Lecture 13

(complex roots)

Review:

$$ay'' + by' + cy = 0$$

roots: $ar^2 + br + c = 0$

e^{rt}

Case(iii): $r_1, r_2 = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$

$ay'' + by' + cy = 0$

$y'' + \left(\frac{b}{a}\right)y' + \frac{c}{a}y = 0$

$$= -p \pm \omega i$$

where $p = \frac{b}{2a}$, $\omega = \sqrt{\frac{c}{a} - \left(\frac{b}{2a}\right)^2}$

$$y(t) = c_1 e^{(-p+i\omega)t} + c_2 e^{(-p-i\omega)t}$$

complex numbers!

Another fundamental basis:

$$e^{-pt}\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right) \quad \frac{e^{(-p+i\omega)t} + e^{(-p-i\omega)t}}{2}, \qquad \frac{e^{(-p+i\omega)t} - e^{(-p-i\omega)t}}{2i}$$

$$\| \qquad\qquad \|$$

$$e^{-pt}\cos(\omega t) \qquad e^{-pt}\sin(\omega t)$$

So general soln. is

$$y(t) = c_1 e^{-pt}\cos(\omega t) + c_2 e^{-pt}\sin(\omega t)$$

$$= \mathrm{Re}\left\{(c_1 - ic_2)\, e^{(-p+i\omega)t}\right\}$$

$$y'(t) = \mathrm{Re}\left\{\underbrace{(c_1 - ic_2)(-p+i\omega)}\, e^{(-p+i\omega)t}\right\}$$

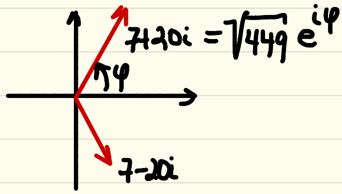
Example. Solve the initial value problem

$$y'' + 4y' + 5y = 0 \quad \begin{cases} y(0) = 7, \\ y'(0) = 6 \end{cases}$$

Soln. Charistic polynomial: $r^2 + 4r + 5 = (r+2)^2 + 1$

$(r+2)^2 = -1 \quad r+2 = \pm i \quad r = -2 \pm i$

roots: $r = -2 \pm i$.



So $y(t) = \mathrm{Re}\left\{ (C_1 - iC_2) e^{(-2+i)t} \right\}$

$y(0) = \mathrm{Re}(C_1 - iC_2) = C_1 = 7$

$\longrightarrow y'(0) = \mathrm{Re}\left\{ (7 - iC_2)(-2+i) \right\}$

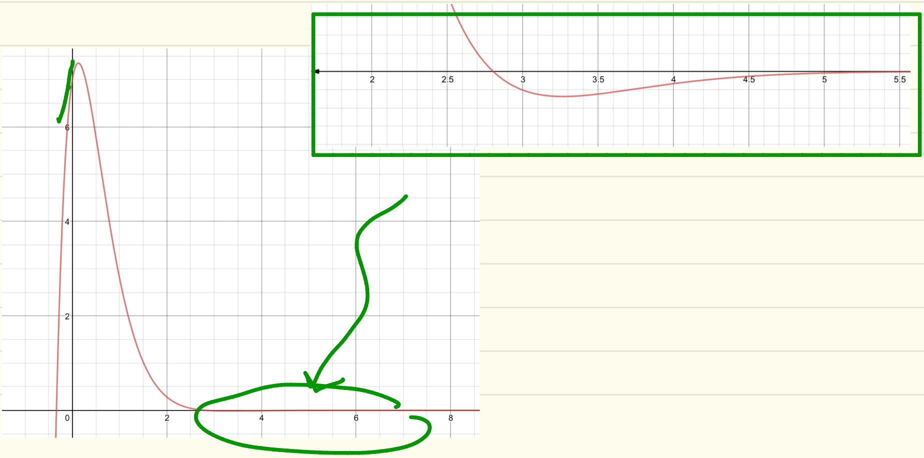
$= -14 + C_2 = 6 \Rightarrow C_2 = 20$

Therefore,

$$y(t) = \mathrm{Re}\left\{ (7 - 20i) e^{(-2+i)t} \right\} = e^{-2t}\left\{ 7\cos(t) + 20\sin(t) \right\}$$

$$7 + 20i = \sqrt{7^2 + 20^2}\, e^{i\tan^{-1}(20/7)} = \sqrt{449}\, e^{i\tan^{-1}(20/7)}$$

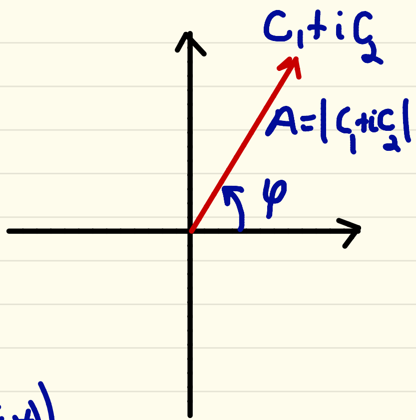
So $\quad y(t) = \sqrt{449}\, e^{-2t} \cos\left(t - \tan^{-1}(20/7)\right)$



In general:

$$L[y] = ay'' + by' + cy = 0$$

$p \pm \omega i$: roots of char. poly.



General solution:

$$\begin{cases} y(t) = \mathrm{Re}\left\{(c_1 - ic_2)\, e^{(p+i\omega)t}\right\} \\ \quad = e^{pt}\left(c_1 \cos(\omega t) + c_2 \sin(\omega t)\right) \\ \quad = A e^{pt} \cos(\omega t - \varphi), \quad A = \sqrt{c_1^2 + c_2^2}, \quad \tan(\varphi) = \dfrac{c_2}{c_1} \end{cases}$$

Soln. to initial value problem: $y(0) = y_0$, $y'(0) = y_0'$

$$y(0) = \mathrm{Re}\,(c_1 - ic_2) = c_1 = y_0$$

$$y'(0) = \mathrm{Re}\left\{(c_1 - ic_2)(p + i\omega)\right\} = c_1 p + c_2 \omega = y_0'$$

$$\Rightarrow \quad c_2 = \frac{y_0' - y_0 p}{\omega}$$

Note If $\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$ write soln in the form

$$y(t) = \mathrm{Re}\left\{(c_1 - ic_2)\, e^{(p+i\omega)(t-t_0)}\right\}$$

$$= e^{p(t-t_0)}\left\{c_1 \cos(\omega(t-t_0)) + c_2 \sin(t-t_0)\right\}$$

$$c_1 = y_0 \quad c_2 = \frac{y_0' - p y_0}{\omega}.$$

Units.

t: time (say in seconds)

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

(assume $a>0$, $b\geq 0$, $c>0$)

Write ODE in form

$$\frac{d^2y}{dt^2} + \left(\frac{b}{a}\right)\frac{dy}{dt} + \left(\frac{c}{a}\right)y = 0$$

Units of $\left(\frac{b}{a}\right)$ are $1/\text{sec}$

Units of $\left(\frac{c}{a}\right)$ are $1/\text{sec}^2$

Let $\omega_0 = \sqrt{\frac{c}{a}} > 0$ (units are $1/\text{sec}$)

Express (b/a) in form

$$(b/a) = 2\zeta\omega_0$$

zeta $\qquad \zeta$ is dimensionless!

Then ODE becomes

$$\frac{d^2y}{dt^2} + 2\zeta\omega_0\frac{dy}{dt} + \omega_0^2 y = 0$$

We can do even better!

Let $\tau = \omega_0 t$. By the chain rule:

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{d\tau}{dt}\dfrac{dy}{d\tau} = \omega_0\dfrac{dy}{d\tau} \\[2ex] \dfrac{d^2y}{dt^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dt}\right) = \omega_0^2\dfrac{d^2y}{d\tau^2} \end{cases}$$

So $\cancel{\omega_0^2}\dfrac{d^2y}{d\tau^2} + 2\zeta\cancel{\omega_0^2}\dfrac{dy}{d\tau} + \cancel{\omega_0^2}y = 0$

$$\Rightarrow \quad \frac{d^2y}{d\tau^2} + 2\zeta\frac{dy}{d\tau} + y = 0$$

Suppose $Y(\tau)$ is a solution of

$$\frac{d^2Y}{d\tau^2} + 2\zeta\frac{dY}{d\tau} + Y = 0$$

Let $y(t) = Y(\omega_0 t)$

Then

$$y'' + 2\zeta\omega_0 y' + \omega_0^2 y$$
$$= \omega_0^2 Y'' + 2\zeta\omega_0^2 Y' + \omega_0^2 Y$$
$$= \omega_0^2\left(Y'' + 2\zeta Y' + Y\right) = 0$$

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$$\frac{d^2y}{dt^2} + 2\zeta\omega_0\frac{dy}{dt} + \omega_0^2 y = 0$$

Char poly: $r^2 + 2\zeta\omega_0 r + \omega^2$
$= (r+\zeta\omega_0)^2 - (\zeta^2-1)\omega_0^2$

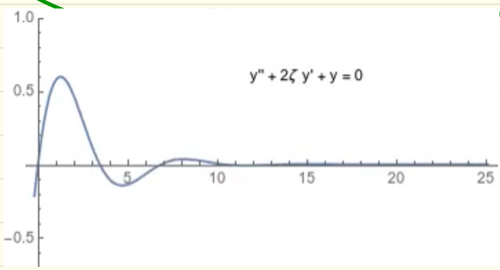
roots: $r = -\zeta\omega_0 \pm \sqrt{\zeta^2-1}\,\omega_0$

$$\begin{cases} \zeta > 1 & \text{2 real} \\ \zeta = 1 & \text{double root} \\ \zeta < 1 & \text{complex roots} \end{cases}$$

Suppose $y(0)=0 \quad y'(0)=1$

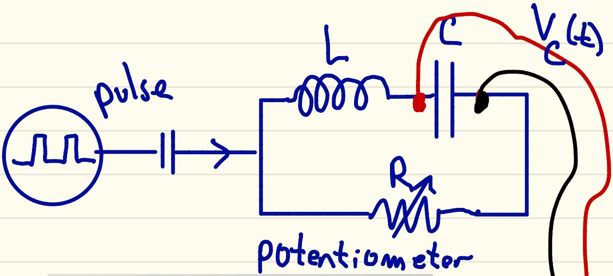
$$\begin{cases} \zeta>0: & y(t) = \dfrac{e^{r_1 t} - e^{r_2 t}}{r_1 - r_2} \\ \zeta = 1 & y(t) = t e^{-r_0 t} \\ \zeta < 1 & y(t) = A e^{-pt}\sin(\omega t) \end{cases}$$

Show animation!



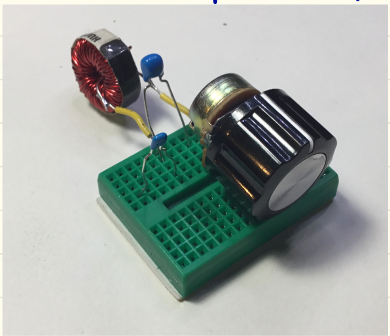
Example

$$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{1}{LC}V_c = 0$$



$$2\zeta\omega = \frac{R}{L} \qquad \omega^2 = \frac{1}{LC}$$

$1/\text{sec}$ $\qquad$ $1/\text{sec}^2$

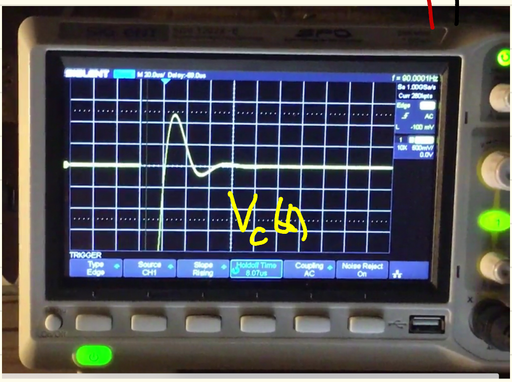


$$\zeta = \frac{R}{2L}\cdot\sqrt{LC} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$\omega = 1/\sqrt{LC}$$

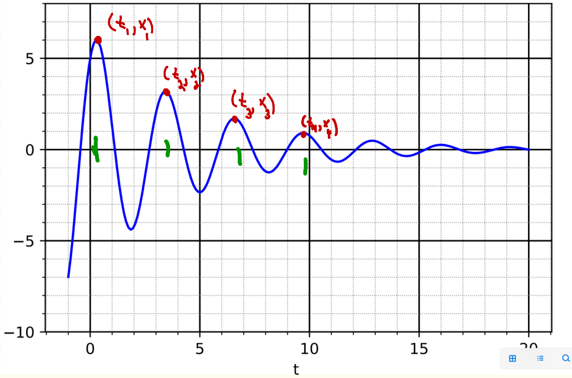
$$\frac{C}{L} \approx \frac{1}{3}$$

Show video



Finding p and ω from the graph of the solution

$$x(t) = A e^{-pt}\cos(\omega t - \varphi)$$



$T = \frac{2\pi}{\omega}$

$\omega = \frac{2\pi}{T}$

Quasi-period: $T = t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \cdots$ can estimate from the graph.

$t_1 \approx 0.25 \quad t_4 \approx 9.75 \quad 3T \approx t_4 - t_1 = 9.50$

So $T \approx \frac{9.50}{3} \approx 3.17$ sec.

Quasi-frequency: $\omega = \frac{2\pi}{T} \approx 1.98$

Note: $\underline{\cos(\omega t_2 - \varphi)} = \cos(\underline{\omega(t_1 + T)} - \varphi)$

$= \cos\left(\omega t_1 + \frac{2\pi}{T}\cdot T - \varphi\right) = \cos(\omega t_1 - \varphi)$

So

$$\frac{x_1}{x_2} = \frac{x(t_1)}{x(t_2)} = \frac{e^{-pt_1}}{e^{-pt_2}} = e^{-p(t_1 - t_2)} = e^{p(t_2 - t_1)} = e^{pT}$$

Take ln: $pT = \ln\left(\frac{x_1}{x_2}\right) \Rightarrow p = \frac{1}{T}\ln\left(\frac{x_1}{x_2}\right)$

$x_1 \approx 6 \quad x_2 \approx 3 \quad$ So $p \approx \frac{1}{3.17}\ln(2) \approx 0.22$

So $x(t) \approx A e^{-0.22t}\cos(2t - \varphi)$

(In fact $x(t) = Ae^{-t/5}\cos(2t - \varphi)$)

$$x'' + bx' + cx = 0$$

$$r^2 + br + c = \left(r + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2$$

$$r = -\frac{b}{2} \pm \sqrt{c - \frac{b^2}{4}}\, i = -p \pm \omega i$$

So $b = 2p \approx 2(0.22)$

$$c - \left(\frac{b}{2}\right)^2 \approx \omega^2 \Rightarrow c = \left(\frac{b}{2}\right)^2 + \omega^2 \approx (0.22)^2 + 4$$

Actually $\left(\frac{1}{5}\right)^2 + 4 = 4.05$

$$\boxed{\text{So } x(t) \text{ is a soln of the ODE} \quad x'' + (0.44)x' + (4.05)x = 0}$$