Lecture 12
Sols to \( L[y] = 0 \)
\( L[y] = ay'' + by' + cy \)
Review

Notation:

\[ L[y] = ay'' + by' + cy \]

Goals:
- Find general soln. to \( L[y] = 0 \)
- Solve the initial value problem \( L[y] = 0, \ \{ y(0) = 0, \ y'(0) = 0 \} \)

Example: Solve the I.V.P.

\[ L[y] = y'' + 2y' - 3y = 0, \ \{ y(0) = 0, \ y'(0) = 1 \} \]

Soln. (1) Find solns. of the form \( e^{rt} \):

\[ L[e^{rt}] = (r^2 + 2r - 3) e^{rt} = 0 \]

\[ \iff \ 
\begin{cases} 
  \iff & r^2 + 2r - 3 = 0 \\
  \iff & (r + 3)(r - 1) = 0 \\
  \iff & r = -3 \text{ or } r = 1 
\end{cases} \]

(2) General soln:

\[ y(t) = c_1 e^{-3t} + c_2 e^t \]

(3) Solve I.V.P.

Better to write:

\[ y(t) = C_1 e^{-3(t-1)} + C_2 e^{(t-1)} \]

\[ = (C_1 e^3) e^{-3t} + (C_2 e^{-1}) e^t \]

So \( c_1 = C_1 e^3 \), \( c_2 = C_2 e^{-1} \)

\[ e^{-3(t-1)} = e^{-3t + 3} \]

\[ e^{(t-1)} = e^{-3t - 2} e^3 \]

Then:

\[ \begin{cases} 
  y(0) = C_1 + C_2 = 0 \\
  y'(0) = -3C_1 + C_2 = 1 
\end{cases} \]

So:

\[ C_2 = -C_1 \]

\[ C_1 = -\frac{1}{4} \]

\[ C_2 = \frac{1}{4} \]

\[ y(t) = -\frac{1}{4} e^{-3t} + \frac{1}{4} e^{t-1} \]

Not as nice:

\[ y(t) = \left( -\frac{e^3}{4} \right) e^{-3t} + \left( \frac{e^{-1}}{4} \right) e^t \]
Example: $y'' - y = 0$

$e^t, e^{-t}$ two "independent" solutions

\[ \sinh(t) = \frac{e^t - e^{-t}}{2}, \quad \cosh(t) = \frac{e^t + e^{-t}}{2}. \]

Another pair of solutions

\[ \cosh'(t) = \sinh(t), \quad \cosh(t) = 1, \quad \sinh(t) = 0, \quad \cosh'(0) = 0, \quad \sinh'(0) = 1. \]

Note

\[ \begin{align*}
\cosh(t) &= \sinh(t), \\
\sinh(t) &= \cosh(t).
\end{align*} \]

What is the solution to the IVP: $y'' - y = 0$, \( \begin{cases} y(0) = y_0 \\ y'(0) = y_0' \end{cases} \)?

\[ y(t) = c_1 \cos(t) + c_2 \sinh(t). \]

\[ \begin{align*}
y(0) &= c_1 = y_0, \\
y'(0) &= c_2 = y_0'.
\end{align*} \]
Independent Solutions

**Cramer's Rule**

**Determinants:**

\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc
\]

\[
\begin{cases}
\begin{align*}
c_1 + c_2 &= 1 \\
-c_2 - 2c &= 2
\end{align*}
\end{cases}
\]

\[
\begin{align*}
C_1 &= \frac{1}{1} = -4 = 4 \\
C_2 &= \frac{1}{1} = \frac{3}{1} = 3
\end{align*}
\]

In general,

\[
\begin{cases}
\begin{align*}
a_1 c_1 + b_1 c_2 &= y_0 \\
a_1 c_1 + a_2 c_2 &= y_0
\end{align*}
\end{cases}
\]

\[
\begin{align*}
C_1 &= \frac{1}{a_1 b_1} \\
C_2 &= \frac{1}{a_1 a_2}
\end{align*}
\]

**Def.** Suppose \( y_1(t) \) and \( y_2(t) \) are two solutions of \( ay'' + by' + cy = 0 \). They are called independent solutions if they are not multiples of each other. The pair \( y_1(t), y_2(t) \) is called a fundamental basis of solutions.

So can solve the I.C.

\[
\begin{cases}
\begin{align*}
C_1 y_1(t_0) + C_2 y_2(t_0) &= y_0 \\
C_1 y_1'(t_0) + C_2 y_2'(t_0) &= y_0'
\end{align*}
\end{cases}
\]

\[
\begin{align*}
C_1 &= \begin{vmatrix} y_0 & y_2(t_0) \\ y_1(t_0) & y_2'(t_0) \end{vmatrix} \\
C_2 &= \begin{vmatrix} y_0 & y_1(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}
\end{align*}
\]
The Characteristic Polynomial

\[ L[y] = ay'' + by' + cy = 0 \]
\[ L[e^{rt}] = (ar^2 + br + c)e^{rt} = 0 \]
\[ \iff ar^2 + br + c = 0 \]

The polynomial \( ar^2 + br + c \) is called the characteristic polynomial of the differential equation.

So we can find solutions of the ODE by finding roots of the char. poly.

Roots are

\[ r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Cases:

\[ \begin{cases} 
  b^2 - 4ac > 0 & \text{get 2 real roots} & (i) \quad r_1, r_2 \neq \frac{b}{2a} \\
  b^2 - 4ac = 0 & \text{get 1 non real root} & (ii) \quad r = \frac{-b}{2a} \\
  b^2 - 4ac < 0 & \text{complex roots} & (iii) \quad \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} 
\end{cases} \]
(i) General solution: $e^{rt}, e^{st}$ is fund. basis.

$y(t) = C_1 e^{rt} + C_2 e^{st}$
Solve the IVP \[ y'' - y = 0, \quad \begin{align*} y(0) &= 6, \\ y'(0) &= 7 \end{align*} \]

**Soln.** Char. poly: \( r^2 - 1 \) roots \( \pm 1 \).

Fundamental basis: \( e^t, e^{-t} \). But \( e^{t-2}, e^{-(t-2)} \) is easier to use.

Com. soln: \[ y(t) = c_1 e^{t-2} + c_2 e^{-(t-2)} \]

**Soln.** Another fundamental basis:

\[ \cosh(t-2), \sinh(t-2) \]

So gen soln can be written as

\[ y(t) = c_1 \cosh(t-2) + c_2 \sinh(t-2) \]

Then

\[ y(0) = c_1, \quad y'(0) = c_2 \] so

Soln of IVP is \[ y(t) = 6 \cosh(t-2) + 7 \sinh(t-2) \]
Another Example. \( y'' + 6y' + 5y = 0 \). \( \begin{cases} y(0) = 1, \\ y'(0) = -9 \end{cases} \)

Char poly: \( r^3 + 6r + 5 = r^3 + 6r + 9 - 4 = (r+3)^2 - 2^2 \)

roots: \( r = -3 \pm 2 \)

Independent basis: \( e^{-t}, e^{-5t} \)

General solution: \( y(t) = C_1 e^{-t} + C_2 e^{-5t} \)

\[ \begin{align*}
    y(0) &= C_1 + C_2 = 1 \quad \Rightarrow -4C = -6 \Rightarrow C = 1 \\
    y'(0) &= -C - 5C_2 = -9 \quad \Rightarrow -C + 2C_2 = -9 \Rightarrow C_2 = -1
\end{align*} \]

So \( y(t) = -e^{-t} + 2e^{-5t} \)

Another independent basis: \( e^{3t} \\cosh(2t), e^{3t} \\sinh(2t) \)

General solution:

\[ y(t) = C_1 e^{3t} \\cosh(2t) + C_2 e^{3t} \\sinh(2t) = e^{3t} \{ C_1 \cosh(2t) + C_2 \sinh(2t) \} \]

Note: \( y(0) = C_1, \quad y'(0) = 3C_1 + 2C_2 \)

So \( \begin{cases} C_1 = 1 \\
    -3C + 2C_2 = -9 \Rightarrow 2C_2 = -6 \Rightarrow C_2 = -6
\end{cases} \)

\( y(t) = e^{3t} \{ \cosh(2t) - 6 \sinh(2t) \} \)
Case (ii): double root.

Example. \( L[y] = y'' + 6y' + 9y = 0 \)
\[ r^2 + 6r + 9 = (r + 3)^2 \quad r = -3 \]
\[ y(t) = (C_1 + C_2 t) e^{-3t} \]

Check: \( (t e^{-3t})'' + 6(t e^{-3t})' + 9(t e^{-3t}) \)
Case (ii) in general:

Suppose \( ar^2 + br + c = q (r - r_0)^2 \).

Then \( y(t) = (c_1 + c_2 t) e^{r_0 t} \).

Verify:

\[
ar^2 + br + c = a (r - r_0)^2 = a (r^2 - 2r_0 r + r_0^2)
\]

So \( L[y] = ay'' + by' + cy \)

\[= a (y'' - 2r_0 y' + r_0^2 y) \]

Compute derivatives:

\[
\begin{align*}
(t e^{r_0 t})' &= (1 + t r_0) e^{r_0 t} \\
(t e^{r_0 t})'' &= ((1 + t r_0) e^{r_0 t})' \\
&= r_0 e^{r_0 t} + (r_0 + t r_0^2) e^{r_0 t} \\
&= (2r_0 + t r_0^2) e^{r_0 t}
\end{align*}
\]

Therefore,

\[
L[te^{r_0 t}] = q \left\{ (2r_0 + t r_0^2) e^{r_0 t} - 2r_0 (1 + t r_0) e^{r_0 t} + r_0^2 t e^{r_0 t} \right\} = 0
\]
Case (iii): \( r_1, r_2 = \frac{-b \pm \sqrt{(\frac{b}{2a})^2 - \frac{c}{a}}}{2a} \)

\[ = -\rho \pm \omega i \]

where \( \rho = \frac{b}{2a}, \ \omega = \sqrt{\frac{c}{a} - (\frac{b}{2a})^2} \)

\[ y(t) = c_1 e^{(-\rho + \omega i)t} + c_2 e^{(-\rho - \omega i)t} \]

**Another fundamental basis:**

\[
\frac{e^{(\rho + \omega i)t} + e^{(\rho - \omega i)t}}{2}, \quad \frac{e^{(\rho + \omega i)t} - e^{(\rho - \omega i)t}}{2i}
\]

\[ e^{\rho t} \cos(\omega t), \quad e^{\rho t} \sin(\omega t) \]

So general solution is

\[ y(t) = c_1 e^{\rho t} \cos(\omega t) + c_2 e^{\rho t} \sin(\omega t) \]

\[ = \text{Re} \left\{ (c_1 - i c_2) e^{(\rho + i\omega) t} \right\} \]