Lecture 22

Laplace Transforms:
Solving I.V.P.s

Note: The cover-up method does not always determine all terms in the partial fractions expansion.

Example.

$$\frac{3s^3-7s^2+6s-3}{(s-2)^2(s^2+1)} = \frac{2}{s-2} + \frac{1}{(s-2)^2} + \frac{s}{s^2+1}$$

$$= \frac{A}{(s-2)} + \frac{B}{(s-2)^2} + \frac{Cs+D}{s^2+1}$$

Cover-up gives $B=1$ $C=1$, $D=0$. Does not determine A!

Need one more condition:

$$\frac{3s^3-7s^2+6s-3}{(s-2)^2(s^2+1)} = \frac{A}{s-2} + \frac{1}{(s-2)^2} + \frac{s}{s^2+1}$$

Let $s=0$:

Example.

$$\frac{3s^3-7s^2+6s-3}{(s-2)^3(s^2+1)} = \frac{A}{(s-2)} + \frac{B}{(s-2)^2} + \frac{C}{(s-2)^3} + \frac{Ds+E}{s^2+1}$$

Cover-up gives C, D, + E but not A and B.

Review.

Solving Initial Value Problems using Laplace transforms

$$\begin{cases} a\,y'' + b\,y' + c\,y = f(t) \\ y(0) = y_0 \quad y'(0) = y_0' \end{cases}$$

(*) $\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)}$

(**) $\boxed{\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)}$

$$\Big\Downarrow \mathcal{L}$$

$$a\,\mathcal{L}\{y''\} + b\,\mathcal{L}\{y'\} + c\,\mathcal{L}\{y\} = F(s)$$

$$a\left(s^2 Y(s) - s\,y_0 - y_0'\right) + b\left(s\,Y(s) - y_0\right) + c\,Y(s) = F(s)$$

$$\left(a s^2 + bs + c\right) Y(s) = F(s) + a y_0 s + a y_0' + b y_0$$

$$Y(s) = \frac{F(s)}{as^2+bs+c} + \frac{(ay_0)s + (ay_0' + by_0)}{as^2+bs+c}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

Example. Solve the I.V.P.

$$\begin{cases} y'' + 6y' + 9y = e^{-3t} \\ y(0) = 1 \quad y'(0) = 2 \end{cases}$$

Soln. Take Laplace transform:

$$\left(s^2 Y(s) - s - 2\right) + 6\left(sY(s) - 1\right) + 9\,Y(s) = \frac{1}{s+3}$$

$$\left(s^2 + 6s + 9\right) Y(s) = s + 8 + \frac{1}{s+3}$$

$$(s+3)^2\, Y(s) = s + 8 + \frac{1}{s+3}$$

So $$Y(s) = \frac{s+8}{(s+3)^2} + \frac{1}{(s+3)^3}$$

$$= \frac{(s+3) + 5}{(s+3)^2} + \frac{1}{(s+3)^3}$$

$$= \frac{1}{s+3} + \frac{5}{(s+3)^2} + \frac{1}{(s+3)^3}$$

So $$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+3} + \frac{5}{(s+3)^2} + \frac{1}{(s+3)^3}\right\}$$

$$= e^{-3t} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{5}{s^2} + \frac{1}{s^3}\right\}$$

$$= e^{-3t}\left(1 + 5t + \frac{t^2}{2}\right)$$

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}$	e^{bt}	$\frac{1}{s-b}$
t^n, $n = 1, 2, 3\ldots$	$\frac{n!}{s^{n+1}}$	$t^n e^{bt}$	$\frac{n!}{(s-b)^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\cos(at)$	$\frac{s}{s^2+a^2}$
$e^{bt}\sin(at)$	$\frac{a}{(s-b)^2+a^2}$	$e^{bt}\cos(at)$	$\frac{(s-b)}{(s-b)^2+a^2}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$\cosh(at)$	$\frac{s}{s^2-a^2}$
$e^{bt}\sinh(at)$	$\frac{a}{(s-b)^2-a^2}$	$e^{bt}\cosh(at)$	$\frac{(s-b)}{(s-b)^2-a^2}$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$\delta(t-c)$	e^{-cs}

General Formulas			
$a\,f(t) + b\,g(t)$	$a\,F(s) + b\,G(s)$	$f(at)$	$\frac{1}{a}F(s/a)$
$e^{bt}f(t)$	$F(s-b)$	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	$u_c(t)f(t)$	$e^{-cs}\mathcal{L}\{f(t+c)\}$
$\int_0^t f(u)\,du$	$\frac{F(s)}{s}$	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$
$f'(t)$	$sF(s) - f(0)$	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f * g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	$f(t+T) = f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$

EXAMPLE 11.15. Solve the initial value problem $y'' + 4y = \cos(4t)$, $y(0) = 0$, $y'(0) = 0$.

SOLUTION. Applying $\mathcal{L}\{-\}$ gives

$$(s^2 + 4)Y(s) = \frac{s}{s^2 + 4}, \text{ or } Y(s) = \frac{s}{(s^2 + 4)^2}.$$

Using the table of Laplace transforms:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2 + 4)^2}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{2 \cdot 2 \cdot s}{(s^2 + 2^2)^2}\right\} = \frac{1}{4}t\sin(2t).$$

EXAMPLE 11.16. Laplace transforms can also be used to solve linear constant coefficient first order initial value problems. For instance, consider the initial value problem

$$y' + 2y = \cos(t)\,, \quad y(0) = 1\,.$$

Computing the Laplace transform of both sides gives

$$sY(s) - 1 + 2Y(s) = \frac{s}{s^2 + 1}\,,$$

which can be solved for $Y(s)$:

$$\begin{aligned} Y(s) &= \frac{s}{(s+2)(s^2+1)} + \frac{1}{s+2} = \frac{(2/5)s + (1/5)}{s^2 + 1} - \frac{2/5}{s+2} + \frac{1}{s+2} \\ &= \frac{(2/5)s}{s^2+1} + \frac{1/5}{s^2+1} + \frac{3/5}{s+2}\,. \end{aligned}$$

Hence,

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\ &= \frac{2}{5}\cos(t) + \frac{1}{5}\sin(t) + \frac{3}{5}e^{-2t} \end{aligned}$$

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}$	e^{bt}	$\frac{1}{s-b}$
t^n, $n = 1, 2, 3\ldots$	$\frac{n!}{s^{n+1}}$	$t^n e^{bt}$	$\frac{n!}{(s-b)^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\cos(at)$	$\frac{s}{s^2+a^2}$
$e^{bt}\sin(at)$	$\frac{a}{(s-b)^2+a^2}$	$e^{bt}\cos(at)$	$\frac{(s-b)}{(s-b)^2+a^2}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$\cosh(at)$	$\frac{s}{s^2-a^2}$
$e^{bt}\sinh(at)$	$\frac{a}{(s-b)^2-a^2}$	$e^{bt}\cosh(at)$	$\frac{(s-b)}{(s-b)^2-a^2}$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$\delta(t-c)$	e^{-cs}

General Formulas			
$a\,f(t) + b\,g(t)$	$a\,F(s) + b\,G(s)$	$f(at)$	$\frac{1}{a}F(s/a)$
$e^{bt}f(t)$	$F(s-b)$	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	$u_c(t)f(t)$	$e^{-cs}\mathcal{L}\{f(t+c)\}$
$\int_0^t f(u)\,du$	$\frac{F(s)}{s}$	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$
$f'(t)$	$sF(s) - f(0)$	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f * g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	$f(t+T) = f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$

Example 11.18. Solve the initial value problem $y'' + 2y' + 2y = \cos(2t)$, $y(0) = 1$, $y'(0) = 0$.

Solution. Proceeding as in the previous example, apply $\mathcal{L}\{-\}$, solve for $Y(s)$, compute the par fractions expansion for $Y(s)$, and finally, compute the inverse Laplace transform.

Here's the (somewhat messy!) computation omitting some algebra:

$$\begin{aligned}(s^2Y(s) - s) + 2(sY(s) - 1) + 2Y(s) &= \frac{s}{s^2+4}\\ (s^2 + 2s + 2)Y(s) &= \frac{s}{s^2+4} + s + 2\,.\end{aligned}$$

Therefore,

$$\begin{aligned}Y(s) &= \frac{s}{(s^2+4)(s^2+2s+2)} + \frac{s+2}{s^2+2s+2}\\ &= \frac{As+B}{s^2+4} + \frac{C(s+1)+D}{(s+1)^2+1} + \frac{(s+1)+1}{(s+1)^2+1}\,.\\ &= \frac{-\frac{1}{10}s + \frac{4}{10}}{s^2+4} + \frac{\frac{1}{10}(s+1) - \frac{3}{10}}{(s+1)^2+1} + \frac{(s+1)+1}{(s+1)^2+1}\\ &= -\frac{1}{10}\frac{s}{s^2+4} + \frac{2}{10}\frac{2}{s^2+4} + \frac{11}{10}\frac{s+1}{(s+1)^2+1} + \frac{7}{10}\frac{1}{(s+1)^2+1}\,.\end{aligned}$$

Using the table of Laplace transforms gives

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{10}\cos(2t) + \frac{1}{5}\sin(2t) + \frac{11}{10}e^{-t}\cos(t) + \frac{7}{10}e^{-t}\sin(t).$$

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}$	e^{bt}	$\frac{1}{s-b}$
t^n, $n = 1, 2, 3 \dots$	$\frac{n!}{s^{n+1}}$	$t^n e^{bt}$	$\frac{n!}{(s-b)^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\cos(at)$	$\frac{s}{s^2+a^2}$
$e^{bt}\sin(at)$	$\frac{a}{(s-b)^2+a^2}$	$e^{bt}\cos(at)$	$\frac{(s-b)}{(s-b)^2+a^2}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$\cosh(at)$	$\frac{s}{s^2-a^2}$
$e^{bt}\sinh(at)$	$\frac{a}{(s-b)^2-a^2}$	$e^{bt}\cosh(at)$	$\frac{(s-b)}{(s-b)^2-a^2}$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$\delta(t-c)$	e^{-cs}

General Formulas			
$a\,f(t) + b\,g(t)$	$a\,F(s) + b\,G(s)$	$f(at)$	$\frac{1}{a}F(s/a)$
$e^{bt}f(t)$	$F(s-b)$	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	$u_c(t)f(t)$	$e^{-cs}\mathcal{L}\{f(t+c)\}$
$\int_0^t f(u)\,du$	$\frac{F(s)}{s}$	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$
$f'(t)$	$sF(s) - f(0)$	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f * g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	$f(t+T) = f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$

EXAMPLE 11.17. Solve the initial value problem

$$y'' - 3y' + 2y = 2e^{-3t}\,, \quad y(0) = 1\,, \quad y'(0) = 0\,.$$

SOLUTION. Applying $\mathcal{L}\{-\}$ and setting $Y(s) = \mathcal{L}\{y\}$ yields the equations

$$(s^2Y(s) - s) - 3(sY(s) - 1) + 2Y(s) = \frac{2}{s+3}\,,$$

which simplifies to

$$(s^2 - 3s + 2)Y(s) - s + 3 = \frac{2}{s+3}\,.$$

Solving for $Y(s)$ yields

$$\begin{aligned} Y(s) &= \frac{2}{(s+3)(s^2-3s+2)} + \frac{s-3}{s^2-3s+2} \\ &= \frac{s^2-7}{(s-1)(s-2)(s+3)} = \frac{3/2}{s-1} + \frac{-3/5}{s-2} + \frac{1/10}{s+3}. \end{aligned}$$

Consequently,

$$y(t) = \frac{3}{2}e^{t} - \frac{3}{5}e^{2t} + \frac{1}{10}e^{-3t}\,.$$

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}$	e^{bt}	$\frac{1}{s-b}$
$t^n,\ n = 1, 2, 3\ldots$	$\frac{n!}{s^{n+1}}$	$t^n e^{bt}$	$\frac{n!}{(s-b)^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\cos(at)$	$\frac{s}{s^2+a^2}$
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$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
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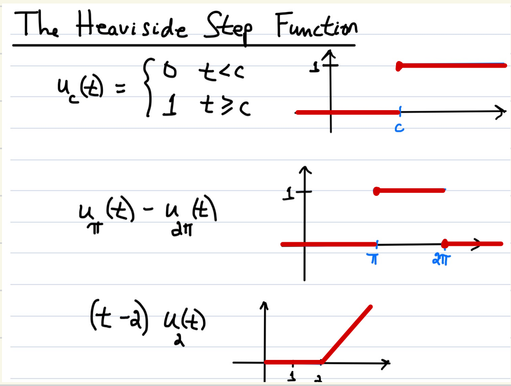
General Formulas			
$a\,f(t) + b\,g(t)$	$a\,F(s) + b\,G(s)$	$f(at)$	$\frac{1}{a}F(s/a)$
$e^{bt}f(t)$	$F(s-b)$	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	$u_c(t)f(t)$	$e^{-cs}\mathcal{L}\{f(t+c)\}$
$\int_0^t f(u)\,du$	$\frac{F(s)}{s}$	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$
$f'(t)$	$sF(s) - f(0)$	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f * g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	$f(t+T) = f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$

But what about $ay'' + by' + cy = f(t)$ when $f(t)$ is not continuous?

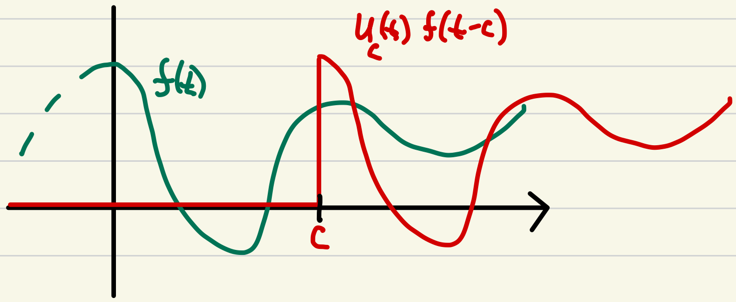
The basic building block:



$$\boxed{\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}}$$

Pf: $\mathcal{L}\{u_c(t)\} = \int_0^\infty u_c(t)e^{-st}\,dt = \int_c^\infty e^{-st}\,dt$

$$= \lim_{T\to\infty} \frac{e^{-st}}{-s}\Big]_c^T = \lim_{T\to\infty} \frac{e^{-cs}}{s} - \frac{e^{-Ts}}{s} = \frac{e^{-cs}}{s}$$



$$\boxed{\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}F(s)}$$

$$= \int_c^\infty e^{-st} f(t-c)\,dt$$

Let $u = t-c$, $du = dt$

$$= \int_0^\infty e^{-s(u+c)} f(u)\,du$$

$$= e^{-cs}\int_0^\infty e^{-su} f(u)\,du$$

$$= e^{-cs}F(s) \quad \blacksquare$$

$$\boxed{\mathcal{L}\{u_c(t) f(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}}$$

Proof. Let $g(t) = f(t+c)$

Then $f(t) = g(t-c)$

By previous identity

$$\mathcal{L}\{u_c(t) f(t)\} = \mathcal{L}\{u_c(t)\, g(t-c)\}$$

$$= e^{-cs}\mathcal{L}\{g(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}.$$