Lecture 15

(Undetermined Coefficients)

Undetermined Coeffs.

Gen. Solution to $L[y] = ay'' + by' + cy = 0$ (homogeneous)

$$y_h(t) = C_1\, y_1(t) + C_2\, y_2(t)$$

What about $L[y] = f(t)$? (non-homogeneous) $\quad f(t)$: "forcing function"

Idea: Suppose $y_p(t)$ is a particular solution.
(say found by guessing)
Suppose $y(t)$ is another solution

So $L[y_p] = f(t)$ and $L[y] = f(t)$

Then $L[y - y_p] = f(t) - f(t) = 0$ (by Linearity!)

So $y(t) - y_p(t)$ is a solution to the homogeneous ODE.

Therefore, $y(t) - y_p(t) = C_1\, y_1(t) + C_2\, y_2(t)$ for some $C_1 + C_2$.

Thus,

$$\boxed{y(t) = y_p(t) + C_1\, y_1(t) + C_2\, y_2(t)}$$

$\Big\{$ We already know how to find $y_1(t) + y_2(t)$.
We only need to find $y_p(t)$!

Special Case:

$$f(t) = \mathrm{Re}\left(p(t)\, e^{(\rho - i\omega)t}\right) \quad \text{for } p(t) \text{ a polynomial}$$

Examples:

$$f(t) = 2 + t + 3t^2 \quad (\rho = 0,\ \omega = 0)$$

$$f(t) = e^{7t}$$

$$f(t) = (2t+1)e^{-3t}$$

$$f(t) = \cos(t) - 3\sin(t) = \mathrm{Re}\left((1+3i)\, e^{it}\right)$$

$$f(t) = \sin(6t) = \mathrm{Re}\left(-i\, e^{6it}\right)$$

$$f(t) = 2t\cos(3t) + 7\sin(3t) + 2\cos(3t) = \mathrm{Re}\left(\left(2t + (2-7i)\right) e^{3it}\right)$$

If $f(t) = \mathrm{Re}\{(p_1 + p_2 t + \cdots + p_{n+1} t^n) e^{rt}\}$

Try $y_p(t) = \mathrm{Re}\{(C_1 + C_2 t + \cdots + C_{n+1} t^n) e^{rt}\}$

as long as $L[e^{rt}] \neq 0$

If $L[e^{rt}] = 0$ try

$t(C_1 + C_2 t + \cdots + C_{n+1} t^n) e^{rt}$ (r single root)

$t^2(C_1 + C_2 t + \cdots + C_{n+1} t^n) e^{rt}$ (r double root)

<u>Examples.</u> $y'' + 2y' + 7y = 3e^{7t}$ Try $y_p = C_1 e^{7t}$

$= (t+7)e^{3t}$ Try $y_p = (C_1 + C_2 t)e^{3t}$

$y'' + 3y' + 2y = e^{-2t}$ Try $y_p = t(C_1 e^{-2t})$

(roots: $-1, -2$)

$y'' + 3y' + 2y = 2te^{-2t}$ Try $y_p = t(C_1 + C_2 t)e^{-2t}$

$y'' + 4y' + 4y = te^{-2t}$ Try $y_p = t^2(C_1 + C_2 t)e^{-2t}$

($r = -2$ is double root!)

$y'' + 2y' + y = 2\cos(4t) = \mathrm{Re}\{2e^{4it}\}$. Try $y_p = \mathrm{Re}\{(C_1 + iC_2)e^{4it}\}$

$y'' + 16y = 2\sin(4t) = \mathrm{Re}\{-2ie^{4it}\}$. Try $y_p = \mathrm{Re}\{t(C_1 + iC_2)e^{4it}\}$

$y'' + 16y = e^{-t}\cos(3t) = \mathrm{Re}\{e^{(-1+3i)t}\}$

Try $y_p = \mathrm{Re}\{(C_1 + iC_2)e^{(-1+3i)t}\}$

Example (a) Find the general solution to $y'' + 2y' + y = 6e^{2t}$

Soln. Char. poly. r^2+2r+1 has double root $r=-1$.

Try $y_p(t) = Ae^{2t}$ $\qquad (Ae^{2t})'' + 2(Ae^{2t})' + Ae^{2t}$

Then $y_p'' + 2y_p' + y_p = ((2)^2 + 2(2) + 1)Ae^{2t} = 6e^{2t}$

Or $9A = 6 \Rightarrow A = 6/9 = 2/3$ $\qquad y_p(t) = \frac{2}{3}e^{2t}$

Gen Soln: $y(t) = \frac{2}{3}e^{2t} + (C_1 + C_2 t)e^{-t}$

(b) Find the general solution to $y'' + 2y' + y = 2\cos(2t) - 3\sin(2t)$

Soln Note: $2\cos(2t) - 3\sin(2t) = \mathrm{Re}\{(2+3i)e^{2it}\}$

Try $y_p(t) = \mathrm{Re}\{Ae^{2it}\}$. Set $z_p = Ae^{2it}$

$z_p'' + 2z_p' + z_p = \{(2i)^2 + 2(2i) + 1\}Ae^{2it} = (-3+4i)Ae^{2it}$

Want $z_p'' + 2z_p' + z_p = (2+3i)e^{2it}$.

Hence $(-3+4i)A = (2+3i) \Rightarrow$

$$A = \frac{2+3i}{-3+4i} = \frac{6}{25} - \frac{17}{25}i = \frac{\sqrt{13}}{5}e^{-i\tan^{-1}(17/6)}$$

So $y_p(t) = \mathrm{Re}\left\{\left(\frac{2+3i}{-3+4i}\right)e^{2it}\right\} = \frac{6}{25}\cos(2t) + \frac{17}{25}\sin(2t) = \frac{\sqrt{13}}{5}\cos(2t - \tan^{-1}(17/6))$

General soln: $y(t) = (C_1 + C_2 t)e^{-t} + \mathrm{Re}\left\{\left(\frac{2+3i}{-3+4i}\right)e^{2it}\right\}$.

Example. Find the general solution to

$$y'' + 2y' + y = te^{2t}$$

Soln. Char. poly. r^2+2r+1 has double root $r=-1$.

Try $y_p(t) = (A+Bt)e^{2t}$

$y_p'(t) = Be^{2t} + 2(A+Bt)e^{2t}$

$= ((2A+B) + 2Bt)e^{2t}$

$y_p''(t) = 2Be^{2t} + 2((2A+B)+2Bt)e^{2t}$

$= ((4A+4B) + 4Bt)e^{2t}$

So $y_p'' + 2y_p' + y_p =$

$$\begin{aligned} &(4A+4B \quad + 4Bt)e^{2t} \\ &+2((2A+B) \quad + 2Bt)e^{2t} \\ &+ (A \quad + Bt)e^{2t} \\ &= (9A+6B + 9Bt)e^{2t} = te^{2t} \end{aligned}$$

So $\begin{cases} 9A+6B = 0 \\ 9B = 1 \end{cases} \Rightarrow \begin{cases} A = -2/27 \\ B = 1/9 \end{cases} \Rightarrow y_p(t) = \left(\frac{-2}{27} + \frac{t}{9}\right)e^{2t}$

Gen Soln is

$$y(t) = (c_1 + c_2 t)e^{-t} + \left(t/9 - 2/27\right)e^{+2t}$$

Example. Find the general solution to

$$L[y] = y'' + 2y' + y = (4 + 8t)e^{-t}$$

Soln. Char. poly. $r^2 + 2r + 1$ has double root $r = -1$.

So the general solution is of the form

$$y(t) = (c_1 + c_2 t)e^{-t} + y_p(t).$$

To find $y_p(t)$ try

$$y_p(t) = t^2(A + Bt)e^{-t}$$

After a bit of algebra, we get

$$L[y_p] = (2A + 6Bt)e^{-t} = (4 + 8t)e^{-t}$$

So $2A = 4$ and $6B = 8$ $\Rightarrow$ $A = 2, B = 4/3$

The general solution is, therefore,

$$y(t) = (c_1 + c_2 t)e^{-t} + t^2(2 + \tfrac{4}{3}t)e^{-t}$$

$$= \{c_1 + c_2 t + 2t^2 + \tfrac{4}{3}t^3\}e^{-t}.$$