Lecture 12

Solns to $L[y]=0$

$L[y] = ay'' + by' + cy$

Review

Notation:

$$L[y] = ay'' + by' + cy$$

Goals
- Find general soln. to $L[y]=0$
- Solve the initial value problem $L[y]=0$, $\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$

Example: Solve the I.V.P.

$$L[y] = y'' + 2y' - 3y = 0 \quad \begin{cases} y(1) = 0 \\ y'(1) = 1 \end{cases}$$

Soln. (1) Find solns of the form e^{rt}:

$$L[e^{rt}] = (r^2+2r-3)e^{rt} = 0 \quad \begin{cases} \Leftrightarrow \ r^2+2r-3=0 \\ \Leftrightarrow \ (r+3)(r-1)=0 \\ \Leftrightarrow \ r=-3 \text{ or } r=1 \end{cases}$$

(2) General soln: $\underline{y(t) = c_1 e^{-3t} + c_2 e^t}$

(3) Solve IVP.

Better to write

$$\boxed{y(t) = C_1 e^{-3(t-1)} + C_2 e^{(t-1)}}$$
$$= \underline{(C_1 e^3)} e^{-3t} + \underline{(C_2 e^{-1})} e^t$$

So $c_1 = C_1 e^3 \quad c_2 = C_2 e^{-1}$

$$e^{-3(t-1)} = e^{-3t+3}$$
$$e^{-3t}e^3$$

$y(1) = 0 \quad y'(1) = 1$

Then $\begin{cases} y(1) = C_1 + C_2 = 0 \\ y'(1) = -3C_1 + C_2 = 1 \end{cases}$

So $\begin{cases} C_2 = -C_1 \\ -4C_1 = 1 \end{cases} \Rightarrow \begin{matrix} C_1 = -1/4 \\ C_2 = 1/4 \end{matrix}$

$$\boxed{y(t) = -\frac{1}{4} e^{-3(t-1)} + \frac{1}{4} e^{(t-1)}}$$

Not as nice: $y(t) = \left(\frac{-e^3}{4}\right)e^{-3t} + \left(\frac{e^{-1}}{4}\right)e^t$!

Example. $y'' - y = 0$

e^t, e^{-t} two "independent" solutions

$\sinh(t) = \frac{e^t - e^{-t}}{2}$ $\cosh(t) = \frac{e^t + e^{-t}}{2}$ another pair of solutions

Note

$\cosh'(t) = \sinh(t)$ $\cosh(0) = 1$ $\sinh(0) = 0$

$\sinh'(t) = \cosh(t)$ $\cosh'(0) = 0$ $\sinh'(0) = 1$

What is the solution to the IVP. $y'' - y = 0, \begin{cases} y(0) = y_0 \\ y'(0) = y_0' \end{cases}$?

$y(t) = c_1 \cosh(t) + c_2 \sinh(t)$

$y(0) = c_1 = y_0$ $y(t) = y_0 \cosh(t) + y_0' \sinh(t)$

$y'(0) = c_2 = y_0'$

Independent Solutions

Cramer's Rule

Determinants: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{cases} c_1 + c_2 = 1 \\ -c_1 - 2c_2 = 2 \end{cases}$$

$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix}} = \frac{-4}{-1} = 4 \qquad c_2 = \frac{\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix}} = \frac{3}{-1} = -3$$

In general,

$$\begin{cases} a\,c_1 + b\,c_2 = y_0 \\ c\,c_1 + d\,c_2 = y_0' \end{cases} \qquad c_1 = \frac{\begin{vmatrix} y_0 & b \\ y_0' & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad c_2 = \frac{\begin{vmatrix} a & y_0 \\ c & y_0' \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Note: Must have $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

Def^n Suppose $y_1(t)$ and $y_2(t)$ are two solutions of $ay'' + by' + cy = 0$. They are called independent solutions if they are not multiples of one another. The pair $y_1(t), y_2(t)$ is called a fundamental basis of solutions

$y_1(t)$ + $y_2(t)$ independent $\Longleftrightarrow \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0$

So can solve the I.C.

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ c_1 y_1'(t_0) + c_2 y_2(t_0) = y_0' \end{cases}$$

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

$$c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

The Characteristic Polynomial

$\cdot\, L[y] = a y'' + b y' + c y = 0$

$$L[e^{rt}] = (a r^2 + b r + c) e^{rt} = 0$$

$$\Longleftrightarrow \quad a r^2 + b r + c$$

The polynomial $ar^2 + br + c$ is called the characteristic polynomial of the differential equation.

$*$ $\boxed{\text{So we can find solns of the ODE by finding roots of the char. poly.}}$

Roots are

$$r_1, r_2 = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

Cases:
$$\begin{cases} b^2 - 4ac > 0 & \text{get 2 real roots} \quad (i) \quad r_1, r_2 \quad r_1 \neq r_2 \\ b^2 - 4ac = 0 & \text{get only one real root} \quad (ii) \quad r_0 = -b/2a \\ b^2 - 4ac < 0 & \text{Complex roots!} \quad (iii) \quad \frac{-b}{2a} \pm \sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}\, i \end{cases}$$

(i) General solution: $e^{r_1 t}, e^{r_2 t}$ is fund. basis.

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Solve the IVP $y'' - y = 0, \begin{cases} y(2) = 6 \\ y'(2) = 7 \end{cases}$

Soln: Char. poly.: $r^2 - 1$ roots ± 1.

Fundamental basis: e^t, e^{-t}. But $e^{(t-2)}$, $e^{-(t-2)}$ is easier to use.

Gen soln: $y(t) = c_1 e^{t-2} + c_2 e^{-(t-2)}$

Soln. Another fundamental basis:

$$\cosh(t-2), \sinh(t-2)$$

So gen soln can be written as

$$y(t) = c_1 \cosh(t-2) + c_2 \sinh(t-2)$$

Then

$y(2) = c_1$, $y'(2) = c_2$ so

Soln of IVP is $y(t) = 6\cosh(t-2) + 7\sinh(t-2)$

Another Example. $y'' + 6y' + 5y = 0.$ $\begin{cases} y(0) = 1, \\ y'(0) = -9 \end{cases}$

Char poly: $r^2 + 6r + 5 = r^2 + 6r + 9 - 4$
$= (r+3)^2 - 2^2$

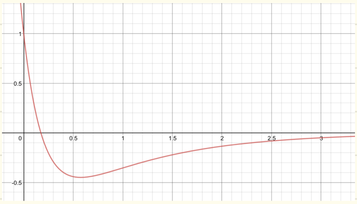
roots: $r = -3 \pm 2$

Independent basis: e^{-t}, e^{-5t}

General Solution: $y(t) = C_1 e^{-t} + C_2 e^{-5t}$

$$\begin{cases} y(0) = C_1 + C_2 = 1 \\ y'(0) = -C_1 - 5C_2 = -9 \end{cases} \Rightarrow \quad -4C_2 = -8 \Rightarrow C_2 = 2$$
$$\Rightarrow C_1 + 2 = 1 \Rightarrow C_1 = -1$$

So $y(t) = -e^{-t} + 2e^{-5t}$



Another independent basis: $e^{-3t}\cosh(2t) \quad e^{-3t}\sinh(2t)$

General Solution:

$$y(t) = C_1 e^{-3t}\cosh(2t) + C_2 e^{-3t}\sinh(2t)$$
$$= e^{-3t}\{C_1\cosh(2t) + C_2\sinh(2t)\}$$

Note: $y(0) = C_1, \quad y'(0) = -3C_1 + 2C_2$

So $\begin{cases} C_1 = 1 \\ -3C_1 + 2C_2 = -9 \end{cases} \Rightarrow \quad -3 + 2C_2 = -9$
$\Rightarrow 2C_2 = -12 \Rightarrow C_2 = -6$

$$y(t) = e^{-3t}\{\cosh(2t) - 6\sinh(2t)\}$$

Case (ii) : double root.

Example. $L[y] = y'' + 6y' + 9y = 0$

$$r^2 + 6r + 9 = (r+3)^2 \quad r_0 = -3$$

$$y(t) = (c_1 + c_2 t)e^{-3t}$$

Check: $(te^{-3t})'' + 6(te^{-3t})' + 9(te^{-3t})$

Case (ii) in general:

Suppose $ar^2+br+c = a(r-r_0)^2$.

Then $y(t) = (c_1+c_2t)e^{r_0t}$

Verify:

$$ar^2+br+c = a(r-r_0)^2 = a(r^2-2r_0r+r_0^2)$$

So $L[y] = ay''+by'+cy$

$$= a(y''-2r_0y'+r_0^2y)$$

Compute derivatives:

$$\begin{cases} (te^{r_0t})' = (1+tr_0)e^{r_0t} \\ (te^{r_0t})'' = ((1+tr_0)e^{r_0t})' \\ \qquad = r_0e^{r_0t}+(r_0+tr_0^2)e^{r_0t} \\ \qquad = (2r_0+tr_0^2)e^{r_0t} \end{cases}$$

Therefore,

$$L[te^{r_0t}] = a\left\{(2r_0+tr_0^2)e^{r_0t}-2r_0(1+tr_0)e^{r_0t}+r_0^2te^{r_0t}\right\}$$

$$\stackrel{\cdot}{=} 0$$

Case (iii): $r_1, r_2 = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$

$$= -p \pm \omega i$$

where $p = \frac{b}{2a}$, $\omega = \sqrt{\frac{c}{a} - \left(\frac{b}{2a}\right)^2}$

$$y(t) = c_1 e^{(-p+i\omega)t} + c_2 e^{(-p-i\omega)t}$$

complex numbers!

Another fundamental basis:

$$\frac{e^{-(p+i\omega)t} + e^{-(p-i\omega)t}}{2} = e^{-pt}\cos(\omega t), \qquad \frac{e^{-(p+i\omega)t} - e^{-(p-i\omega)t}}{2i} = e^{-pt}\sin(\omega t)$$

So general soln. is

$$y(t) = c_1 e^{pt}\cos(\omega t) + c_2 e^{pt}\sin(\omega t)$$

$$= \mathrm{Re}\left\{ (c_1 - i c_2)\, e^{(p+i\omega)t} \right\}$$