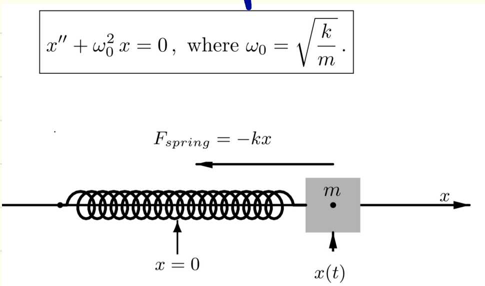
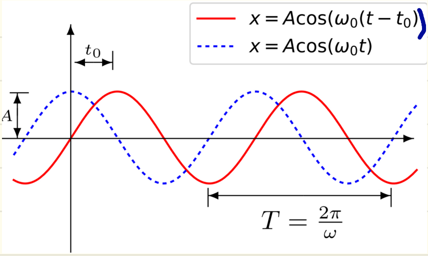
Lecture 14

The harmonic Oscillator

The Harmonic Oscillator

The Undamped case:



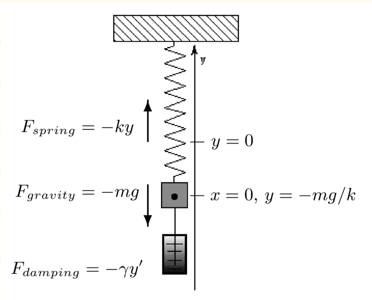


General solution:

$$x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$
$$= A \cos(\omega_0 t - \varphi)$$
$$= A \cos(\omega_0 (t - t_0)), \quad t_0 = \frac{\varphi}{\omega_0}$$

ω_0 is called the <u>natural frequency</u>

The damped case:



"$F = ma$":

$$my'' = -mg - ky - \gamma y'$$
$$\Rightarrow my'' + \gamma y' + ky = -mg$$

At equilibrium: $y = y_{eq} = -mg/k$

Set $y = y_{eq} + x$:

$$mx'' + \gamma x' + kx = 0$$

(show demo.)

$$x'' + \frac{k}{m} x = 0$$

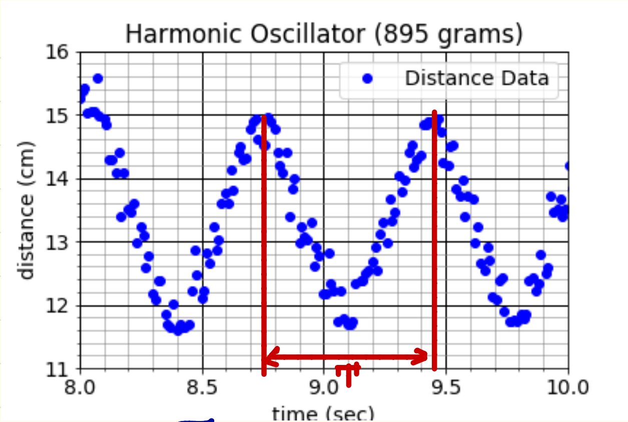
$$\omega_0 = \sqrt{k/m}$$

$$T = \text{Period} = \frac{2\pi}{\omega_0}$$

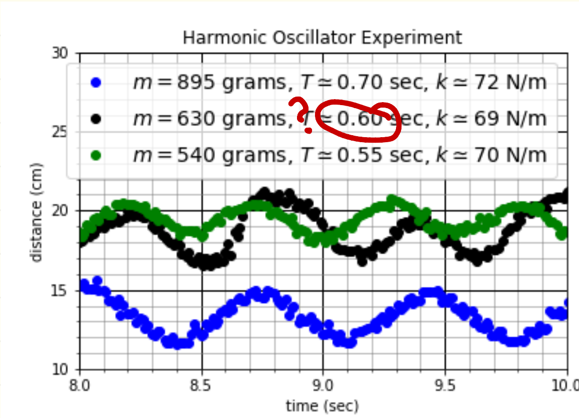
So $\frac{k}{m} = \omega_0^2 = \left(\frac{2\pi}{T}\right)^2$

$$k = \left(\frac{2\pi}{T}\right)^2 m$$

$$\approx \left(\frac{2\pi}{0.70}\right)^2 (0.895) \approx 72 \text{ N/m}$$



$T \approx 9.45 - 8.75 \approx 0.70$ sec



Predictions of mass:

$$m \approx (0.72)\left(\frac{T}{2\pi}\right)^2$$

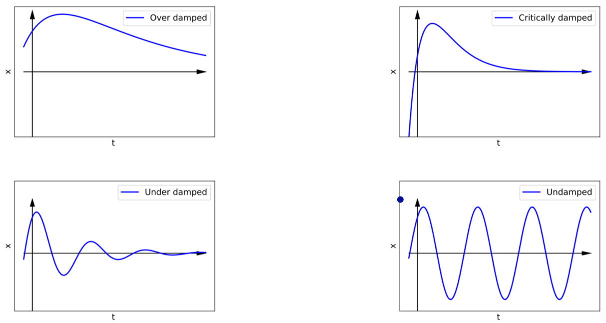
$T = 0.55$ sec $\Rightarrow$

$m \approx 552$ grams

Four cases:

$$mx'' + \gamma x' + kx = 0$$

roots: $\frac{-\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m} = \frac{-\gamma}{2m} \pm \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}} = \frac{-\gamma}{2m} \pm \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \omega_0^2}$



Over-damped $x(t) = A\,e^{-\left(\frac{\gamma}{2m} + \frac{\sqrt{\gamma^2 - 4mk}}{2m}\right)t} + B\,e^{-\left(\frac{\gamma}{2m} - \frac{\sqrt{\gamma^2 - 4mk}}{2m}\right)t}$	Critically damped $x(t) = (A + Bt)\,e^{-\frac{\gamma}{2m}\,t}$
Under-damped $x(t) = A\,e^{-\frac{\gamma}{2m}\,t}\cos(\omega t - \phi)$	Undamped $x(t) = A\cos(\omega_0 t - \phi)$

Note: In both overdamped + critically damped case,

$$x(t) = 0$$

for exactly one value of t.

- Over damped:

$$Ae^{-at} + Be^{-bt} = 0$$

$$= e^{bt}e^{-at} = -\frac{B}{A}$$

$$\Rightarrow e^{(b-a)t} = B/A$$

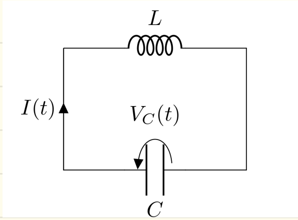
$$t = \frac{\ln(B/A)}{b-a}$$

- Critically damped:

$$(A+Bt)e^{-ct} = 0 \Rightarrow t = -A/B.$$

Electrical Circuits

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0 \qquad LC\frac{d^2V_C}{dt^2} + V_C = 0$$



$$\Rightarrow \boxed{\frac{d^2V_C}{dt^2} + \frac{1}{LC}V_C = 0} \qquad \omega_0 = \sqrt{\frac{1}{LC}}$$

Example.

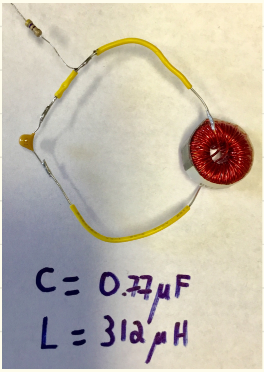
$L \approx 312\mu H = 312\times10^{-6} H$

$C \approx 0.77\mu F = 0.77\times10^{-6} F$

$\omega = \frac{1}{\sqrt{LC}} = 6.5\times10^4 \, 1/sec$

$f = \omega/2\pi = 10.3 \text{ kHz}$

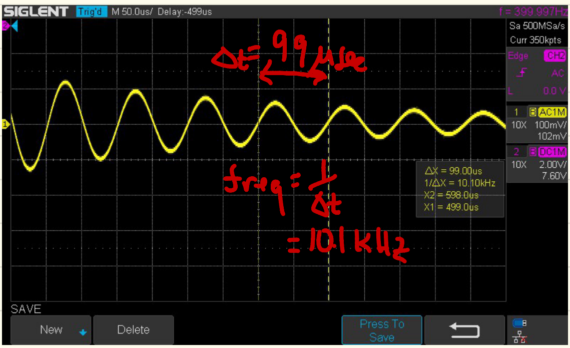
$(10.3\times10^3 \text{ Hz})$



Measured frequency:

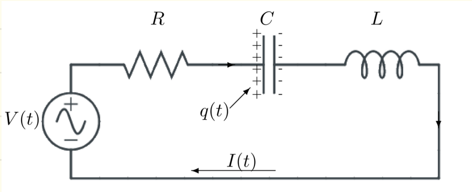
10.1 kHz

Note the decreasing amplitude of the oscillations!



RLC- circuits

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V(t)$$



$$LC\frac{d^2V_C}{dt^2} + RC\frac{dV_C}{dt} + V_C = V(t)$$

$$V_C = \frac{q}{C}$$

$$\boxed{\frac{d^2V_C}{dt^2} + \frac{R}{L}\frac{dV_C}{dt} + \frac{V_C}{LC} = \frac{V(t)}{LC}}$$

$$\frac{L}{R}\frac{d^2V_R}{dt^2} + \frac{dV_R}{dt} + \frac{V_R}{RC} = V'(t)$$

$$V_R = R\frac{dq}{dt}$$

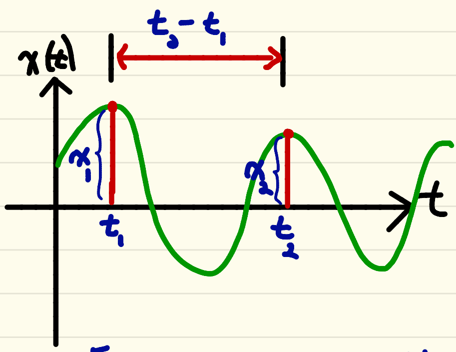
$$\boxed{\frac{d^2V_R}{dt^2} + \frac{R}{L}\frac{dV_R}{dt} + \frac{V_R}{LC} = \frac{R}{L}V'(t)}$$

Inverse Problem:

Find $\frac{\gamma}{m}$ and $\frac{k}{m}$ from solution of the diff. eq.

$$m x'' + \gamma x' + k x = 0$$

(underdamped case).



Soln:

$$x(t) = A e^{-\frac{\gamma}{2m} t} \cos(\omega_d t - \varphi)$$

$$\omega_d = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2} \qquad \omega_0 = \sqrt{k/m}$$

From graph, can find

t_1, $x_1 = x(t_1)$

t_2, $x_2 = x(t_2)$

$$t_2 - t_1 = \frac{2\pi}{\omega_d} \Rightarrow \boxed{\omega_d = \frac{2\pi}{t_2 - t_1}}$$

$$\omega_d t_2 - \varphi = \omega_d t_1 - \varphi + 2\pi$$

$$\Rightarrow \frac{x_2}{x_1} = \frac{A \exp\left(-\frac{\gamma}{2m} t_2\right) \cos(\omega_d t_2 - \varphi)}{A \exp\left(-\frac{\gamma}{2m} t_1\right) \cos(\omega_d t_1 - \varphi)} = \frac{\exp\left(-\frac{\gamma}{2m} t_2\right)}{\exp\left(-\frac{\gamma}{2m} t_1\right)} = \exp\left(-\frac{\gamma}{2m}(t_2 - t_1)\right)$$

So $-\frac{\gamma}{2m}(t_2 - t_1) = \ln\left(\frac{x_2}{x_1}\right) \Rightarrow \boxed{\frac{\gamma}{2m} = \frac{1}{t_2 - t_1} \ln\left(\frac{x_1}{x_2}\right)}$

$$\boxed{\frac{k}{m} = \omega_0^2 = \omega_d^2 + \left(\frac{\gamma}{2m}\right)^2}$$

$\Delta = \ln\left(\frac{x_1}{x_2}\right)$: <u>logrithmic decrement</u>

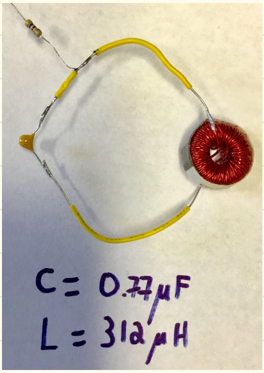
<u>Note</u> Can write

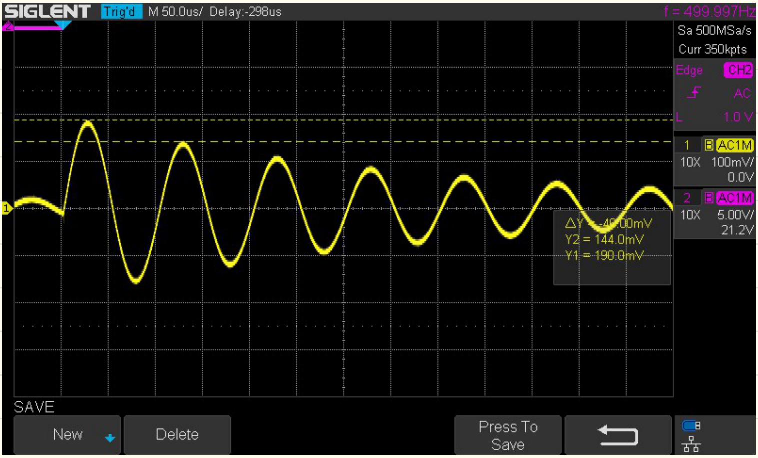
$$\frac{\gamma}{2m} = \zeta \omega_0$$

ζ called <u>damping ratio</u> (dimensionless)

$$\omega_d^2 = \omega_0^2 - \zeta^2 \omega_0^2 = (1 - \zeta^2)\omega_0^2$$

Example.





$$L v_c'' + R v_c' + \frac{1}{C} v_c = 0$$

Goal: Find R.

measured values via multimeter:
  C = 0.77e-6 #Farads

  L = 312e-6 #Henries (from manufacturer)

  w0 = 1/sqrt(L*C)   (1/sec)   (theoretical value)
  f0 = w0/(2*pi) (= 10.27 kHz) (theoretical value)

  # measured frequency and damping
   (from oscilloscope):

  fd = 10.10 kHz
  y1 = 190.0e-3 # volts
  y2 = 144.0e-3 # volts

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \omega_d^2 = \omega_0^2 - \left(\frac{R}{2L}\right)^2$$

$$\frac{\omega_d}{2\pi} = \frac{1}{t_2 - t_1}, \quad \Delta = \ln\left(\frac{y_1}{y_2}\right)$$

$$\frac{R}{2L} = \frac{1}{t_2 - t_1}\Delta$$

$$R = 2L \cdot \frac{\Delta}{t_2 - t_1} = 1.75\,\Omega$$