Lecture 18

Applications

Example.

Let $T_A(t) = 10\cos\left(\frac{2\pi}{24}t\right)$ be the outside temperature in °C, t in hours after noon on Jan 1. Let $T(t)$ be the temperature of an insulated object placed outside.

Several days later, the maximum temperature is reached at 2:00 pm every day. What is that temperature? Assume Newton's Law of Cooling.

Recall:

$$\frac{dT}{dt} = -k(T - T_A(t))$$

$$\frac{dT}{dt} + kT = k\,10\cos\left(\frac{\pi}{12}t\right)$$

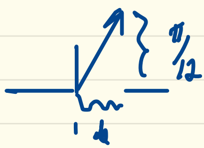
$$= 10k \,\mathrm{Re}\, \exp\left(\frac{\pi}{12}t\, i\right)$$

$$T = \mathrm{Re}\left(G e^{i\omega t}\right) \qquad \omega = \frac{\pi}{12}$$

$$(i\omega + k) G e^{i\omega t} = 10k e^{i\omega t}$$

$$G = \frac{10k}{k + i\omega} = \frac{10k}{k + i\,\pi/12} = \frac{10k}{\sqrt{}\; e^{i\varphi}}$$

$$|G| = \frac{10k}{\sqrt{k^2 + (\pi/12)^2}}$$



$$\underbrace{\varphi = \tan^{-1}\left(\pi/12 \,/\, k\right)}$$

$$T = |G|\cos\left(\frac{\pi}{12}t - \varphi\right)$$

$$T(2) = |G|\cos\left(\frac{\pi}{12}2 - \varphi\right)$$

$$\frac{\pi}{12}2 - \varphi = 0$$

$$\frac{\pi}{6} = \varphi = \tan^{-1}\left(\frac{\pi/12}{k}\right)$$

$$\tan(\pi/6) = \pi/12 \,/\, k$$

$$k = \frac{\pi/12}{\tan(\pi/6)}$$

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Soln

We can ignore "transients" (Why?) and look for the periodic solution.

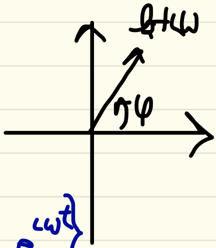
Let $T(t) = \mathrm{Re}\{Re^{i\omega t}\}$, $\omega = \frac{2\pi}{24} = \frac{\pi}{12}$

Find R $\quad (Re^{i\omega t})' + k(Re^{i\omega t}) = k\,10e^{i\omega t}$

So $(k+i\omega)R = 10k \Rightarrow R = \dfrac{10k}{k+i\omega}$

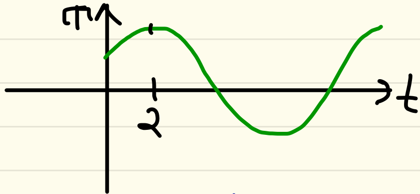
In polar form

$$k+i\omega = \sqrt{k^2+\omega^2}\,e^{i\varphi}$$



where $\tan\varphi = \dfrac{\omega}{k}$

So $$T(t) = \mathrm{Re}\left\{\frac{10k}{k+i\omega}e^{i\omega t}\right\} = \frac{10k}{\sqrt{k^2+\omega^2}}\cos(\omega t - \varphi)$$



Because the max temp is reached at 2:00 PM, we know that

$$\omega 2 - \varphi = 0 \quad \text{or}$$

$$\varphi = 2\omega = \frac{4\pi}{24} = \pi/6$$

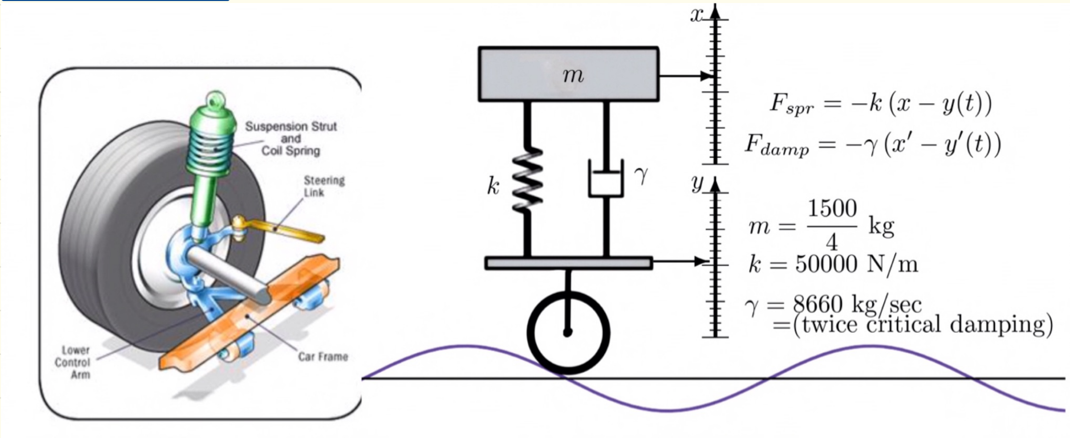
Hence, $\dfrac{\omega}{k} = \tan(\pi/6) = \dfrac{1}{\sqrt{3}}$

Therefore, $k = \sqrt{3}\,\omega = \dfrac{\sqrt{3}\,\pi}{12} \approx 0.453\,\dfrac{1}{hr}$

and the max temperature is

$$\frac{10k}{\sqrt{k^2+\omega^2}} \approx 8.66\,°C$$

Automobile Struts



$$m x'' = -k(x - y(t)) - \gamma(x' - y'(t))$$

$$m x'' + \gamma x' + k x = k y(t) + \gamma y'(t)$$

$$y(t) = \cos(\omega t) = \mathrm{Re}\, e^{i\omega t}$$

$$k y(t) + \gamma y'(t) = \mathrm{Re}\{(k + \gamma\omega i) e^{i\omega t}\}$$

$$x = \mathrm{Re}\{G e^{i\omega t}\}$$

$$x(t) = R(\omega)\cos(\omega t - \varphi)$$

$$\{m(-\omega^2) + \gamma\omega i + k\} G e^{i\omega t}$$

$$= (k + \gamma\omega i) e^{i\omega t}$$

$$G = \frac{k + \gamma\omega i}{(k - m\omega^2) + \gamma\omega i}$$ "transfer function"

$$= R(\omega) e^{-i\varphi}$$

$$R(\omega) = |G| = \frac{\sqrt{k^2 + (\gamma\omega)^2}}{\sqrt{(k - m\omega^2)^2 + (\gamma\omega)^2}}$$

$$= \sqrt{\frac{k^2 + (\gamma\omega)^2}{(k - m\omega^2)^2 + (\gamma\omega)^2}}$$

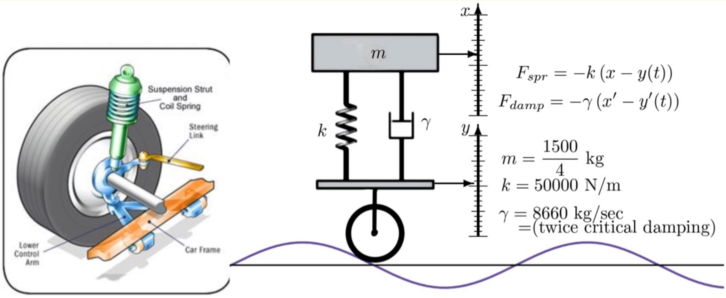


FIGURE 9.5. Left: sketch of strut on an automobile. Right: Simplified model of the system. The values of m, k, and γ in the figure are similar to those found in automobiles. The mass is divided by four because the weight of an automobile is distributed over four wheels.

EXAMPLE 9.3. (AUTOMOBILE STRUTS) A similar analysis can be done for the mechanical system modeling the struts on an automobile. Place the x-axis and the y-axis so that $x = y = 0$ at equilibrium, so the forces of gravity and the spring cancel—for this reason we make no mention of the force of gravity. Then, as shown in Figure 9.5, the spring force F_{spr} and the damping force F_{damp} both depend on the relative values of x and y. Newton's second law of motion then implies that the function $x = x(t)$ is a solution of the differential equation

$$mx'' = -\gamma(x' - y'(t)) - k(x - y(t)) \text{ or } m\,x'' + \gamma\,x' + k\,x = \gamma\,y'(y) + k\,y(t)\,.$$

Assume that the automobile is moving at a constant speed along a straight (but not flat!) road. For simplicity, also assume that the rise and fall of the road is given by the sine function

$$y(t) = a\cos(\omega t)\,,$$

where $a > 0$ is a constant and ω depends on the speed of the car. The steady-state solution is then

$$x(t) = aR(\omega)\cos(\omega t - \phi)\,,$$

where both $R(\omega)$ and ϕ have yet to be determined. Since both $R(\omega)$ and ϕ are independent of a, there is no loss of generality in setting $a = 1$.

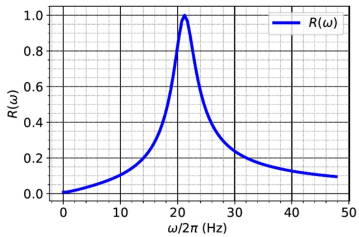


FIGURE 9.6. Notice, that the maximum response is only slightly above 1. This implies that the amplitude of oscillations in the road is never amplified by the struts, and, in fact, it is reduced except at frequencies of around 20 Hz (cycles per second).

As in earlier examples, set $x(t) = \operatorname{Re}(z(t))$, where $z(t) = Ae^{i\omega t}$ is a solution of the complex differential equation

$$m\,z'' + \gamma\,z' + kz = k\,e^{i\omega t} + \gamma\,(e^{i\omega t})' = (k + i\gamma\omega)e^{i\omega t}$$

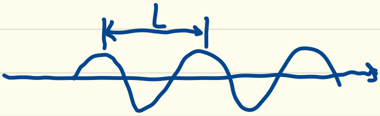
Proceeding as above we arrive at the equation $\{(-m\omega^2 + k) + \gamma\omega\,i\}\,Ae^{i\omega t} = (k + \gamma\omega i)e^{i\omega t}$. Hence,

$$z(t) = \frac{k + \gamma\,\omega\,i}{m(\omega_0^2 - \omega^2) + \gamma\omega i}e^{i\omega t} = \frac{\omega_0^2 + (\gamma/m)\,\omega\,i}{(\omega_0^2 - \omega^2) + (\gamma/m)\omega i}e^{i\omega t}$$

and

$$R(\omega) = \left|\frac{\omega_0^2 + (\gamma/m)\,\omega\,i}{(\omega_0^2 - \omega^2) + (\gamma/m)\omega i}\right| = \sqrt{\frac{\omega_0^4 + (\gamma/m)^2\omega^2}{(\omega_0^2 - \omega^2)^2 + (\gamma/m)^2\omega^2}}\,.$$

Figure 9.6 shows the graph of $R(\omega)$ for the numerical values of m, γ, and k given in Figure 9.5.



$$\cos\left(2\pi \frac{x}{L}\right) \qquad \cos\left(2\pi \frac{v_s t}{L}\right)$$

$x = v_s t$

$L = 0.10$ meters

$$\omega = \frac{2\pi v_s}{L} = (21)\, 2\pi$$

7.74 km/hr