Lecture 07

Modeling Examples

Review:

- General Solution to $y' + p(t)y = f(t)$

$$y(t) = \underbrace{e^{-P(t)}\int e^{P(t)} f(t)\,dt}_{y_p(t)} + C\underbrace{e^{-P(t)}}_{y_h(t)}$$

where $P(t) = \int p(t)\,dt$.

- Special Case: $y' + ky = e^{bt}$ $b \neq -k$

Try $y_p(t) = Ae^{bt}$: $(b+k)Ae^{bt} = e^{bt}$

$\Rightarrow A = \frac{1}{b+k}$ So $y(t) = \frac{1}{b+k}e^{bt} + Ce^{-kt}$

Examples.

- Gen. Soln to $y' + 5y = e^{7t}$ $y = \frac{e^{7t}}{7+5} + Ce^{-5t}$
- Gen. Soln to $y' + 5y = 8$ $y = \frac{8}{5} + Ce^{-5t}$
- Gen Soln to $y' + 5y = e^{-5t}$

Try $y_p(t) = Ate^{-5t}$ $(Ate^{-5t})' + 5(Ate^{-5t})$

$= Ae^{-5t} = e^{-5t} \Rightarrow A = 1$

Gen Soln: $y(t) = te^{-5t} + Ce^{-5t}$

Example 3. $y' + 3y = \cos(\omega t)$

Try $y_p(t) = A\cos(\omega t) + B\sin(\omega t)$

Then $y_p' + 3y_p =$

$$(-\omega A\sin(\omega t) + \omega B\cos(\omega t)) + 3(A\cos(\omega t) + B\sin(\omega t))$$

$$= (3A + \omega B)\cos(\omega t) + (3B - \omega A)\sin(\omega t) = \cos(\omega t)$$

So $\begin{cases} 3A + \omega B = 1 \\ -\omega A + 3B = 0 \end{cases} \Rightarrow A = \frac{3}{9+\omega^2} \quad B = \frac{\omega}{9+\omega^2}$

So $\boxed{y(t) = \frac{3}{9+\omega^2}\cos(\omega t) + \frac{\omega}{9+\omega^2}\sin(\omega t) + Ce^{-3t}}$

$$= \frac{1}{\sqrt{9+\omega^2}}\cos(\omega t - \phi) + Ce^{-3t}, \quad \tan\phi = \frac{\omega}{3}$$

Example. $\frac{dy}{dt} + ky = \cos(\omega t)$

Example

$$\frac{dy}{dt} + ky = \cos(\omega t)$$

$$y(t) = y_p(t) + Ce^{-kt}$$

Try $y_p(t) = A\cos(\omega t) + B\sin(\omega t)$

Then

$$y_p'(t) + k\,y_p(t) = \omega(-A\sin(\omega t) + B\cos(\omega t)) + k(A\cos(\omega t) + B\sin(\omega t))$$

$$= (\omega B + kA)\cos(\omega t) + (-\omega A + kB)\sin(\omega t) = \cos(\omega t)$$

$$\Rightarrow A = \frac{k}{k^2+\omega^2}, \quad B = \frac{\omega}{k^2+\omega^2}$$

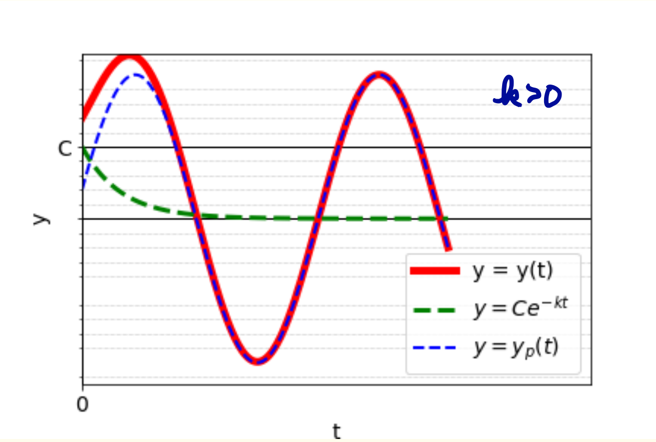
So $y_p(t) = \frac{k}{k^2+\omega^2}\cos(\omega t) + \frac{\omega}{k^2+\omega^2}\sin(\omega t) = \frac{1}{\sqrt{k^2+\omega^2}}\cos(\omega t - \varphi)$,

where $\tan\varphi = \frac{\omega}{k}$.

Consequently,

$$\boxed{y(t) = \frac{1}{\sqrt{k^2+\omega^2}}\cos(\omega t - \varphi) + Ce^{-kt}.}$$

transient



Steady State Soln.

Note If $\frac{dy}{dt} + k y = \underline{a_1 \cos(\omega_1 t)} + \underline{a_2 \cos(\omega_2 t)}$

then
$$y(t) = C e^{-kt} + \underline{\frac{a_1}{\sqrt{k^2+\omega_1^2}} \cos(\omega_1 t - \varphi_1)} + \underline{\frac{a_2}{\sqrt{k^2+\omega_2^2}} \cos(\omega_2 t - \varphi_2)}$$

where $\begin{cases} \varphi_1 = \tan^{-1}(\omega_1/k) \\ \varphi_2 = \tan(\omega_2/k) \end{cases}$

Example. Find a particular solution of
$$y' + k y = \sin(\omega t)$$

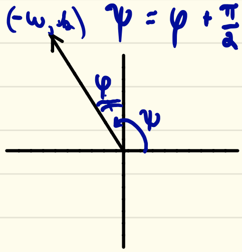
Soln. Try $y_p = A\cos(\omega t) + B\sin(\omega t)$

Then
$$\begin{aligned} y_p' + k y_p &= \omega(-A\sin(\omega t) + B\cos(\omega t)) \\ &\quad + k(B\sin(\omega t) + A\cos(\omega t)) \\ &= \sin(\omega t) \end{aligned}$$

$$\Rightarrow \begin{cases} -\omega A + kB = 1 \\ kA + \omega B = 0 \end{cases} \Rightarrow \begin{cases} A = \dfrac{-\omega}{k^2+\omega^2} \\ B = \dfrac{k}{k^2+\omega^2} \end{cases}$$

So
$$\begin{aligned} y_p &= \frac{-\omega}{k^2+\omega^2}\cos(\omega t) + \frac{k}{k^2+\omega^2}\sin(\omega t) \\ &= \frac{1}{\sqrt{k^2+\omega^2}} \cos(\omega t - \psi) \\ &= \frac{1}{\sqrt{k^2+\omega^2}} \sin(\omega t - \varphi) \end{aligned}$$

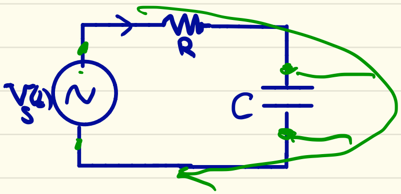
$\psi = \varphi + \frac{\pi}{2}$

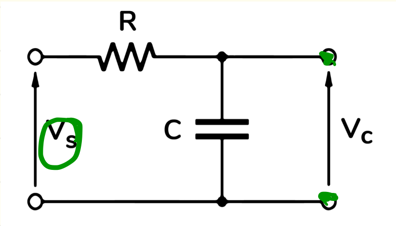


Recall
$\cos(\alpha) = \sin(\alpha - \frac{\pi}{2})$

$$\tan\varphi = \frac{\omega}{k}$$

Low pass filter





$$V_s(t) = RI + \frac{q}{C}$$

$$I = \frac{dq}{dt} \qquad V_c = q/C$$

$$q = C \cdot V_c$$

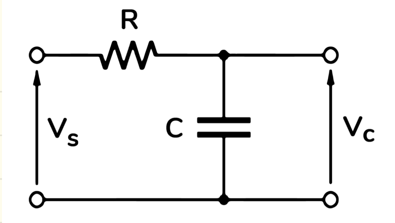
$$V_s(t) = R\frac{dq}{dt} + \frac{q}{C}$$

$$V_s(t) = R\frac{d(CV_c)}{dt} + V_c$$

$$RC\frac{dV_c}{dt} + V_c = V_s(t)$$

$$\frac{dV_c}{dt} + \frac{1}{RC}V_c = \frac{1}{RC}V_s$$

$$\frac{dV_c}{dt} + \frac{1}{RC}V_c = \frac{1}{RC}V_S(t)$$



$$k = \frac{1}{RC}$$

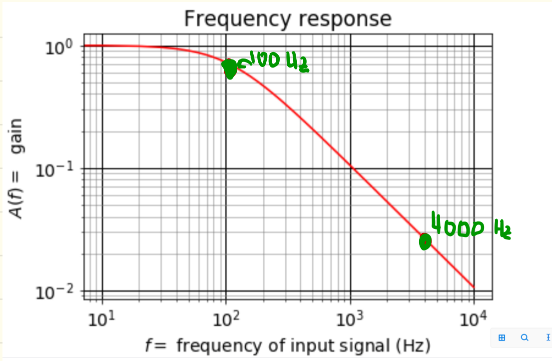
$$\frac{1}{\sqrt{k^2+\omega^2}} = \frac{RC}{\sqrt{1+(RC\omega)^2}}$$

If $V_S(t) = \cos(\omega t)$ then

$$V_c(t) = Ce^{-t/RC} + \frac{1}{\sqrt{1+(RC\omega)^2}}\cos(\omega t - \varphi)$$

$A(\omega) = 1/\sqrt{1+(RC\omega)^2}$ called "gain"

Example. $R = 150\,\Omega$
$C = 10 \times 10^{-6}$ F



Input signal:

$$V_S(t) = \overset{2.5}{\cancel{1}} * \cos(\omega_1 t) + \overset{0.5}{\cancel{1}} * \cos(\omega_2 t)$$

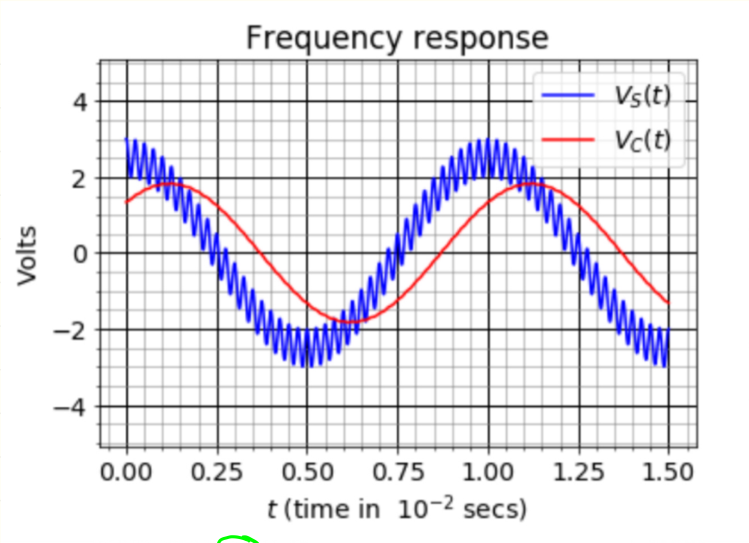
$$f_1 = \omega_1/2\pi = 100\text{Hz},\ f_2 = \omega_2/2\pi = 4000\text{Hz}$$

$V_c(t) =$

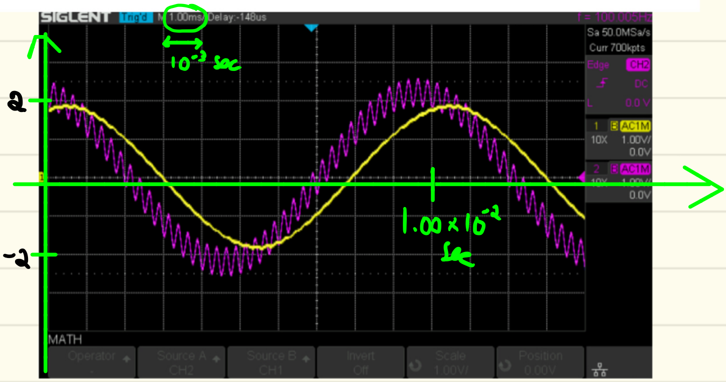
$$Ce^{-t/RC} + \left(\frac{2.5}{RC}\right)\frac{1}{\sqrt{k^2+\omega_1^2}}\cos(\omega_1 t - \varphi_1) + \left(\frac{0.5}{RC}\right)\frac{1}{\sqrt{k^2+\omega_2^2}}\cos(\omega_2 t - \varphi_2)$$

$$= Ce^{-t/RC} + \frac{2.5}{\sqrt{1+(RC\omega_1)^2}}\cos(\omega_1 t - \varphi_1) + \frac{0.5}{\sqrt{1+(RC\omega_2)^2}}\cos(\omega_2 t - \varphi_2)$$

$$\approx 1.82\cos(\omega_1 t - \varphi_1) + (0.03)\cos(\omega_2 t - \varphi_2)$$



theoritical



observed

Changing Units (Examples)

e^{-kt}

- Newton's Law of Cooling.

$$\frac{dT}{dt} + kT = kT_A(t)$$

t : time in seconds

T : temperature (in degrees C)

$T_A(t)$: ambient temperature.

units of k : $\frac{1}{\text{sec.}}$

$1/k$ natural time scale.

"time constant"

$$\frac{dT}{dt} = \frac{d\tau}{dt}\frac{dT}{d\tau} = k\frac{dT}{d\tau}$$

So $k\frac{dT}{d\tau} + kT = kT_A(t)$

or

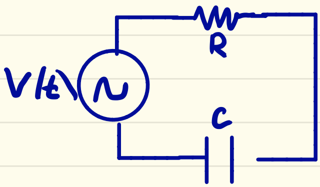
$$\frac{dT}{d\tau} + T = T_A(\tau/k)$$

Let $\tau = kt = t/(1/k)$ (dimensionless)

$t = \tau\cdot(1/k)$ "time in multiples of $1/k$"

- RC - circuit

$$\frac{dV_C}{dt} + \frac{1}{RC}V_C = \frac{1}{RC}V(t)$$



RC is "time constant"

units of RC = seconds

Let $\tau = t/RC$ or $t = \tau\cdot RC$

$$\frac{dV_C}{dt} = \frac{d\tau}{dt}\frac{dV_C}{d\tau} = \frac{1}{RC}\frac{dV_C}{dt}$$

$$\frac{1}{RC}\frac{dV_C}{d\tau} + \frac{1}{RC}V_C = \frac{1}{RC}V(t)$$

$t = \tau(RC)$

or

$$\frac{dV_C}{d\tau} + V_C = V(\tau RC)$$

$$\frac{dT}{d\tau} + T = T_A(\tau/k)$$