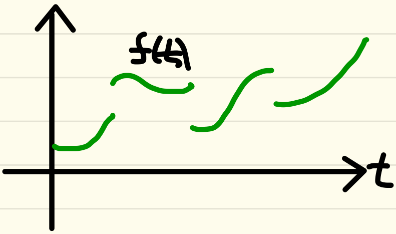
Lecture 20

Into. to
Laplace
Transforms

Laplace Transforms

$$f(t) \begin{cases} \cdot \text{ piecewise continuous} \\ \cdot \text{ subexponential growth} \end{cases}$$



For some M, a, and K,

$|f(t)| \leq Me^{at}$ for all $t \geq K$.

We say that f has **subexponential growth**

Examples.

$\mathcal{L}\{1\} = ?$

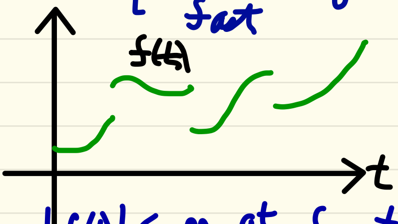
$\mathcal{L}\{e^{at}\} = ?$

$\mathcal{L}\{\cos(bt)\} = ?$

$\mathcal{L}\{\sin(bt)\} = ?$

Laplace Transforms

Assume $f(t)\begin{cases}\bullet \text{ piecewise continuous} \\ \bullet \text{ doesn't grow too fast}\end{cases}$



$$|f(t)| \leq M e^{at} \text{ for } t \geq K$$

Then, the Laplace transform of $f(t)$ is

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

Makes sense for s sufficiently large

Examples

(1) $\mathcal{L}\{1\} = \int_0^\infty 1\, e^{-st}\, dt$

$$= \lim_{A\to\infty} \int_0^A e^{-st}\, dt$$

$$= \lim_{A\to\infty} \left(-\frac{e^{-sA}}{s}\right) - \left(-\frac{1}{s}\right) = \frac{1}{s}$$

for $s > 0$

So $\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$

- $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} e^{-st}\, dt$

$$= \lim_{A\to\infty} \int_0^A e^{-(s-a)t}\, dt$$

$$= \lim_{A\to\infty} \left(-\frac{e^{-(s-a)A}}{s-a}\right) - \left(\frac{-1}{s-a}\right) = \frac{1}{s-a}$$

for $s > a$

So $\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}}$

- $e^{ibt} = \cos(bt) + i\sin(bt)$

So

$$\mathcal{L}\{e^{ibt}\} = \mathcal{L}\{\cos(bt)\} + i\,\mathcal{L}\{\sin(bt)\}$$

$$= \int_0^\infty e^{ibt} e^{-st}\, dt = \lim_{A\to\infty} \int_0^A e^{-(s-ib)t}\, dt$$

$$= \lim_{A\to\infty} \left(\frac{-e^{-(s-ib)A}}{s-ib}\right) + \frac{1}{s-ib} = \frac{1}{s-ib}$$

(for $s > 0$)

$$= \frac{s}{s^2+b^2} + i\,\frac{b}{s^2+b^2}$$

So $\boxed{\begin{aligned}\mathcal{L}\{\cos(bt)\} &= \frac{s}{s^2+b^2} \\ \mathcal{L}\{\sin(bt)\} &= \frac{b}{s^2+b^2}\end{aligned}}$

Linearity

(*) $$\boxed{\mathcal{L}\{a f(t) + b g(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}}$$

- **Derivatives**

(*) $$\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)}$$

(*) $$\boxed{\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s f(0) - f'(0)}$$

Solving Initial Value Problems using Laplace transforms

$$\begin{cases} a\,y'' + b\,y' + c\,y = f(t) \\ y(0) = y_0 \quad y'(0) = y_0' \end{cases}$$

$(*)$ $\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)}$

$(**)$ $\boxed{\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)}$

Example. $\begin{cases} y'' - 3y' + 2y = 0 \\ y(0) = 2 \quad y'(0) = 1 \end{cases}$

$(*)$ $\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)}$

$(**)$ $\boxed{\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)}$

- Derivatives

(*) $\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)}$

Proof $\int_0^\infty f'(t)e^{-st}\,dt = \lim_{A\to\infty}\int_0^A f'(t)e^{-st}\,dt$

(Integrate by parts)

$$= \lim_{A\to\infty} f(t)e^{-st}\Big]_0^A - \int_0^A f(t)\left(-se^{-st}\right)dt$$

$$= -f(0) + s\int_0^\infty f(t)e^{-st}\,dt$$

$$= s\,\mathcal{L}\{f(t)\} - f(0) \quad \blacksquare$$

- $\boxed{\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s f(0) - f'(0)}$

Pf Use (*) twice as follows

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{(f'(t))'\} =$$

$$s\,\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\left\{s\,\mathcal{L}\{f(t)\} - f(0)\right\} - f'(0) \quad \blacksquare$$

- Application to initial value problem

$$\begin{cases} ay'' + by' + cy = f(t) \\ y(0) = y_0 \\ y'(0) = y_0' \end{cases}$$

Let $F(s) = \mathcal{L}\{f(t)\}$

$Y(s) = \mathcal{L}\{y(t)\}$

Then

$$a\{s^2 Y(s) - s y_0 - y_0'\} + b\{sY(s) - y_0\} + cY(s) = F(s)$$

Solve for $Y(s)$

$$\boxed{Y(s) = \frac{F(s)}{as^2+bs+c} + \frac{a y_0 s + a y_0' + b y_0}{as^2+bs+c}}$$

So $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

Conclusion We can solve

the IVP if

① We can compute $F(s) = \mathcal{L}\{f(t)\}$

and

② We can compute $\mathcal{L}^{-1}\{Y(s)\}$