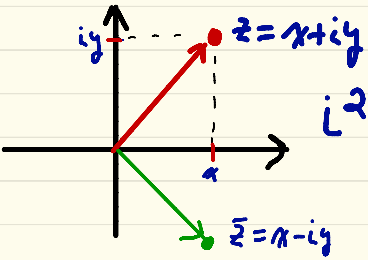
Lecture 09

(Complex Numbers)

Mon, Oct 14

Complex Numbers.



(complex conjugate)

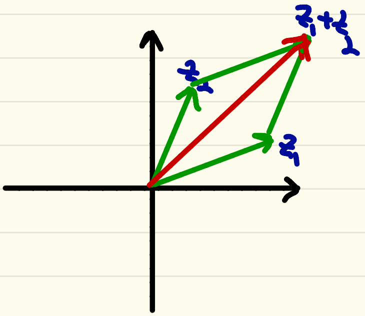
$x = \mathrm{Re}(z)$ real part

$y = \mathrm{Im}(z)$ imaginary part

$i^2 = -1$

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 \cdot z_2 = (x_1 + iy_1)\cdot(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$



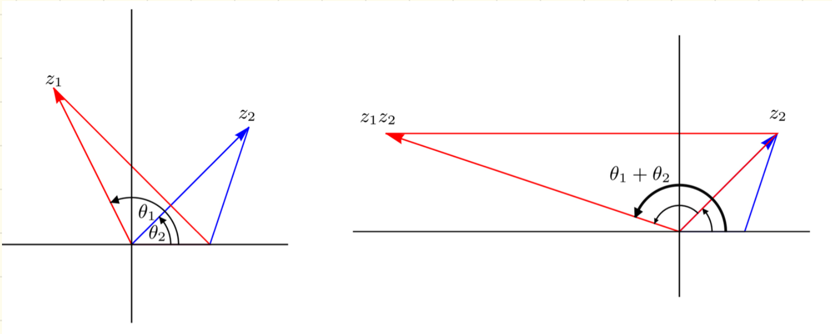
$$\frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Note $z \cdot \bar{z} = x^2 + y^2$, $\quad \frac{1}{z} = \frac{\bar{z}}{(z\bar{z})}$

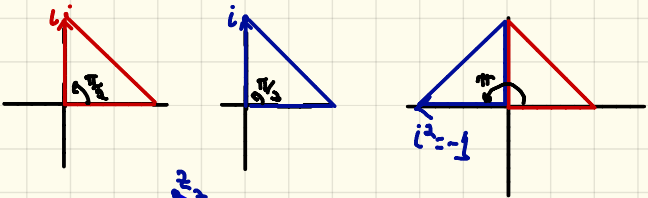
$$\frac{1}{i} = -i$$

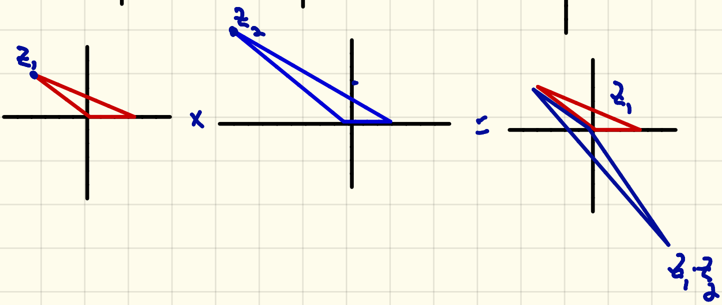
$$\begin{cases} \mathrm{Re}(z) = \dfrac{z + \bar{z}}{2} \\ \mathrm{Im}(z) = \dfrac{z - \bar{z}}{2i} \end{cases}$$

Complex Multiplication



Examples $i \cdot i = -1$





Examples (complex arithmetic)

$(3+2i)+(4-5i)=?$

$(3+2i)\cdot(4-5i)=?$

$\dfrac{1}{4-5i}=?$

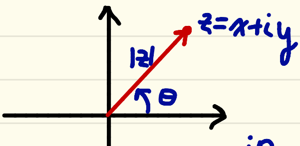
$\dfrac{3+2i}{4-5i}=?$

$\overline{\left(\dfrac{3+2i}{4-5i}\right)}=?$

$\dfrac{1}{i}=?$

$(3+2i)\,\overline{(3+2i)}=?$

Polar Form



$|z| = \sqrt{x^2+y^2}$ <u>modulus</u> of z

$\sqrt{z\bar{z}}$

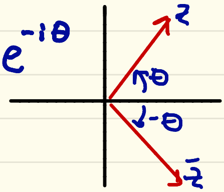
$\theta = \arg(z)$ <u>argument</u> of z

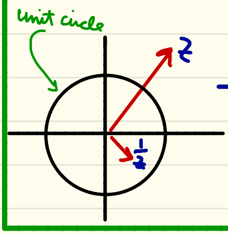
$\tan(\theta) = y/x$

$z = |z|e^{i\theta} = |z|\left(\cos(\theta) + i\sin(\theta)\right)$

Why this?

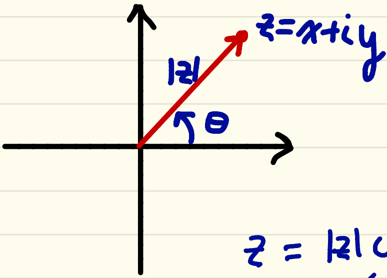
$\bar{z} = |z|e^{-i\theta}$





$\dfrac{1}{z} = \dfrac{1}{|z|}e^{-i\theta}$

Polar Form



$|z| = \sqrt{x^2+y^2}$ <u>modulus</u> of z

$\theta = \arg(z)$ <u>argument</u> of z

<u>Note</u>: $|z| = \sqrt{z\bar{z}}$

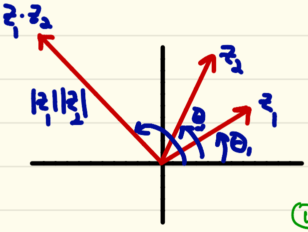
$$z = |z|\cos(\theta) + i\,|z|\sin(\theta)$$
$$= |z|\left(\cos(\theta) + i\sin(\theta)\right) = |z|e^{i\theta}$$
$$\bar{z} = |z|e^{-i\theta}$$

<u>Note</u>: $\begin{cases} z_1 = |z_1|\left(\cos\theta_1 + i\sin\theta_1\right) \\ z_2 = |z_2|\left(\cos\theta_2 + i\sin\theta_2\right) \end{cases} \Rightarrow$ $z_1 \cdot z_2 =$

$$|z_1||z_2| \cdot \left\{\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right) + i\left(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2\right)\right\}$$

$$\boxed{z_1 \cdot z_2 = |z_1||z_2|\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)}$$

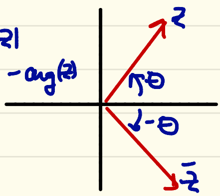
$$\boxed{\left(|z_1|e^{i\theta_1}\right)\cdot\left(|z_2|e^{i\theta_2}\right) = |z_1||z_2|e^{i(\theta_1+\theta_2)}}$$

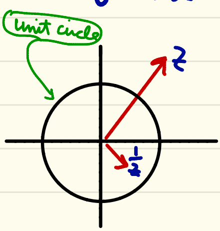


$$|z_1 z_2| = |z_1||z_2|$$
$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$|\bar{z}| = |z|$

$\arg(\bar{z}) = -\arg(z)$



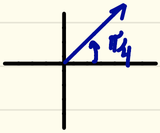


$$\left|\frac{1}{z}\right| = \frac{1}{|z|}$$

$$\arg\left(\frac{1}{z}\right) = -\arg(z)$$

Examples.

① $2+2i = \sqrt{2^2+2^2}\, e^{i\tan^{-1}(1)}$



$= 2\sqrt{2}\, e^{i\pi/4}$

$$\frac{1}{2+2i} = \frac{1}{2\sqrt{2}} e^{-i\pi/4}$$

$= ?$

$$(2+2i)^2 = (2\sqrt{2})^2 e^{i(\pi/4+\pi/4)}$$
$$= 8 e^{i\pi/2} = 8i$$

Check: $(2+2i)\cdot(2+2i) =$

$$= 2\cdot 2 + (2i)(2i) \quad + (2\cdot 2i + 2\cdot 2i)$$
$$= 4 - 4 \quad + 4i + 4i = 8i$$

② $5e^{i\pi/3} = 5\left(\cos(\pi/3) + i\sin(\pi/3)\right)$

$$= 5\left(\frac{1}{2}\right) + \frac{5\sqrt{3}}{2} i$$

$$\left(5e^{i\pi/3}\right)^3 = 5^3 e^{i\frac{\pi}{3}\cdot 3} = 125 e^{i\pi} = -125$$

$$\left(\frac{5}{2} + \frac{5\sqrt{3}}{2} i\right)^3 = \left(\frac{5}{2}\right)^3 (1+\sqrt{3} i)^3$$

$$= \frac{125}{8}(1+\sqrt{3}i)^3$$

<u>Note:</u>

$$\begin{cases} \cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin(\theta) = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} \end{cases}$$

Why?

Two trig Identities

$$\begin{cases} \cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right) \\ \sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \end{cases}$$

Proof. Let $\alpha = A+B$ $\quad \beta = A-B$

Then $A = \frac{\alpha+\beta}{2}$ $\quad B = \frac{\alpha-\beta}{2}$

So
$$e^{iA} + e^{iB} = e^{i\frac{\alpha+\beta}{2}} + e^{i\frac{\alpha-\beta}{2}} = e^{i\frac{\alpha}{2}}\left(e^{i\beta/2} + e^{-i\beta/2}\right)$$

What does this say?

$$e^{iA} + e^{iB} = \left(\cos(A)+\cos(B)\right) + i\left(\sin(A)+\sin(B)\right)$$

$$e^{i\beta/2} + e^{-i\beta/2} = \left(\cos(\beta/2) + i\sin(\beta/2)\right) + \left(\cos(-\beta/2) + i\sin(-\beta/2)\right)$$
$$= 2\cos(\beta/2)$$

So
$$e^{i\alpha/2}\left(e^{i\beta/2} + e^{-i\beta/2}\right) = \left(\cos(\alpha/2) + i\sin(\alpha/2)\right)2\cos(\beta/2)$$
$$= 2\cos(\alpha/2)\cos(\beta/2) + 2\sin(\alpha/2)\cos(\beta/2)$$

Compare:
$$\begin{cases} \cos(A) + \cos(B) = 2\cos(\alpha/2)\cos(\beta/2) \\ \sin(A) + \sin(B) = 2\sin(\alpha/2)\cos(\beta/2) \end{cases}$$

So:
$$\begin{cases} \cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\ \sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \end{cases}$$