Lecture 11

Intro to 2nd order ODE's

2^{nd} Order ODEs.

$$\frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right)$$

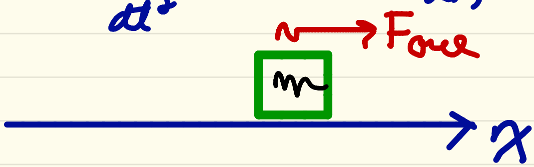
Initial conditions: $\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$

Most important example:

Newton's 2^{nd} Law of Motion:

mass × acceleration = Force

$$m \frac{d^2x}{dt^2} = F(t, x, dx/dt)$$



Note: position and velocity at time t_0 uniquely determines $x(t)$.

Special Cases:

i) $\frac{d^2y}{dt^2} = F\left(t, \frac{dy}{dt}\right)$

Let $u = \frac{dy}{dt}$:

$$\begin{cases} \frac{du}{dt} = F(t, u) & 1^{st} \text{ Order} \\ u(t_0) = u_0 \end{cases}$$

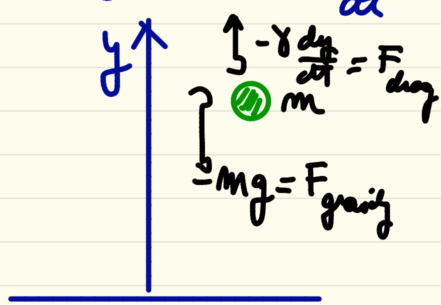
$$\Rightarrow u = U(t)$$

Then $\begin{cases} \frac{dy}{dt} = U(t) \\ y(t_0) = y_0 \end{cases}$

$$\Rightarrow y(t) = y_0 + \int_{t_0}^{t} U(s)\,ds$$

Example. Free fall with air resistance:

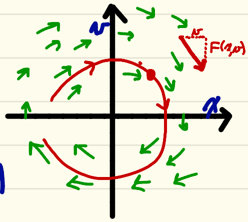
$$m\frac{d^2y}{dt^2} = -mg - \gamma\frac{dy}{dt}$$



(2) Autonomous 2^{nd} order ODEs

$$\frac{d^2x}{dt^2} = F\left(x, \frac{dx}{dt}\right) \Longleftrightarrow \begin{cases} \dfrac{dx}{dt} = v \\ \dfrac{dv}{dt} = F(x,v) \end{cases}$$

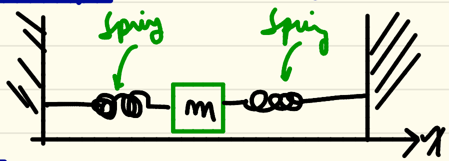
Phase plane



Solution $\begin{cases} x = x(t) \\ v = v(t) \end{cases}$

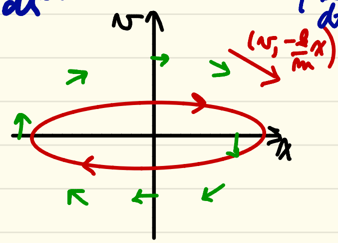
Key idea: View v as a function of x

Example (Harmonic Oscillator)



$$\text{Force} = -kx$$

$$m\frac{d^2x}{dt^2} = -kx \Longleftrightarrow \begin{cases} \dfrac{dx}{dt} = v \\ \dfrac{dv}{dt} = -kx \end{cases}$$



Conservation of Energy

$$m\frac{d^2x}{dt^2} = -kx$$

$$\Longleftrightarrow \quad m\frac{dv}{dt} = -kx$$

Chain Rule: $\dfrac{dv}{dt} = \dfrac{dx}{dt} \cdot \dfrac{dv}{dx}$

So

$$mv\frac{dv}{dx} = -kx$$

Integrate with respect to x:

$$\int mv\frac{dv}{dx}\,dx = -\int kx\,dx$$

$$\Rightarrow \quad \frac{1}{2}mv^2 = -\frac{1}{2}kx^2 + E$$

constant of integration

$$\Rightarrow \quad \boxed{\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E}$$

kinetic energy + potential energy = constant.

Check: $\dfrac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = mv\dfrac{dv}{dt} + kx\dfrac{dx}{dt} = (-kx)v + (kx)v = 0$

Can use conservation of energy to solve ODE

$$\begin{cases} \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E \quad (\text{constant}) \\ v = \frac{dx}{dt} \end{cases}$$

So $\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 = E$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{2E - kx^2}{m}$$

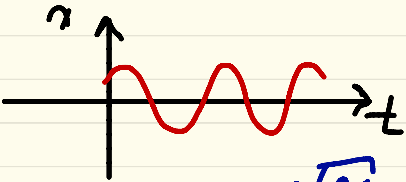
$$\left(\frac{dx}{dt}\right)^2 = k/m\left(\frac{2E}{k} - x^2\right)$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{k}{m}}\sqrt{\frac{2E}{k} - x^2}$$

$$\Rightarrow \int \frac{dx}{\sqrt{2E/k - x^2}} = \pm\sqrt{\frac{k}{m}}\,t + C$$

$$\Rightarrow \sin^{-1}\left(x/\sqrt{2E/k}\right) = \pm\sqrt{\frac{k}{m}}\,t + C$$

Or $\boxed{x(t) = \sqrt{\frac{2E}{k}}\sin\left(\sqrt{\frac{k}{m}}\,t - \beta\right)}$



$\varphi = \beta + \frac{\pi}{2}$

Or $x(t) = \sqrt{\frac{2E}{k}}\cos\left(\sqrt{\frac{k}{m}}\,t - \varphi\right)$

$\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$

Linear ODEs

$$\frac{dy}{dt} + p(t)\,y = \begin{cases} 0 & (\text{Homogeneous}) \\ f(t) & (\text{nonhomogeneous}) \end{cases}$$

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = \begin{cases} 0 & (\text{Homogeneous}) \\ f(t) & (\text{nonhomogeneous}) \end{cases}$$

Shorthand notation:

$$L[y] = y'' + p(t)y' + q(t)y$$

L is an example of a linear operator:

$$L[c_1 y_1(t) + c_2 y_2(t)] = c_1 L[y_1(t)] + c_2 L[y_2(t)]$$

Example If $L[y] = y'' + 4y$

Then $L[\sin(2t)]$

$$= (\sin(2t))'' + 4(\sin(2t))$$

$$= -4\sin(2t) + 4\sin(2t) = 0$$

So $y(t) = \sin(2t)$ is a solution of the ODE

$$L[y] = y'' + 4y = 0$$

Theorem Suppose $p(t)$, $q(t)$, and $f(t)$ are continuous on the interval $a<t<b$. If $a<t_0<b$, then the initial value problem

$$L[y] = y'' + p(t)y' + q(t)y = f(t)$$

$$y(t_0) = y_0 \quad y'(t_0) = y_0'$$

Has a unique solution $y(t)$ defined for all $a<t<b$.

Bad news: Theorem does not give a way to find the solution!

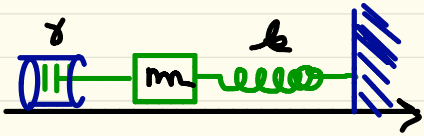
Good News: If $p(t)=b$ and $q(t)=c$ (i.e. constants), so

$$L[y] = y'' + by' + cy$$

Then there <u>are</u> methods for solving the I.V.P.

Main Example:

The (damped) harmonic oscillator:



$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0$$

Plan for the next 2 weeks:

(1) Study the homogeneous case:

$$L[y] = ay'' + by' + cy = 0$$

(2) Study the nonhomogeneous case

$$L[y] = f(t)$$

in special cases:

Example. Solve the I.V.P.

$$L[y] = y'' + 3y' + 2y = 0 \qquad y(0) = 1,\ y'(0) = 2$$

Solution:

$$L[e^{rt}] = (r^2 + 3r + 2)e^{rt} = 0$$

$$\Leftrightarrow \quad r^2 + 3r + 2 = 0$$

$$\Leftrightarrow \quad (r+2)(r+1) = 0$$

$$\Rightarrow \quad r = -2 \text{ or } r = -1$$

So $y(t) = c_1 e^{-t} + c_2 e^{-2t}$

is a solution for any c_1 & c_2

$$\begin{cases} y(0) = c_1 + c_2 = 1 \\ y'(0) = -c_1 - 2c_2 = 2 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = 4 \\ c_2 = -3 \end{cases}$$

$$\Rightarrow y(t) = 4e^{-t} - 3e^{-2t}$$