Lecture 17

(The driven harmonic oscillator)

Review

Example Find the general soln of

$$L[y] = 9y'' + 6y' + 10y = \underbrace{e^{t}(2\cos(3t) + 6\sin(3t))}$$

$L[y]=0 \qquad c_1 y_1(t) + c\, y_2(t) \qquad \mathrm{Re}\{(2-6i)\, e^{t} e^{3it}\}$

$L[y]=f(t) \qquad c_1 y_1(t) + c_2 y_2(t) + \underline{y_p(t)} \qquad \mathrm{Re}\{(2-6i)\, e^{(1+3i)t}\}$

$$y_p = \mathrm{Re}\{\underbrace{G e^{(1+3i)t}}_{z_p}\}$$

$$9(z_p)'' + 6(z_p)' + 10 z_p = (2-6i)e^{(1+3i)t}$$

$$9\left((1+3i)^2 G e^{(1+3i)t}\right) + 6\left((1+3i) G e^{(1+3i)t}\right) + 10\left(G e^{(1+3i)t}\right)$$

$$\{9(1+3i)^2 + 6(1+3i) + 10\}\, G e^{(1+3i)t} = (2-6i)\, e^{(1+3i)t}$$

$$G = \frac{(2-6i)}{9(1+3i)^2 + 6(1+3i) + 10}$$

Review

Example Find the general soln of

$$L[y] = 9y'' + 6y' + 10y = e^{t}(2\cos(3t) + 6\sin(3t))$$

Solution

$$e^{t}(2\cos(3t)+6\sin(3t)) = \mathrm{Re}\{(2-6i)e^{(1+3i)t}\}$$

Let

$$z_p(t) = Ge^{(1+3i)t}$$

Want

$$L[z_p] = \{9(1+3i)^2 + 6(1+i) + 10\}Ge^{(1+3i)t}$$

$$= (2-6i)e^{(1+3i)t}$$

So

$$G = \frac{2-6i}{\{9(1+3i)^2 + 6(1+i) + 10\}}$$

$$= \frac{2-6i}{56+72i} \approx \begin{matrix} -0.065+0.023i \\ 0.069e^{2.8i} \end{matrix}$$

$$y_p(t) = \mathrm{Re}(z_p(t)) = 0.069e^{t}\cos(3t+2.8)$$

$$y(t) = Ae^{-t/3}\cos(t-\varphi) + 0.069e^{t}\cos(3t+2.8)$$

$$= \mathrm{Re}\{(c_1 - ic_2)e^{(-\frac{1}{3}+i)t} + (-0.065+0.023i)e^{(1+3i)t}\}$$

transient quasi-periodic

Suppose $\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$

then

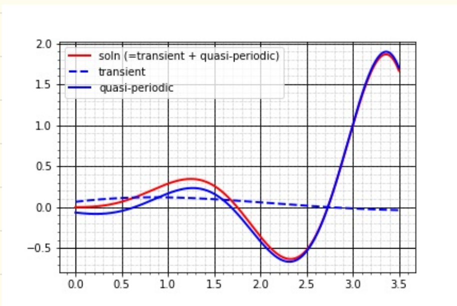
$$y(0) = c_1 - 0.065 = 0$$

$$y'(0) = \mathrm{Re}\{(c_1 - ic_2)(-\tfrac{1}{3}+i) + (-0.065+0.023i)(1+3i)\} = 0$$

$$= -\frac{1}{3}c_1 + c_2 - 0.13 = 0$$

$$c_1 = 0.065$$

$$c_2 = 0.156$$



Beats

- A trig identity:

$$\boxed{\cos(A) + \cos(B) = 2\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)}$$

Proof:

Let $a = \frac{A+B}{2}$, $b = \frac{A-B}{2}$

Then $A = a+b$, $B = a-b$

$$\begin{cases} \cos(A) = \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \\ \cos(B) = \cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b) \end{cases}$$

So $\cos(A) + \cos(B) = 2\cos(a)\cos(b)$

$$= 2\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right) \blacksquare$$

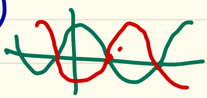
Special cases

- $\cos(A) - \cos(B) = \cos(A) + \cos(B+\pi)$

$$= 2\cos\left(\frac{A-(B+\pi)}{2}\right)\cos\left(\frac{A+B+\pi}{2}\right)$$

$$= -2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)$$

$$= 2\sin\left(\frac{B-A}{2}\right)\sin\left(\frac{A+B}{2}\right)$$



- $\sin(A) - \sin(B) = \cos\left(A - \frac{\pi}{2}\right) + \cos\left(B + \frac{\pi}{2}\right)$

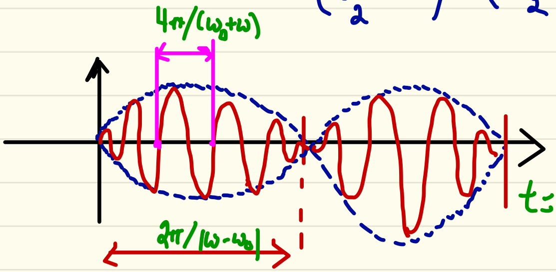
$$= 2\cos\left(\frac{B-A+\pi}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$= -2\sin\left(\frac{B-A}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$= 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

Application. $\cos(\omega t) - \cos(\omega_0 t) = 2\sin\left(\frac{\omega_0-\omega}{2}t\right)\sin\left(\frac{\omega_0+\omega}{2}t\right)$

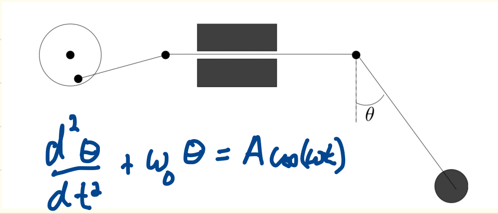
$\omega - \omega_0$ small



$$t = \frac{2\pi}{\frac{\omega-\omega_0}{2}} = \frac{4\pi}{\omega_0-\omega}$$

The driven harmonic oscillator

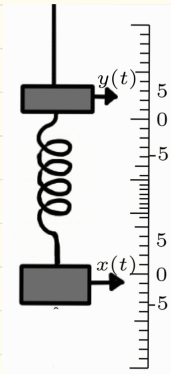
Various ways to apply a force.



$$\frac{d^2\theta}{dt^2} + \omega_0 \theta = A\cos(\omega t)$$

$$\rightarrow m\frac{d^2x}{dt^2} + kx = ky(t)$$

$$m\frac{d^2x}{dt^2} = -k(x - y)$$



Resonance

$$m x'' + kx = F_0 \cos(\omega t)$$

$$\begin{cases} x'' + \omega_0^2 x = (F_0/m)\cos(\omega t) \\ x(0) = 0, \quad x'(0) = 0 \end{cases}$$

Solve the IVP

$$x_p(t) = Re\{G e^{i\omega t}\}$$

$$\underbrace{(G e^{i\omega t})''}_{(-\omega^2 G e^{i\omega t})} + \omega_0^2 (G e^{i\omega t}) = F_0/m \, e^{i\omega t}$$

$$\Rightarrow G = \frac{F_0/m}{\omega_0^2 - \omega^2} \Rightarrow$$

$$x_p(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos(\omega t)$$

Therefore,

$$x(t) = Re\{(c_1 - i c_2) e^{i\omega_0 t}\} + \frac{F_0/m}{\omega_0^2 - \omega^2}\cos(\omega t)$$

$$x(0) = 0 \Rightarrow c_1 + \frac{F_0/m}{\omega_0^2 - \omega^2} = 0$$

$$x'(0) = 0 \Rightarrow c_2 \omega_0 + 0 = 0$$

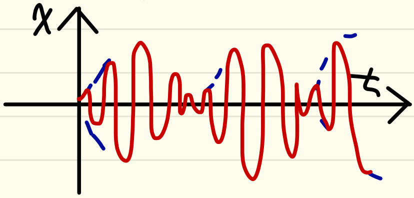
Hence,

$$x(t) = F_0/m \, \frac{\cos(\omega t) - \cos(\omega_0 t)}{\omega_0^2 - \omega^2}$$

$$\cos(a) - \cos(b) = 2\sin\left(\frac{b-a}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

So

$$x(t) = \frac{2F_0/m}{\omega_0^2 - \omega^2} \sin\left(\frac{\omega_0 - \omega}{2} t\right) \sin\left(\frac{\omega_0 + \omega}{2} t\right)$$



$$\begin{cases} \frac{\omega_0 - \omega}{2} & \text{low frequency} \\ \frac{\omega_0 + \omega}{2} & \text{high frequency} \end{cases}$$

What happens when $\omega = \omega_0$?

$$\lim_{\omega \to \omega_0} \frac{\cos(\omega t) - \cos(\omega_0 t)}{\omega_0^2 - \omega^2} = \lim_{\omega \to \omega_0} \frac{-t \sin(\omega t)}{-2\omega} = \frac{t \sin(\omega_0 t)}{2\omega_0}$$

So

$$x(t) \to F_0/m \, \frac{t \sin(\omega_0 t)}{2\omega_0},$$

which satisfies the IVP

$$m x'' + kx = F_0 \cos(\omega_0 t),$$

$$x(0) = 0 \quad x'(0) = 0$$

What happens when $\gamma \neq 0$?

$$m x'' + \gamma x' + k x = F_0 \cos(\omega t) = \mathrm{Re}\{F_0 e^{i\omega t}\}$$

Let

$$z_p(t) = G e^{i\omega t} \qquad x_p(t) = \mathrm{Re}\{z_p(t)\}$$

Want $m z_p'' + \gamma z_p' + k z_p = F_0 e^{i\omega t}$

$$\{m(i\omega)^2 + \gamma(i\omega) + k\} G e^{i\omega t} = F_0 e^{i\omega t}$$

So

$$G = \frac{F_0}{k - m\omega^2 + i\gamma\omega} = \frac{F_0}{m(\omega_0^2 - \omega^2) + i\gamma\omega}$$

(remember $\omega_0 = \sqrt{k/m}$)

$$= \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}} e^{-i\alpha},$$

$$\tan\alpha = \frac{\gamma\omega}{m(\omega_0^2 - \omega^2)}$$

Therefore

$$x(t) = \underbrace{A e^{-\frac{\gamma}{2m}t} \cos(\omega_d t - \varphi)}_{\text{transient}}$$

$$+ \underbrace{\frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}} \cos(\omega t - \alpha)}_{\text{periodic}}$$

To maximize $|G|$,

minimize

$$f(\omega) = m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2$$

$$f'(\omega) = 2m^2(\omega_0^2 - \omega^2)(-2\omega) + 2\gamma^2\omega = 0$$

$$\Rightarrow -4m^2(\omega_0^2 - \omega^2) + 2\gamma^2 = 0$$

$$\Rightarrow \omega_{res} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2m^2}}$$

Note

$$\omega_{res} < \omega_d = \sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}} < \omega_0 = \sqrt{\frac{k}{m}}$$

$$\boxed{m x'' + \gamma x' + k x = F_0 \cos(\omega t)}$$

$$m x'' = -kx - \gamma x' + F_0 \cos(\omega t)$$

$x_p(t)$ steady state $\quad \underbrace{R(\omega)}_{\text{gain}} \cos(\omega t - \underset{\text{phase shift}}{\varphi})$

$$z_p = G e^{i\omega t}$$

$$m z_p'' + \gamma z_p' + k z_p = F_0 e^{i\omega t}$$

$$\underbrace{(-m\omega^2 + \gamma\omega i + k)}\, G e^{i\omega t} = F_0 e^{i\omega t}$$

$$G = \frac{F_0}{k - m\omega^2 + \gamma\omega i} \qquad R(\omega) = |G(\omega)| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{\gamma}{m}\omega\right)^2}}$$

$$= \frac{F_0/m}{k/m - \omega^2 + \frac{\gamma}{m}\omega i}$$

$$= \frac{F_0/m}{(\omega_0^2 - \omega^2) + \frac{\gamma}{m}\omega i} = R e^{-i\varphi}$$