t-a)}$$

Examples.

$$\mathcal{L}\{u_3(t)(7t-4)\}$$

$$= e^{-3s}\mathcal{L}\{7(t+3)-4\}$$

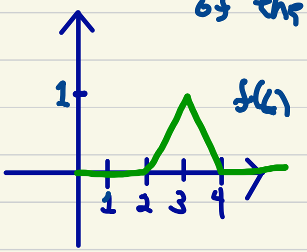
$$= e^{-3s}\mathcal{L}\{7t+17\}$$

$$= e^{-3s}\left(\frac{7}{s^2} + \frac{17}{s}\right)$$

$$\mathcal{L}^{-1}\left\{e^{-6s}\left(\frac{s}{s^2+4}\right)\right\} =$$

$$u_6(t)\cos(2(t-6))$$

Example Compute the Laplace transform of the function shown:



$$f(t) = \left(u_2(t)(t-2) - u_3(t)(t-2)\right) + \left(u_3(t)(4-t) - u_4(t)(4-t)\right)$$

$$= u_2(t)(t-2) + u_3(t)(6-2t) + u_4(t)(t-4)$$

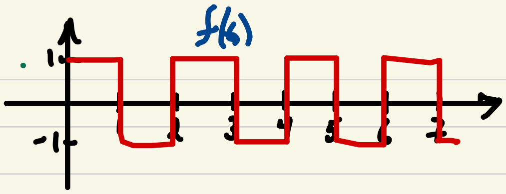
$$\mathcal{L}\{f(t)\} = e^{-2s}\mathcal{L}\{t\} + e^{-3s}\mathcal{L}\{6-2(t+3)\} + e^{-4s}\mathcal{L}\{t\}$$

$$= e^{-2s}\mathcal{L}\{t\} + e^{-3s}\mathcal{L}\{-2t\} + e^{-4s}\mathcal{L}\{t\}$$

$$= e^{-2s}\frac{1}{s^2} - 2e^{-3s}\frac{1}{s^2} + e^{-4s}\frac{1}{s^2}$$

$$= \left(e^{-2s} - 2e^{-3s} + e^{-4s}\right)\frac{1}{s^2}$$

Example.



$$f(t) = u_0(t) - 2u_1(t) + 2u_2(t) - 2u_3(t) + \cdots$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s} - 2\frac{e^{-s}}{s} + 2\frac{e^{-2s}}{s} - 2\frac{e^{-3s}}{s} + \cdots$$

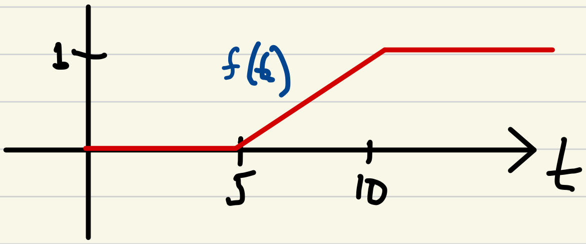
$$= \frac{1}{s} - \frac{2}{s}\left(e^{-s} - e^{-2s} + e^{-3s} - e^{-4s} + \cdots\right)$$

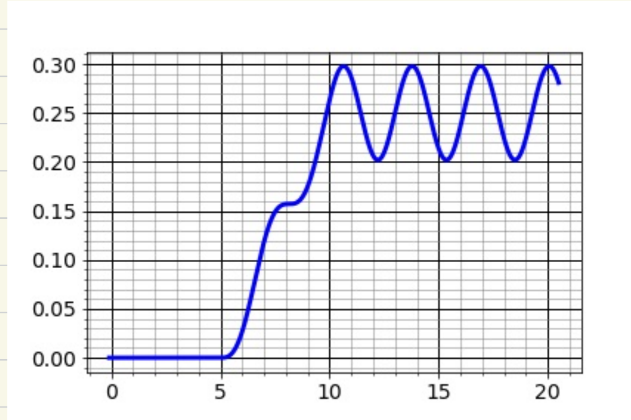
Example.

Solve the initial value problem

$$y'' + 4y = f(t)$$
$$y(0) = y'(0) = 0$$

where $f(t) = \begin{cases} 0 & \text{for } t \le 5 \\ \frac{1}{5}(t-5) & \text{for } 5 \le t \le 10 \\ 1 & \text{for } t \ge 10 \end{cases}$





Soln: $f(t) = \frac{1}{5} u_5(t)(t-5) - \frac{1}{5} u_{10}(t)(t-5) + u_{10}(t)$

$$F(s) = \mathcal{L}\{g(t)\} =$$
$$= \frac{1}{5} e^{-5s}\mathcal{L}\{t\} - \frac{1}{5} e^{-10s}\mathcal{L}\{t+10-5\} + \mathcal{L}\{u_{10}(t)\}$$
$$= \frac{1}{5} e^{-5s} \frac{1}{s^2} - \frac{1}{5} e^{-10s}\left(\frac{1}{s^2} + \frac{5}{s}\right) + e^{-10s}\frac{1}{s}$$
$$= \left(e^{-5s} - e^{-10s}\right) \frac{1}{5s^2}$$

$$(s^2+4)\, Y(s) = F(s)$$
$$\Rightarrow Y(s) = \frac{F(s)}{s^2+4}$$
$$Y(s) = \left(e^{-5s} - e^{-10s}\right) \frac{1}{5} \frac{1}{(s^2+4)s^2}$$

Let $H(s) = \frac{1}{5} \frac{1}{(s^2+4)s^2}$

$$= \frac{1}{20}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)$$

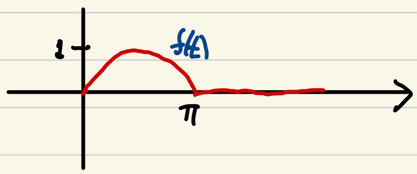
$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{t}{20} - \frac{1}{40}\sin(2t)$$

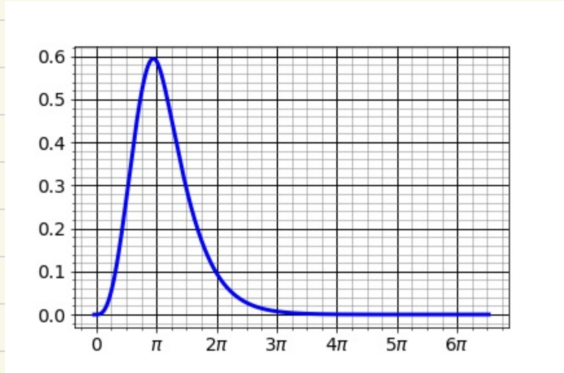
Then $y(t) = u_5(t)\, h(t-5) - u_{10}(t)\, h(t-10)$

Example. Solve the initial value problem

$$y'' + 2y' + y = f(t) = \begin{cases} \sin(t) & \text{for } 0 \le t \le \pi \\ 0 & \text{for } t \ge \pi \end{cases}$$

$$y(0) = y'(0) = 0$$





Solution. $f(t) = \sin(t) - u_\pi(t)\sin(t)$

Take the Laplace transform

$$F(s) = \frac{1}{s^2+1} - e^{-\pi s}\mathcal{L}\{\underbrace{\sin(t+\pi)}_{-\sin(t)}\} = (1 - e^{-\pi s})\frac{1}{s^2+1}$$

$$(s^2+2s+1)Y(s) = F(s) \Rightarrow (s+1)^2 Y(s) = (1 - e^{-\pi s})\frac{1}{s^2+1}$$

$$\Rightarrow Y(s) = (1 - e^{-\pi s})\frac{1}{(s+1)^2(s^2+1)}$$

Let $H(s) = \dfrac{1}{(s+1)^2(s^2+1)}$ $\quad h(t) = \mathcal{L}^{-1}\{H(s)\}$

Then $y(t) = h(t) - u_\pi(t)h(t-\pi)$

What is $h(t)$? Use Cover-up method

$$H(s) = \frac{1}{(s+1)^2(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$$

$$B = \frac{1}{(-1)^2+1} = \frac{1}{2} \qquad C(i)+D = \frac{1}{(i+1)^2} = \frac{1}{2i} = -\frac{1}{2}i$$

$$\Rightarrow D = 0,\ C = -\frac{1}{2}$$

$$H(s) = \frac{\frac{1}{2}}{(s+1)^2} - \frac{\frac{1}{2}}{s^2+1} + \frac{A}{s+1} = \frac{1}{(s+1)^2(s^2+1)}$$

Set $s = 0$: $A = 1$ $\quad$ So $H(s) = \dfrac{1}{s+1} + \dfrac{1}{2}\dfrac{1}{(s+1)^2} - \dfrac{1}{2}\dfrac{1}{s^2+1}$

Let $h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{2}\sin(t)$