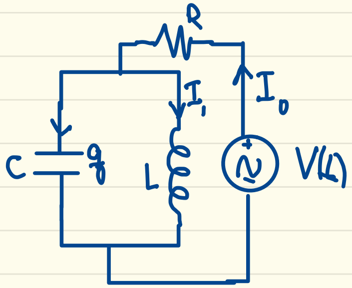
Lecture 19

Resonance in Electrical Circuits

Example: LC- circuit



$$\begin{cases} R I_0 + \frac{q}{C} = V(t) \\ R I_0 + L \frac{dI_1}{dt} = V(t) \\ I_1 = I_0 - dq/dt \end{cases}$$

$$\Swarrow$$

$$\begin{cases} I_0 = \frac{1}{R}\left\{V(t) - q/C\right\} \\ R I_0 + L \frac{dI_1}{dt} = V(t) \\ I_1 = I_0 - dq/dt \end{cases} \Longrightarrow \begin{cases} R I_0 + L \frac{d}{dt}\left(I_0 - dq/dt\right) = V(t) \\ I_0 = \frac{1}{R}\left\{V(t) - q/C\right\} \end{cases}$$

$$\Downarrow$$

$$R \frac{1}{R}\left\{V(t) - q/C\right\} + L \frac{d}{dt}\left(\frac{1}{R}\left\{V(t) - q/C\right\} - \frac{dq}{dt}\right) = V(t)$$

$$\Downarrow$$

$$\cancel{R \frac{1}{R}}\left\{\cancel{V(t)} - q/C\right\} + L \frac{d}{dt}\left(\frac{1}{R}\left\{V(t) - q/C\right\} - \frac{dq}{dt}\right) = \cancel{V(t)}$$

$$\Downarrow$$

$$- q/C + \frac{L}{R}\frac{dV(t)}{dt} - \frac{L}{RC}\frac{dq}{dt} - \frac{L}{R}\frac{d^2q}{dt^2} = 0$$

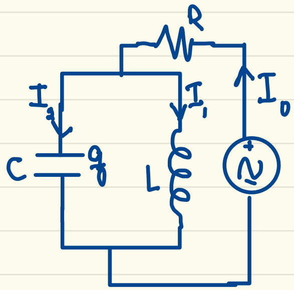
$$\Downarrow$$

$$L \frac{d^2q}{dt^2} + \frac{L}{RC}\frac{dq}{dt} + q/C = \frac{L}{R}\frac{dV(t)}{dt} \qquad q = C V_c$$

$$LC \frac{d^2V_c}{dt^2} + \frac{L}{R}\frac{dV_c}{dt} + V_c = \frac{L}{R}\frac{dV(t)}{dt}$$

$$\boxed{\frac{d^2V_c}{dt^2} + \frac{1}{RC}\frac{dV_c}{dt} + \frac{1}{LC} V_c = \frac{1}{RC}\frac{dV(t)}{dt}}$$

Example: LC-circuit (continued)



$$\boxed{\frac{d^2 V_c}{dt^2} + \frac{1}{RC}\frac{dV_c}{dt} + \frac{1}{LC} V_c = \frac{1}{RC}\frac{dV(t)}{dt}}$$

$V(t) = \cos(\omega t)$

Frequency response:

$$\left\{-\omega^2 + \frac{1}{RC} i\omega + \omega_0^2\right\} G e^{i\omega} = \frac{i\omega e^{i\omega}}{RC}$$

So $$G(i\omega) = \frac{i\omega/RC}{(\omega_0^2 - \omega^2) + i\omega/RC}$$

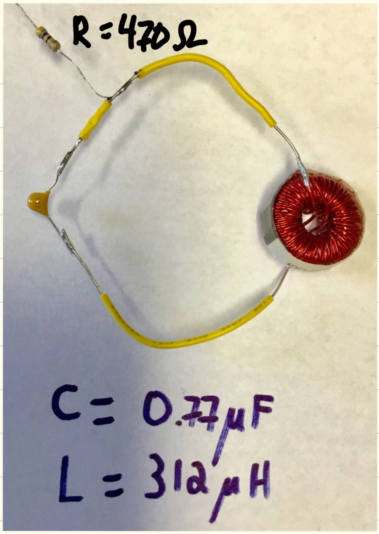
$$= \frac{1}{\frac{\omega_0^2 - \omega^2}{i\omega/RC} + 1} = \frac{1}{1 + RC\frac{(\omega^2 - \omega_0^2)}{\omega} i}$$

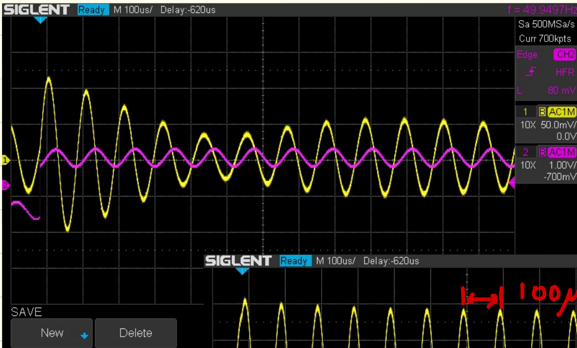
$$\boxed{\text{Resonance at } \omega = \omega_0 = 1/\sqrt{LC},\ \varphi = 0}$$

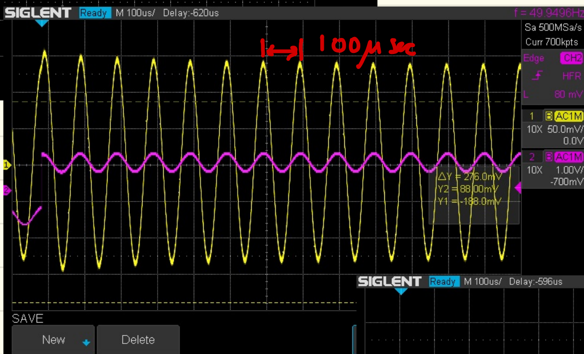
$$\omega_0 = 1/\sqrt{LC} \approx 6.45\times10^4$$

$$f_0 = \frac{\omega_0}{2\pi} \approx 10.27\times10^3 \approx 10.3\ kHz$$

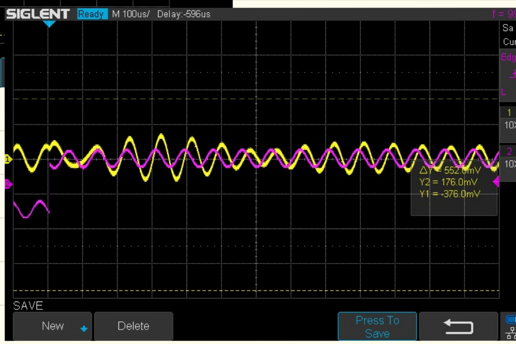
$$\text{Period} = \frac{1}{f_0} \approx 97\times10^{-6}\ \text{sec.}$$



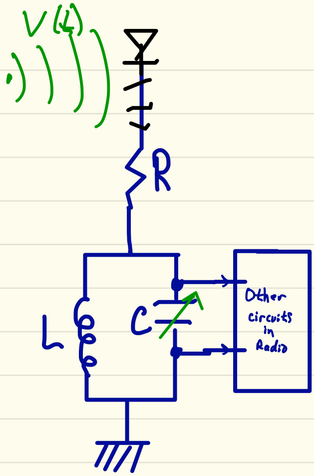




Response (in yellow)
with increasing ω
(in purple)



A.M. Radio



$$V_C'' + \frac{1}{RC} V_C' + \frac{1}{LC} V_C = \frac{1}{RC} V'(t)$$

$$\omega_0 = 1/\sqrt{LC}$$

Frequency response:

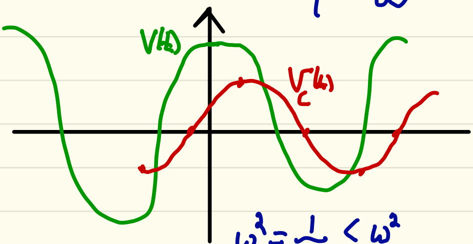
$$\left\{-\omega^2 + \frac{1}{RC} i\omega + \omega_0^2\right\} G e^{i\omega} = \frac{i\omega e^{i\omega}}{RC}$$

So $$G(i\omega) = \frac{i\omega/RC}{(\omega_0^2-\omega^2) + i\omega/RC} = \frac{1}{\frac{\omega_0^2-\omega^2}{i\omega/RC}+1} = \frac{1}{1 + RC\frac{(\omega^2-\omega_0^2)}{\omega} i}$$

$$\boxed{\text{Resonance at } \omega = \omega_0 = 1/\sqrt{LC} \quad \varphi = 0}$$

$$V_C(t) = R(\omega)\cos(\omega t - \varphi)$$

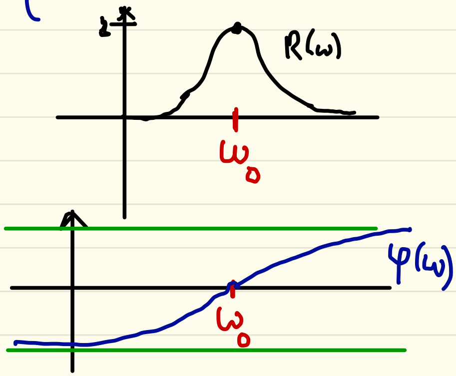
$$\varphi = \tan^{-1}\left\{\frac{(\omega^2-\omega_0^2)RC}{\omega}\right\}$$

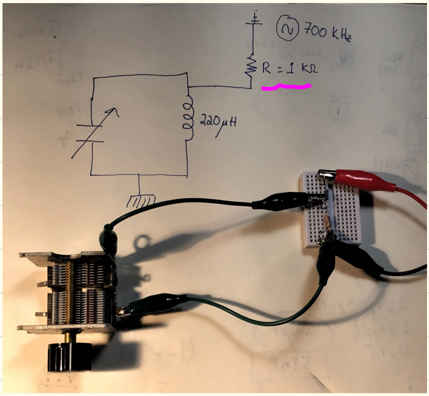


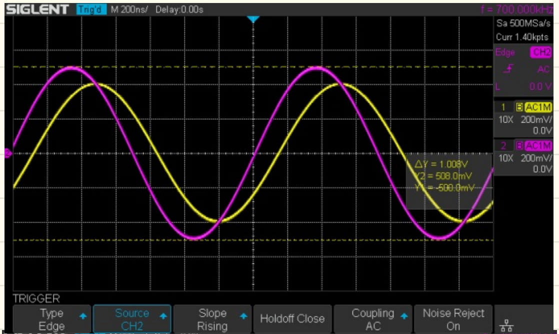
$$\omega_0^2 = \frac{1}{LC} < \omega^2$$

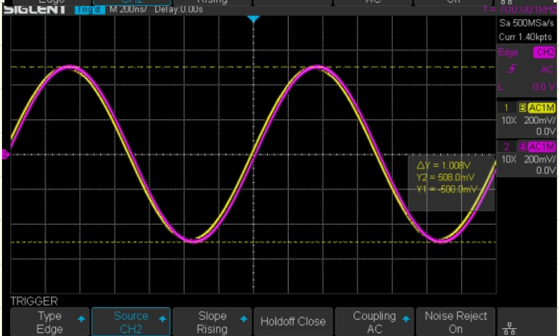
$$V_c(t) = R(\omega)\cos(\omega t - \varphi)$$

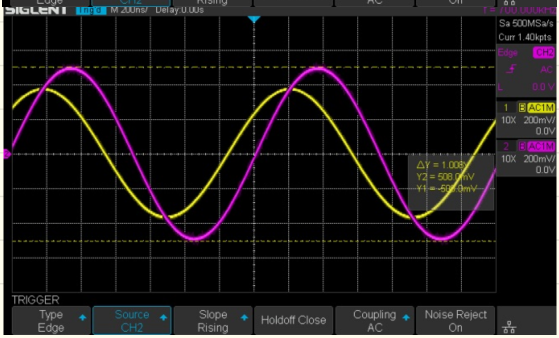
$$\begin{cases} R(\omega) = \dfrac{1}{\sqrt{1 + \left(\dfrac{\omega^2 - \omega_0^2}{\omega} RC\right)^2}} \\[2ex] \varphi = \tan^{-1}\left\{\dfrac{(\omega^2 - \omega_0^2)RC}{\omega}\right\} \end{cases}$$











$L = 220\,\mu H$

$24\,pF \leq C \leq 383\,pF \; (383 \times 10^{-12} F)$

$R = 1000$ Ohm

$f = \dfrac{\omega}{2\pi} = 700$ kHz

Note $\dfrac{1}{LC} = \omega_0^2$ so at resonance

$$C = \frac{1}{L \cdot \omega_0^2} \approx 2.35 \times 10^{-10}\ F$$

$$\approx 235\ pF$$

$$R(\omega) = \frac{1}{\sqrt{1 + \left(\frac{\omega^2 - \omega_0^2}{\omega} RC\right)^2}}$$ Gain