Lecture 08

Modeling (continued)

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Solution. (a) Label quantities.

Let t = time in years since start of loan.

$P(t)$ = amount owed at time t.

P_0 = \$250,000

T = 30 years (duration of loan)

k = repayment rate (in \$/year)

(b) Construct a model:

$$\begin{cases} \dfrac{dP}{dt} = rP - k \\ P(0) = P_0 \end{cases} \quad \text{I.V.P.}$$

(c) Solve the I.V.P. to find $P(t)$

$$\begin{cases} \dfrac{dP}{dt} - rP = k \\ P(0) = P_0 \end{cases} \quad \text{Linear!}$$

$$P(t) = Ce^{rt} + \frac{k}{r}$$

$$P(0) = P_0 \Rightarrow P_0 = C + k/r$$

So

$$P(t) = (P_0 - k/r)e^{rt} + k/r$$

(d) Determine k.

$$P(T) = 0 \Rightarrow (P_0 - k/r)e^{rT} + k/r = 0$$

$$\Rightarrow (1 - e^{rT})k/r + P_0 e^{rT} = 0$$

$$\Rightarrow k = \frac{-rP_0 e^{rT}}{1 - e^{rT}} = \frac{re^{rT}}{e^{rT} - 1} P_0$$

$$\Rightarrow k = \frac{rP_0}{1 - e^{-rT}} \text{ \$/year}$$

Monthly payments $= k/12$.

$r = 0.04$ 1/yr $P_0 =$ \$250,000 $T = 30$ yr

$$\frac{k}{12} \approx \$1192.51/\text{mo.}$$

Problem. You place a hot brick (temp $50°C$) outside at noon. The outside temperature is $10°C$. At 1:00 pm the temperature of the brick is $40°C$. When will the temperature of the brick be $20°C$?

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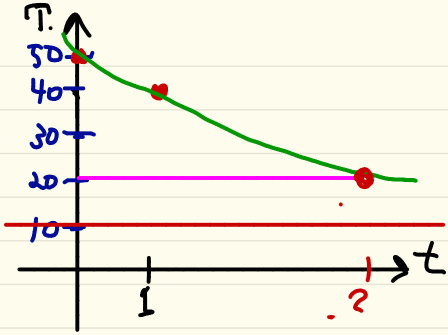


Solution. Assume Newton's Law of cooling!

Let t = time in hours after noon.
T = temp. of brick
$T_A = 10°C$ ambient temp.
$T_0 = 50°C$ initial temp of brick.

Then have IVP

$$\begin{cases} \frac{dT}{dt} = -k(T-10) \\ T(0) = 50 \end{cases}$$

k: to be determined

Solve the IVP:

$$\frac{dT}{dt} + kT = k10 \quad T(0) = 50$$

$$T(t) = 10 + Ce^{-kt}$$

$$50 = 10 + C \Rightarrow C = 40$$

So $T(t) = 10 + 40e^{-kt}$

Determine k:

$$T(1) = 40 \Rightarrow 40 = 10 + 40e^{-k(1)}$$

$$\Rightarrow e^{-k} = \frac{30}{40} \Rightarrow -k = \ln(3/4)$$

$$\Rightarrow k = \ln(4/3)$$

So $T(t) = 10 + 40e^{-\ln(4/3)t}$

Find time when $T = 20$.

$$20 = 10 + 40e^{-\ln(4/3)t}$$

$$\Rightarrow \frac{1}{4} = e^{-\ln(4/3)t}$$

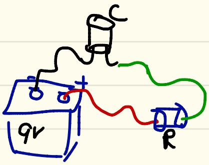
$$\Rightarrow -\ln(4) = -\ln(4/3)t$$

$$\Rightarrow t = \ln(4)/\ln(4/3) \approx 4.8$$

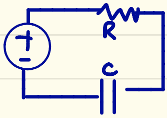
0.8 hr = 0.8 × 60 = 49 min.

$$\boxed{\text{So } T = 20°C \text{ at } 4{:}49 \text{ pm}}$$

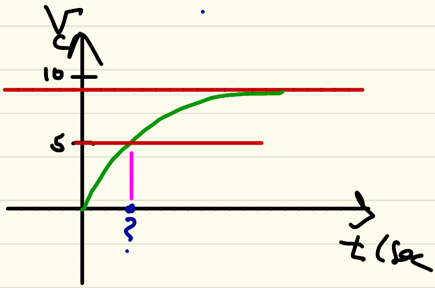
Problem. A 9 volt battery is connected to a 1.0 μF uncharged capacitor and a 100 Ω resistor, as shown.



How long will it take for the voltage drop across the capacitor to reach 5 volts?



Solution.



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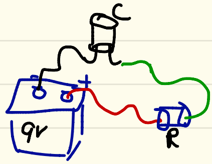

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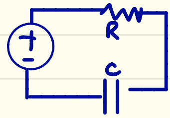
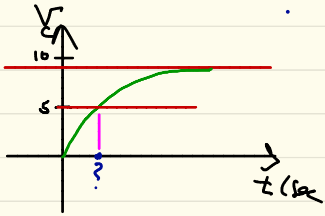


Solution.

Recall I = current, q = charge on capacitor

$$V_R = IR \qquad V_C = \frac{q}{C}$$



Kirchoff Voltage Law: $V_R + V_C = 9 \qquad IR + \frac{q}{C} = 9$

$$I = \frac{dq}{dt} \text{ so } R\frac{dq}{dt} + \frac{q}{C} = 9$$

$$RC\frac{dV_c}{dt} + V_C = 9 \text{ so } \frac{dV_c}{dt} + \frac{1}{RC}V_C = \frac{9}{RC}$$

General solution:

$$V_C(t) = 9 + c_1 e^{-t/RC}$$

$$V_C(0) = 0 \Rightarrow V_C(t) = 9\left(1 - e^{-t/RC}\right)$$

Need to solve $9(1-e^{-t/RC}) = 5$ for t.

$$1 - e^{-t/RC} = 5/9 \Rightarrow$$

$$e^{-t/RC} = 1 - 5/9 = 4/9 \Rightarrow$$

$$-t/RC = \ln(4/5) \Rightarrow$$

$$t/RC = \ln(5/4) \Rightarrow$$

$$t = RC\ln(5/4)$$

$$RC = 100 \times 1.0\times 10^{-6} = 10^{-4}\text{ sec}$$

So $t = \ln(5/4)\times 10^{-4} \approx$

$$0.81\times 10^{-4} = \underline{\underline{8.1\times 10^{-3}\text{ sec}}}$$

Example 3.6. (The Logistic Equation for Population Growth)

In the late 1920's Raymond Pearl analyzed data collected by to determine how well the logistic equation predicted the population growth of yeast.[8] In Carlson's experiments a small number of yeast cells were placed into a jar containing sugar, and the approximate number of yeast cells were counted each hour. Here is some of the data from the experiment:

$t =$	0	1	2	3	4	5	6	7	8	9
$Y(t) =$	10	18	19	47	71	119	175	257	351	441
$t =$	10	11	12	13	14	15	16	17	18	
$Y(t) =$	513	560	595	629	641	651	656	660	661	

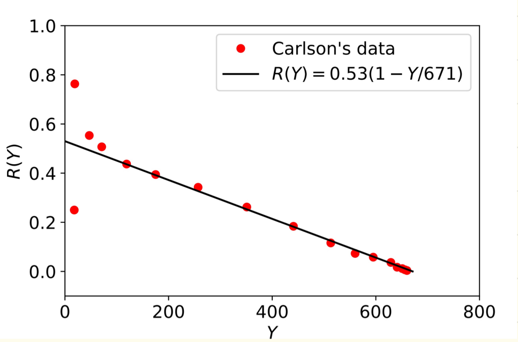
Pearl conjectured that the number of yeast $Y(t)$ after t hours obeyed a differential equation of the form

$$\frac{1}{Y}\frac{dY}{dt} = R(Y),$$

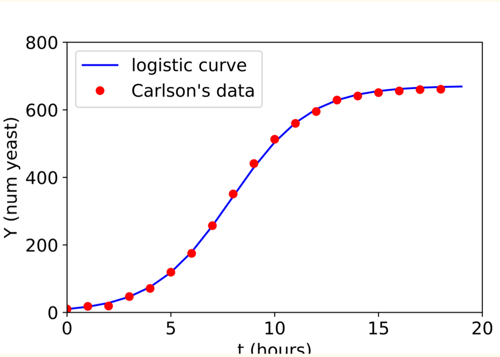
where $R(Y)$ is a function involving only the number of yeast. Notice that the left hand side is the "fractional rate of growth" (or the *logarithmic growth rate*). The right hand side is called the *reproduction function*. The object of Pearl's analysis was to determine $R(Y)$.

At any t, the derivative can be approximated by a "difference quotient" $\Delta Y/\Delta t$. A better estimate can be obtained by averaging successive quotients. For example, $Y'(5)$ can be estimated by

$$Y'(5) \approx \frac{1}{2}\left(\frac{Y(6)-Y(5)}{1} + \frac{Y(5)-Y(4)}{1}\right) = \frac{Y(6)-Y(4)}{2}.$$

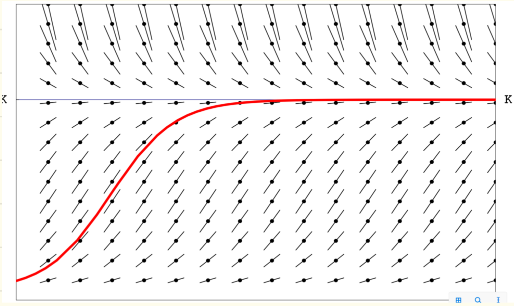


So $$\boxed{\frac{dY}{dt} = 0.53\left(1-\frac{Y}{671}\right)\cdot Y}$$



In general $\boxed{\frac{dP}{dt} = r(1 - P/K)P}$

Direction Field:



General Soln:

Because the logistic equation. is separable, it can be solved by separation of variables:

$$\int \frac{dP}{(1-P/K)P} = \int r dt\,.$$

The left hand side can be integrated by partial fractions:

$$\frac{1}{(1-P/K)P} = \left(\frac{1}{K-P} + \frac{1}{P}\right) \implies \int \frac{dP}{P(1-P/K)} = \int \frac{dP}{K-P} + \int \frac{dP}{P}$$

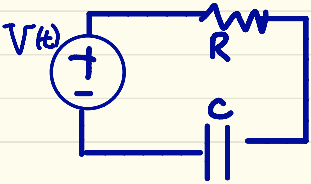
So

$$\int \frac{dP}{P(1-P/K)} = \int \frac{dP}{K-P} + \int \frac{dP}{P} = -\ln|K-P| + \ln|P| + C = \ln\left|\frac{P}{K-P}\right| + C$$

Thus $\ln\left|\frac{P}{K-P}\right| = rt + C \implies \left|\frac{P}{K-P}\right| = e^{rt+C} \implies \frac{P}{K-P} = Ae^{rt}$. where $A = \pm e^C$. Solving for P yields ans explicit formula for $P(t)$:

$$P = Ae^{rt}(K-P) \implies P(t) = \frac{AKe^{rt}}{1+Ae^{rt}} = \frac{K}{1+e^{-kt}/A}\,.$$

RC-circuit



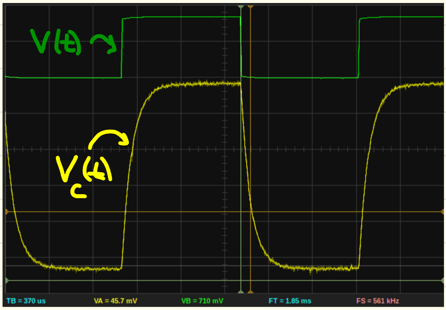
Review

$$RI + \frac{q}{C} = V(t)$$

$$\Rightarrow RC\,\frac{d(q/C)}{dt} + q/C = V(t)$$

$$\Rightarrow RC\,\frac{dV_c}{dt} + V_c = V(t)$$

$$\boxed{\frac{dV_c}{dt} = -\frac{1}{RC}\left(V_c - V(t)\right)}$$



Note: Units of RC : seconds

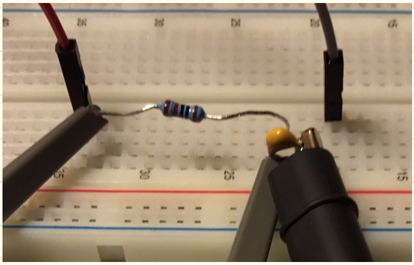
$\tau = RC$ is called the time constant

What about V_R?

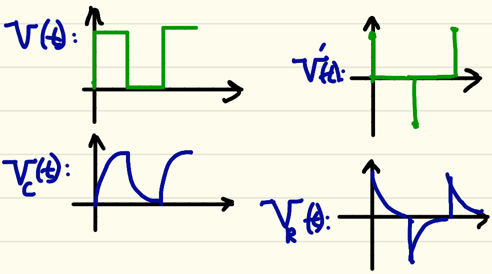
$$\frac{d}{dt}\left(RI + q/C\right) = \frac{dV(t)}{dt}$$

$$\Rightarrow \frac{d}{dt}(RI) + \frac{I}{C} = \frac{dV(t)}{dt}$$

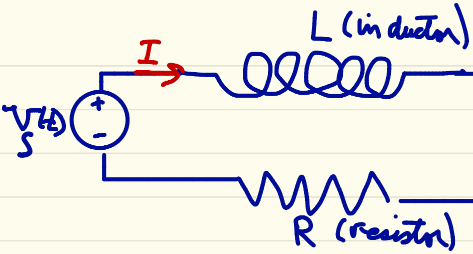
$$\Rightarrow \boxed{\frac{dV_R}{dt} + \frac{V_R}{RC} = \frac{dV(t)}{dt}}$$



Example



Example. RL-Circuits



Kirchoff's Voltage Law:

$$V_L + V_R = V_S(t)$$

$$(*)\quad L \cdot \frac{dI}{dt} + RI = V_S(t)$$

$V_R = RI$ so

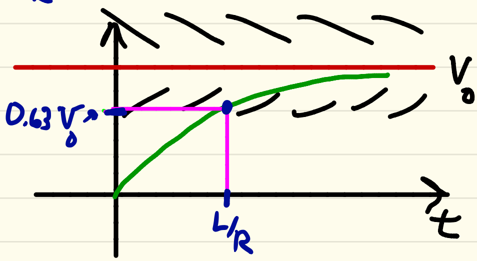
$$L \frac{d}{dt}\left(\frac{V_R}{R}\right) + V_R = V_S(t)$$

$$\Longleftrightarrow \quad \boxed{\frac{dV_R}{dt} = -\frac{R}{L}\left(V_R - V_S(t)\right)}$$

Note: Units of R/L are $1/\text{sec}$. "time constant" $= L/R$

If $V_S(t) = V_0$ (a constant) and $V_R(0) = 0$

$$\frac{dV_R}{dt} = -\frac{R}{L}\left(V_R - V_0\right)$$



$V_R = V_0$ stable fixed point

$$V_R(t) = V_0\left(1 - e^{-\frac{R}{L}t}\right)$$

Notice: $V_R(L/R) = V_0(1 - e^{-1}) \approx (0.63)\,V_0$