Lecture 16

$$L[y] = \cos(\omega t)$$

the steady state solution

Review.

To solve $y'' + 2y' + y = \cos(3t)$ (*)

- Look for a solution $z_p = Ae^{2it}$

of $\quad z'' + 2z' + z = e^{2it}$

Then $y_p = \mathrm{Re}\{Ae^{2it}\}$ is a particular solution of (*).

The general solution of (*) is

$$y(t) = (C_1 + C_2 t)e^{-t} + \mathrm{Re}\{Ae^{2it}\}$$

To find A, compute as follows:

$$z_p'' + 2z_p' + z_p = \{(2i)^2 + 2(2i) + 1\}Ae^{2it}$$
$$= (-3+4i)Ae^{2it} = e^{2it}$$

So

$$A = \frac{1}{-3+4i} = \frac{-3}{25} - \frac{4}{25}i$$

and

$$y_p(t) = \frac{-3}{25}\cos(2t) + \frac{4}{25}\sin(2t)$$

Initial Value Problem.

Suppose $y(0) = 0$ and $y'(0) = 0$.

Then
$$y(0) = C_1 - \frac{3}{25}\cos(0) + \frac{4}{5}\sin(0) = 0$$
$$y'(0) = C_2 + \frac{6}{25}\sin(0) + \frac{8}{25}\cos(0) = 0$$
$$\Rightarrow \begin{cases} C_1 = 3/25 \\ C_2 = -8/25 \end{cases}$$

So

$$y(t) = \left(3/25 - 8/25\, t\right)e^{-t} - \frac{3}{25}\cos(2t) + \frac{4}{25}\sin(2t)$$

$$y(t) = \left(\tfrac{3}{25} - \tfrac{8}{25}t\right)e^{-t} - \frac{3}{25}\cos(2t) + \frac{4}{25}\sin(2t)$$

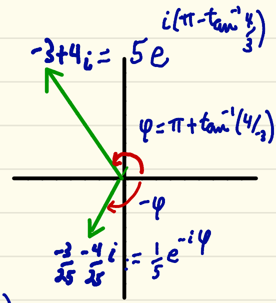
$$= \left(\tfrac{3}{25} - \tfrac{8}{25}t\right)e^{-t} + \mathrm{Re}\left\{\left(\frac{-3}{25} - \frac{4}{25}i\right)e^{2it}\right\}$$

$$= \left(\tfrac{3}{25} - \tfrac{8}{25}t\right)e^{-t} + \mathrm{Re}\left\{\frac{1}{-3+4i}\,e^{2it}\right\}$$

$$= \underbrace{\left(\tfrac{3}{25} - \tfrac{8}{25}t\right)e^{-t}}_{\text{transient}} + \underbrace{\frac{1}{5}\cos\Big(2t - \underbrace{(\pi - \tan^{-1}(4/3))}_{\varphi}\Big)}_{\text{steady state solution}}$$

transient ≈ 0 for t large

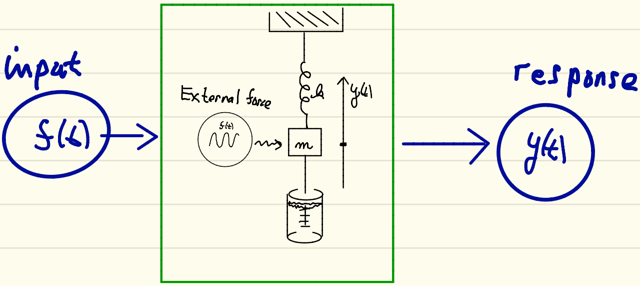
steady state solution

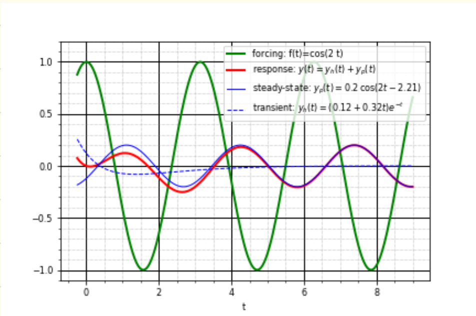


input

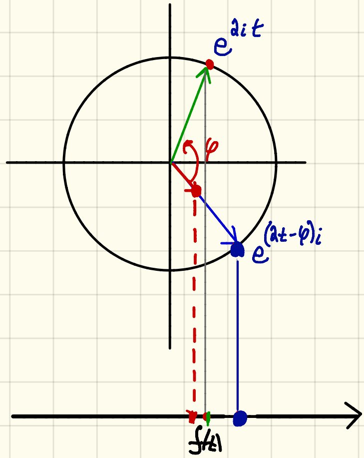
$\cos(2t)$ $\longrightarrow$ $\boxed{\begin{array}{c} y'' + 2y' + y = f(t) \\ y(0) = 0,\ y'(0) = 0 \end{array}}$ $\longrightarrow$ $y(t)$

response





Geometric Picture



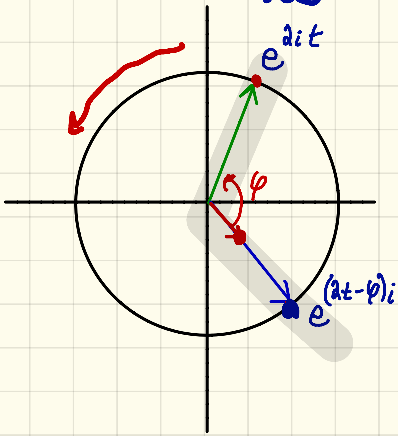
$$f(t) = \mathrm{Re}\{e^{2it}\}$$

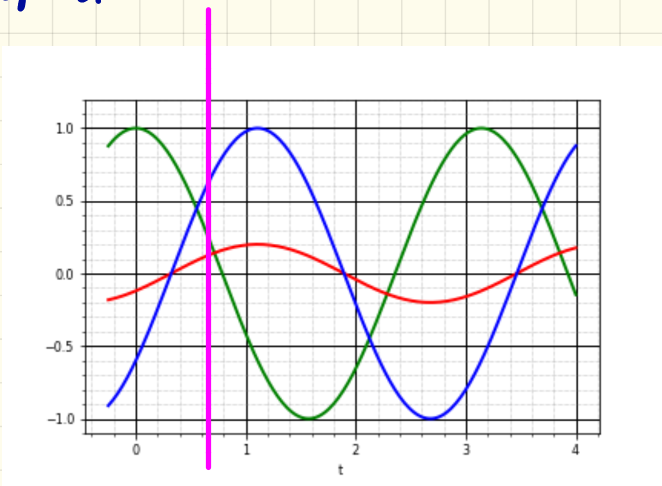
$$y(t) = \mathrm{Re}\{A e^{2it}\}$$

$$A = \frac{1}{5} e^{-i\varphi}$$

$\varphi \approx 2{,}2$ or $\approx 126^\circ$

Rotating at angular speed
$\omega = 2$ rad/sec





Example. Solve the I.V.P.

$$9y'' + 6y' + 10y = 410\cos(2t)$$

$$y(0) = 0, \quad y'(0) = 0$$

$y_p(t) = \mathrm{Re}\{z(t)\}$ where

$$9z'' + 6z' + 10z = 410e^{2it}$$

Try

$z = G \cdot e^{2it}$, G (complex number)

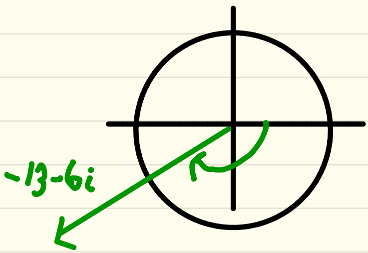
$$\{9(2i)^2 + 6(2i) + 10\}Ge^{2it} = 410e^{2it}$$

$$\Rightarrow (-26+12i)\cdot G = 410$$

$$\Rightarrow G = \frac{205}{-13+6i} = \frac{205(-13-6i)}{169+36}$$

$$= -13-6i$$

$$y_p(t) = \mathrm{Re}\{-(13+6i)e^{2it}\}$$



$-(13+6i)$

$\approx 14.3e^{-2.7i}$

$$y(t) = \mathrm{Re}\{Ce^{(-\frac{1}{3}+i)t}\} - \mathrm{Re}\{(13+6i)e^{2it}\}$$

$$= \mathrm{Re}\{Ce^{(-\frac{1}{3}+i)t} - (13+6i)e^{2it}\}$$

$$C = c_1 - ic_2$$

$$y(0) = \mathrm{Re}\{C - (13+6i)\} = 0$$

$$\Rightarrow c_1 = 13$$

$$y'(0) = \mathrm{Re}\{C(-\tfrac{1}{3}+i) - (13+6i)(2i)\} = 0$$

$$-\tfrac{1}{3}c_1 + c_2 + 12 = 0$$

$$-\frac{13}{3} + c_2 + 12 = 0$$

$$\Rightarrow c_2 = -\frac{23}{3}$$

So

$$\boxed{y(t) = \mathrm{Re}\left\{\left(13+\tfrac{23}{3}i\right)e^{(-\frac{1}{3}+i)t} - (13+6i)e^{2it}\right\}}$$

$$\boxed{y(t) = y_h(t) + y_p(t) = \mathrm{Re}\left\{\left(13+\tfrac{23}{3}i\right)e^{\left(-\frac{1}{3}+i\right)t} - (13+6i)\,e^{2it}\right\}}$$

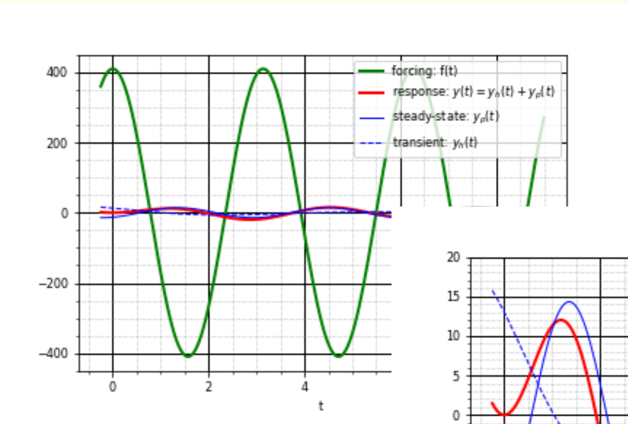
Other ways to write $y(t)$

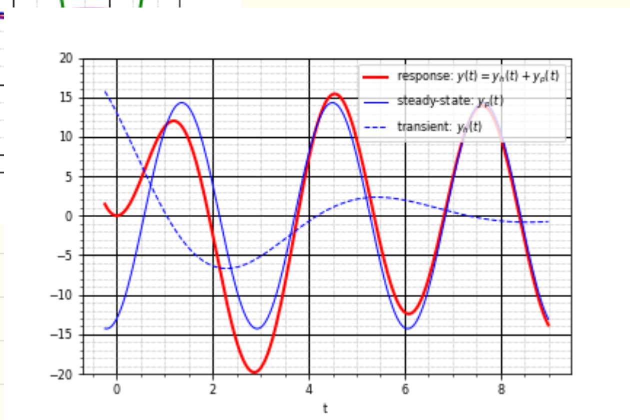
$$y(t) = e^{-t/3}\left(13\cos(t) - \frac{23}{3}\sin(t)\right) - \left(-13\cos(2t) + 6\sin(2t)\right)$$

$$= \sqrt{(13)^2+\left(\tfrac{23}{3}\right)^2}\, e^{-t/3}\cos\left(t + \tan^{-1}\left(\tfrac{23}{39}\right)\right)$$

$$+ \sqrt{(13)^2+(6)^2}\,\cos\left(2t - \left(\pi - \tan^{-1}\left(\tfrac{6}{13}\right)\right)\right)$$

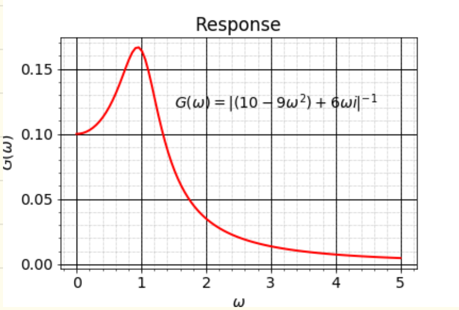
$$\approx 15.09\, e^{-t/3}\cos(t + 0.53) + 14.3\cos(2t - 2.7)$$





Can we get a larger response?

$$9y'' + 6y' + 10y = \cos(\omega t)$$



$$z = G e^{i\omega t}$$

$$\{(10 - 9\omega^2) + 6\omega i\} G(\omega i) = 1$$

$$G(\omega i) = \frac{1}{(10 - 9\omega^2) + 6\omega i}$$

Idea: Choose ω to minimize the modulus of the denominator.

$$|(10 - 9\omega^2) + 6\omega i| = \sqrt{(10 - 9\omega^2)^2 + 36\omega^2}$$

let $x = \omega^2$ $\quad f(x) = (10 - 9x)^2 + 36x$

$$f'(x) = -18(10 - 9x) + 36 = 0$$

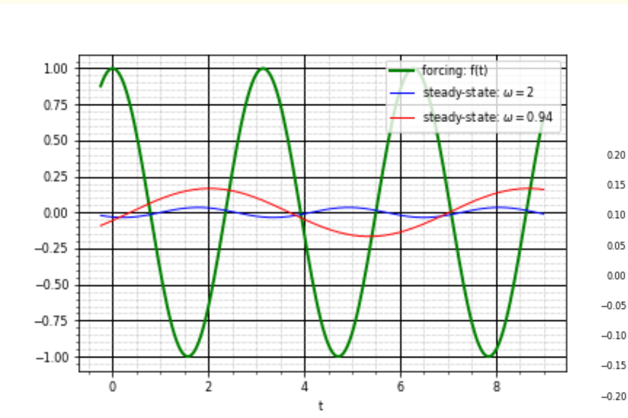
$$\Rightarrow \quad 10 - 9x = \frac{36}{18} = 2$$

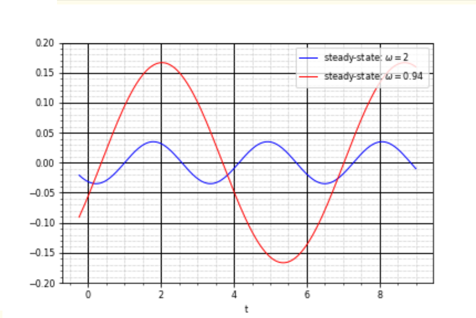
$$\Rightarrow \quad 9x = 8 \quad x = 8/9$$

$$\omega = \sqrt{8/9} = \frac{2\sqrt{2}}{3} \approx 0.94$$

$$|G(2i)| \cong 0.035$$

$$|G(0.94i)| \cong 0.16\overline{6}$$





Some Terminology

$G(\omega i) = \frac{1}{(10-9\omega^2)+6\omega i}$ is the transfer function

$= R(\omega)e^{i\varphi(\omega)}$ of the system

$R(\omega) = |G(\omega i)|$ is called the gain

$\varphi(\omega)$ is called the phase (or phase shift)

Useful observation. Consider the ODE

$$y'' + 3y' + 2y = \cos(2t) = \mathrm{Re}\{e^{2it}\}$$

The steady state solution is $y_p(t) = \mathrm{Re}\{G(2i)e^{2it}\}$

where $G(2i) = \frac{1}{-2+6i}$

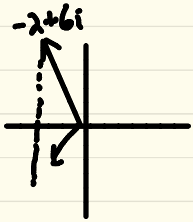
Since $2\cos(2t) + 7\sin(2t) = \mathrm{Re}\{(2-7i)e^{2it}\}$

the steady state solution to

$$y'' + 3y' + 2y = 2\cos(2t) + 7\sin(2t) = f(t)$$

is $y_p(t) = \mathrm{Re}\left\{\frac{2-7i}{-2+6i}e^{2it}\right\}$

$$G(2i) = \frac{1}{-2+6i} = \frac{1}{\sqrt{40}} e^{-i\varphi}$$



$$\varphi = \pi - \tan^{-1}(3) \approx 1.9$$
$$\approx 108^\circ$$

$$2-7i = \sqrt{53}\, e^{-i\tan^{-1}(7/2)}$$
$$\approx 7.28 e^{-1.3i}$$
$$= A e^{-\alpha i}$$
$$\alpha \approx 74^\circ$$