Lecture 21

Inverse
Laplace
Transforms

Review

Definition.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

- Linearity

$$\boxed{\mathcal{L}\{af(t)+bg(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}}$$

- Derivatives

(*) $$\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)}$$

- $$\boxed{\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s f(0) - f'(0)}$$

Some Laplace transforms:

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}}$$

$$\boxed{\begin{aligned}\mathcal{L}\{\cos(bt)\} &= \frac{s}{s^2+b^2}\\ \mathcal{L}\{\sin(bt)\} &= \frac{b}{s^2+b^2}\end{aligned}}$$

More Identies

$$\boxed{\mathcal{L}\{\cosh(bt)\} = \frac{s}{s^2-b^2}}$$

Pf $\mathcal{L}\{\cosh(bt)\} = \mathcal{L}\left\{\frac{e^{bt}+e^{-bt}}{2}\right\}$

$$= \frac{1}{2}\left(\frac{1}{s-b} + \frac{1}{s+b}\right) = \frac{s}{s^2-b^2} \blacksquare$$

$$\boxed{\mathcal{L}\{\sinh(bt)\} = \frac{b}{s^2-b^2}}$$

Pf $\mathcal{L}\{\sinh(bt)\} = \mathcal{L}\left\{\frac{e^{bt}-e^{-bt}}{2}\right\}$

$$= \frac{1}{2}\left(\frac{1}{s-b} - \frac{1}{s+b}\right) = \frac{b}{s^2-b^2} \blacksquare$$

- Suppose $\mathcal{L}\{f(t)\} = F(s)$. Then

$$\boxed{\mathcal{L}\{e^{at}f(t)\} = F(s-a)}$$

$$\boxed{\mathcal{L}\{f(bt)\} = \frac{1}{b}F(s/b) \quad b>0}$$

$$\boxed{\mathcal{L}\{t\,f(t)\} = -F'(s)}$$

$$\boxed{\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}F(s)}$$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}}$$

Pf

- $\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{at} f(t) e^{-st}\,dt = \int_0^\infty f(t) e^{-(s-a)t}\,dt = F(s-a)$

- $\mathcal{L}\{f(bt)\} = \int_0^\infty f(bt) e^{-st}\,dt$ Let $u = bt$ $du = b\,dt$
  $= \int_0^\infty f(u) e^{-\frac{s}{b}u} \frac{du}{b} = \frac{1}{b} F(s/b)$

- $F'(s) = \int_0^\infty \frac{d}{ds} f(t) e^{-st}\,dt = \int_0^\infty -t f(t) e^{-st}\,dt = -\mathcal{L}\{t f(t)\}$

- $F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\frac{d}{dt}\int_0^t f(u)\,du\}$
  $= s\,\mathcal{L}\{\int_0^t f(u)\,du\} - \int_0^0 f(u)\,du$
  $= s\,\mathcal{L}\{\int_0^t f(u)\,du\}$ So $\mathcal{L}\{\int_0^t f(u)\,du\} = \frac{1}{s} F(s)$

- $\mathcal{L}\{t\} = \mathcal{L}\{t\,1\} = -\frac{d}{ds}(\mathcal{L}\{1\}) = -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2}$

- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

  $\mathcal{L}\{t^2\} = \mathcal{L}\{t \cdot t\} = -\frac{d}{ds}\mathcal{L}\{t\} = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$

  $\mathcal{L}\{t^3\} = -\frac{d}{ds}\left(\frac{2}{s^3}\right) = \frac{3!}{s^4}$ ...

C. Table of Laplace Transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}$	e^{bt}	$\frac{1}{s-b}$
$t^n,\ n = 1, 2, 3 \ldots$	$\frac{n!}{s^{n+1}}$	$t^n e^{bt}$	$\frac{n!}{(s-b)^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\cos(at)$	$\frac{s}{s^2+a^2}$
$e^{bt}\sin(at)$	$\frac{a}{(s-b)^2+a^2}$	$e^{bt}\cos(at)$	$\frac{(s-b)}{(s-b)^2+a^2}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$\cosh(at)$	$\frac{s}{s^2-a^2}$
$e^{bt}\sinh(at)$	$\frac{a}{(s-b)^2-a^2}$	$e^{bt}\cosh(at)$	$\frac{(s-b)}{(s-b)^2-a^2}$
$\cancel{u_c(t)}$	$\cancel{\frac{e^{-cs}}{s}}$	$\cancel{\delta(t-c)}$	$\cancel{e^{-cs}}$

later

General Formulas			
$a\,f(t) + b\,g(t)$	$a\,F(s) + b\,G(s)$	$f(at)$	$\frac{1}{a}F(s/a)$
$e^{bt}f(t)$	$F(s-b)$	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$\cancel{u_c(t)f(t-c)}$	$\cancel{e^{-cs}F(s)}$	$\cancel{u_c(t)f(t)}$	$\cancel{e^{-cs}\mathcal{L}\{f(t+c)\}}$
$\int_0^t f(u)\,du$	$\frac{F(s)}{s}$	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$
$f'(t)$	$sF(s) - f(0)$	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$\cancel{f * g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau}$	$\cancel{F(s)G(s)}$	$\cancel{f(t+T) = f(t)}$	$\cancel{\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}}$

Computing Inverse Laplace Transforms

Important special case:

$$\mathcal{L}^{-1}\left\{\frac{As+B}{as^2+bs+c}\right\}$$

Examples.

$$\mathcal{L}^{-1}\left\{\frac{3s}{s^2-2s-6}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3(s-1)+3}{(s-1)^2-7}\right\} = e^t\,\mathcal{L}^{-1}\left\{\frac{3s+3}{s^2-7}\right\}$$

$$= e^t\left(3\,\mathcal{L}^{-1}\left\{\frac{s}{s^2-7}\right\} + 3\,\mathcal{L}^{-1}\left\{\frac{1}{s^2-7}\right\}\right)$$

$$= e^t\left(3\cosh(\sqrt{7}t) + \frac{3}{\sqrt{7}}\sinh(\sqrt{7}t)\right)$$

$$\boxed{\begin{aligned}
\mathcal{L}\{e^{bt}f(t)\} &= F(s-b)\\
\mathcal{L}\{e^{bt}\} &= \frac{1}{s-b}\\
\mathcal{L}\{\cos(at)\} &= \frac{s}{s^2+a^2}\\
\mathcal{L}\{\sin(at)\} &= \frac{a}{s^2+a^2}\\
\mathcal{L}\{\cosh(at)\} &= \frac{s}{s^2-a^2}\\
\mathcal{L}\{\sinh(at)\} &= \frac{a}{s^2-a^2}
\end{aligned}}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+11}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+2}\right\}$$

$$= e^{3t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\}$$

$$= e^{3t}\left(\frac{1}{\sqrt{2}}\sin(\sqrt{2}\,t)\right)$$

$$\boxed{\begin{aligned}
&* \ \mathcal{L}\{e^{bt}f(t)\} = F(s-b)\\
&\mathcal{L}\{e^{bt}\} = \frac{1}{s-b}\\
&\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}\\
&\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}\\
&\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2-a^2}\\
&\mathcal{L}\{\sinh(at)\} = \frac{a}{s^2-a^2}
\end{aligned}}$$

$$\mathcal{L}^{-1}\left\{\frac{3s-4}{2s^2+4s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{3s-4}{2(s^2+2s+4)}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{3(s+1)-7}{(s+1)^2+3}\right\}$$

$$= \frac{e^{-t}}{2}\mathcal{L}^{-1}\left\{\frac{3s-7}{s^2+3}\right\}$$

$$= \frac{e^{-t}}{2}\left(3\mathcal{L}^{-1}\left\{\frac{s}{s^2+3}\right\} - 7\mathcal{L}^{-1}\left\{\frac{1}{s^2+3}\right\}\right)$$

$$= \frac{e^{-t}}{2}\left(3\cos(\sqrt{3}t) - \frac{7}{\sqrt{3}}\sin(\sqrt{3}t)\right)$$

$$\boxed{\begin{aligned} \mathcal{L}\{e^{bt}f(t)\} &= F(s-b) \\ \mathcal{L}\{e^{bt}\} &= \frac{1}{s-b} \\ \mathcal{L}\{\cos(at)\} &= \frac{s}{s^2+a^2} \\ \mathcal{L}\{\sin(at)\} &= \frac{a}{s^2+a^2} \\ \mathcal{L}\{\cosh(at)\} &= \frac{s}{s^2-a^2} \\ \mathcal{L}\{\sinh(at)\} &= \frac{a}{s^2-a^2} \end{aligned}}$$

$$\bullet\ F(s) = \frac{2s^2+3s+1}{s(s-1)(s+4)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+4}$$

"Cover-up" method: $sF(s) = \dfrac{2s^2+3s+1}{(s-1)(s+4)} = A + s\left(\dfrac{B}{s-1} + \dfrac{C}{s+4}\right)$

Now set $s=0$: $\dfrac{2(0)^2+3(0)+1}{(0-1)(0+4)} = \dfrac{1}{-4} = -$

$$A = \frac{2(0)^2+3(0)+1}{(0-1)(0+4)} = \frac{1}{-4} \qquad \frac{2s^2+3s+1}{\cancel{s}(s-1)(s+4)}$$

$$B = \frac{2(1)^2+3(1)+1}{(1)(1+4)} = \frac{6}{5} \qquad \frac{2s^2+3s+1}{s\cancel{(s-1)}(s+4)}$$

$$C = \frac{2(-4)^2+3(-4)+1}{(-4)(-4-1)} = \frac{21}{20} \qquad \frac{2s^2+3s+1}{s(s-1)\cancel{(s+4)}}$$

So $\mathcal{L}^{-1}\{F(s)\} =$

$$-\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{21}{20}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{1}{4} + \frac{6}{5}e^{t} + \frac{21}{20}e^{-4t}$$

$$\bullet\ F(s) = \frac{8s^2-4s+12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$A = \frac{8(0)^2-4(0)+12}{(0)^2+4} = 3$$

$$\frac{8s^2-4s+12}{s} = Bs+C + (s^2+4)\frac{A}{s}$$

Set $s = 2i$: $\dfrac{8(2i)^2-4(2i)+12}{(2i)} = B(2i)+C$

$$= \frac{-32-8i+12}{2i} = \frac{-20-8i}{2i}$$

$$= -4+10i = C+2Bi$$

So $C=-4$ and $B=5$

So

$$\mathcal{L}^{-1}\left\{\frac{8s^2-4s+12}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s} + \frac{5s-4}{s^2+4}\right\}$$

$$= 3 + 5\cos(2t) - 2\sin(2t)$$

$$F(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

$$= \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$A(i) + B = \frac{2(i)^3 + (i)^2 + 8(i) + 6}{(i)^2+4}$$

$$= \frac{5+6i}{3} = \frac{5}{3} + 2i$$

So $B = 5/3$ $A = 2$

$$C(2i) + D = \frac{2(2i)^3 + (2i)^2 + 8(2i) + 6}{(2i)^2+1}$$

$$D + 2Ci = \frac{2+0i}{-3}$$

So $C = 0$ $D = -2/3$

$$\mathcal{L}^{-1}\left\{\frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}\right\} =$$

$$\mathcal{L}^{-1}\left\{\frac{2s + 5/3}{s^2+1} - \frac{1}{3}\,\frac{2}{s^2+4}\right\}$$

$$= 2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t)$$