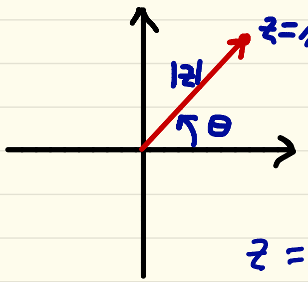
Lecture 10

(Complex Calculus)

Polar Form



$|z|=\sqrt{x^2+y^2}$ modulus of z

$\theta=\arg(z)$ argument of z

Note: $|z|=\sqrt{z\bar{z}}$

$$z=|z|\cos(\theta)+i\,|z|\sin(\theta)$$
$$=|z|\left(\cos(\theta)+i\sin(\theta)\right)=|z|e^{i\theta}$$

Note: $\begin{cases} z_1=|z_1|\left(\cos\theta_1+i\sin\theta_1\right) \\ z_2=|z_2|\left(\cos\theta_2+i\sin\theta_2\right)\end{cases} \Rightarrow$ $z_1\cdot z_2=$

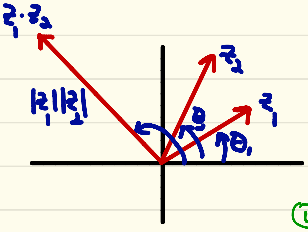
$|z_1||z_2|\cdot\{(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)+$
$i(\cos\theta_1\sin\theta_2+\sin\theta_1\cos\theta_2)\}$

$$\boxed{z_1\cdot z_2=|z_1||z_2|\left(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right)}$$

$$\boxed{\left(|z_1|e^{i\theta_1}\right)\cdot\left(|z_2|e^{i\theta_2}\right)=|z_1||z_2|e^{i(\theta_1+\theta_2)}}$$

$\bar{z}=|z|e^{-i\theta}$

$\frac{1}{z}=\frac{1}{|z|}e^{-i\theta}$

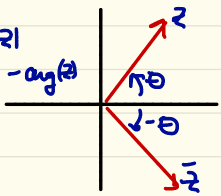


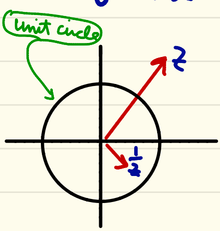
$|z_1z_2|=|z_1||z_2|$

$\arg(z_1z_2)=\arg(z_1)+\arg(z_2)$

$|\bar{z}|=|z|$

$\arg(\bar{z})=-\arg(z)$





$\left|\frac{1}{z}\right|=\frac{1}{|z|}$

$\arg\left(\frac{1}{z}\right)=-\arg(z)$

Examples. Suppose $\begin{cases} z_1 = 4e^{3i} \\ z_2 = 7e^{6i} \end{cases}$

Then $z_1 \cdot z_2 = 4 \cdot 7 e^{(3+6)i} = 28 e^{9i}$

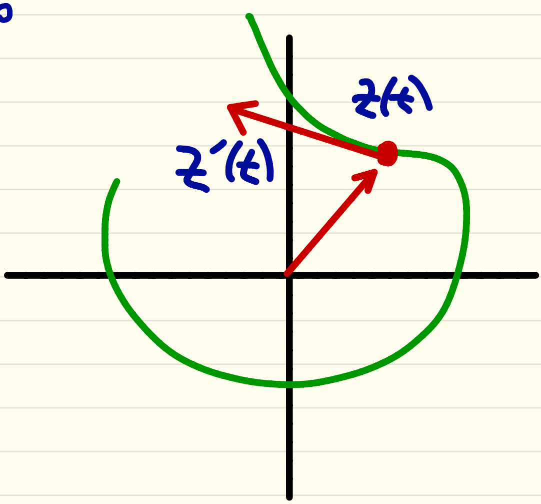
$\frac{1}{z_1} = \frac{1}{4} e^{-3i}$

$\frac{z_1}{z_2} = \frac{4}{7} e^{(3-6)i} = \frac{4}{7} e^{-3i}$

$z_1 \cdot \bar{z}_1 = (4e^{3i})(4e^{-3i}) = 16 e^{0i} = 16.$

$$\begin{cases} e^{\pi/2 i} = \\ e^{\pi i} = \\ e^{2\pi i} = \end{cases}$$

Complex-Valued Functions



$$z(t) = x(t) + i\,y(t)$$

$$z'(t) = x'(t) + i\,y'(t)$$

Properties of derivative

(same as for real-valued functions):

(i) $(C\,z(t))' = C\,z'(t)$

(ii) $(z(t)+w(t))' = z'(t) + w'(t)$

(iii) $(z(t)\,w(t))' = z'(t)\,w(t) + z(t)\,w'(t)$

(iv) $\left(\dfrac{z(t)}{w(t)}\right)' = \dfrac{z'(t)\,w(t) - z(t)\,w'(t)}{w(t)^2}$

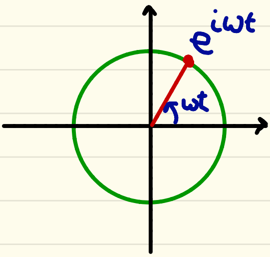
The exponential function

Recall: $(e^{rt})' = r\,e^{rt}$

for r real. What if r is complex?

Special case: $r = i\omega$

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$



$$(e^{i\omega t})' = i\omega\, e^{i\omega t}$$

General case: $r = \rho + i\omega$

$$e^{(\rho+i\omega)t} = e^{\rho t} e^{i\omega t}$$

$$= e^{\rho t}\cos(\omega t) + i\,e^{\rho t}\sin(\omega t)$$

$$(e^{(\rho+i\omega)t})' = (\rho e^{\rho t})e^{i\omega t} + e^{\rho t}(i\omega e^{i\omega t})$$

$$= (\rho + i\omega t)e^{(\rho+i\omega)t}$$

Computation. $\frac{d}{dt}\left\{e^{(3+2i)t}\right\} = ?$

$$\frac{d}{dt} e^{(p+i\omega)t} = ?$$

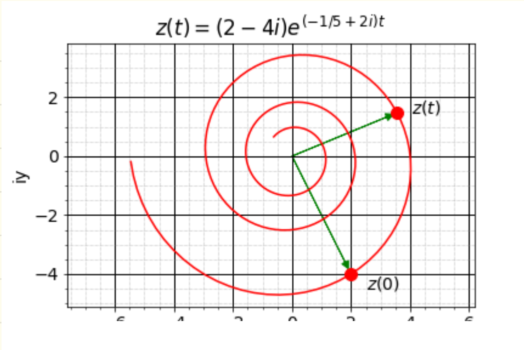
Example

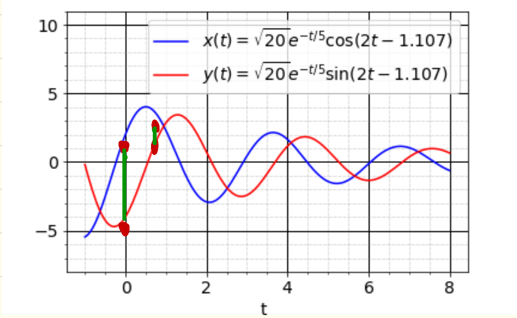
$$z(t) = (2-4i)\,e^{\left(-\frac{1}{5}+2i\right)t}$$

$$= x(t) + i\,y(t)$$

$$= e^{-\frac{t}{5}}\left\{2\cos(2t) + 4\sin(2t)\right\}$$
$$+ i\,e^{-\frac{t}{5}}\left\{2\sin(2t) - 4\cos(2t)\right\}$$

$$\tan^{-1}\left(\frac{-4}{2}\right) \simeq 1.107 \;(\simeq 63^\circ)$$





$$\frac{dx(t)}{dt} = \frac{d}{dt}\left\{e^{-t/5}\left(2\cos(2t) + 4\sin(2t)\right)\right\} = ?$$

$$\int x(t)\,dt = \int e^{-t/5}\left(2\cos(2t) + 4\sin(2t)\right)dt = ?$$

Example

$$z(t) = (2-4i)e^{\left(-\frac{1}{5}+2i\right)t}$$

$$= x(t) + iy(t)$$

$$= e^{-\frac{t}{5}}\{2\cos(2t) + 4\sin(2t)\}$$
$$+ i e^{-\frac{t}{5}}\{2\sin(2t) - 4\cos(2t)\}$$

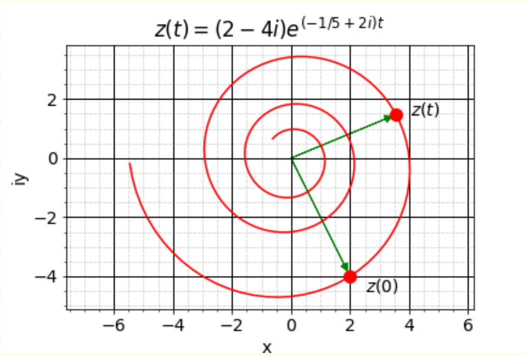
$$2-4i = \sqrt{20}\, e^{-i\varphi}$$
$$\approx 4.47\, e^{-i(1.11)}$$

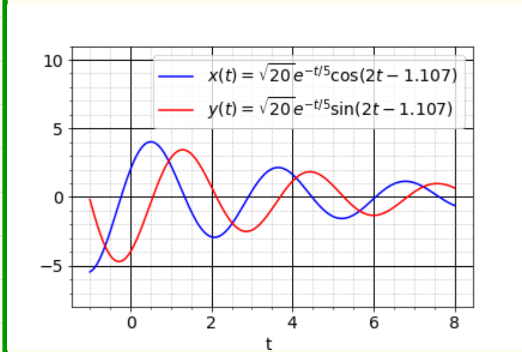
$$\varphi = \arctan(2) \approx 1.11$$

$$z(t) = \sqrt{20}\, e^{-\frac{t}{5}} e^{i(2t-\varphi)}$$

$$= x(t) + iy(t)$$

$$\sqrt{20}\, e^{-\frac{t}{5}} \cos(2t-\varphi) +$$
$$i\sqrt{20}\, e^{-\frac{t}{5}} \sin(2t-\varphi)$$





$$z'(t) = (2-4i)\cdot\left(-\frac{1}{5}+2i\right)e^{\left(-\frac{1}{5}+2i\right)t}$$
$$= (7.6 + 4.8i)\, e^{\left(-\frac{1}{5}+2i\right)t}$$
$$= x'(t) + iy'(t)$$

$$z''(t) = (2-4i)\left(-\frac{1}{5}+2i\right)^2 e^{\left(-\frac{1}{5}+2i\right)t}$$
$$\simeq (-11.12 + 14.24i)\, e^{\left(-\frac{1}{5}+2i\right)t}$$

Example. Find the general solution to

$$x' + 3x = 7\cos(2t) + 3\sin(2t).$$

Solution. $x(t) = Ce^{-3t} + x_p(t)$.

Find $x_p(t)$.