Lecture 06

First Order Linear ODE's

Linear Differential Equations

$$\begin{cases} (a) & \dfrac{dy}{dt} + p(t)\,y = 0 \quad 1^{st} \text{ order homogeneous} \\ (b) & \dfrac{dy}{dt} + p(t)\,y = f(t) \quad 1^{st} \text{ order non homogeneous} \end{cases}$$

Key observation: Suppose $\begin{cases} y_h(t) \text{ is a solution to (a)} \\ y_p(t) \text{ is a solution to (b)} \end{cases}$

Then the general solution to (b) is $\boxed{y(t) = C\,y_h(t) + y_p(t)}$

Check this:

$$y'(t) + p(t)\,y(t) = (C\,y_h(t) + y_p(t))' + p(t)\,(C\,y_h(t) + y_p(t))$$
$$= C\,(y_h'(t) + p(t)\,y_h(t)) + y_p'(t) + p(t)\,y_p(t)$$
$$= C \cdot (0) + f(t) = f(t)$$

Another important observation:

If $\begin{cases} y_{p_1}' + p(t)\,y_{p_1} = f_1(t) \\ y_{p_2}' + p(t)\,y_{p_2} = f_2(t) \end{cases}$ Then the general solution to

$$y' + p(t)\,y = a_1 f_1(t) + a_2 f_2(t)$$

a_1 and a_2 constants

is

$$y(t) = C\,y_h(t) + a_1\,y_{p_1}(t) + a_2\,y_{p_2}(t)$$

$$y' + p(t)\,y =$$

Strategy for solving $y' + p(t)\,y = f(t)$:

① Solve $y' + p(t)\,y = 0$. (any particular solution $y_h(t)$ will suffice)

② Find a particular solution $y_p(t)$ to
$$y' + p(t)\,y = f(t).$$
(can often guess $y_p(t)$.)

③ General solution is
$$y(t) = C\,y_h(t) + y_p(t)$$

Step 1: solve $y' + p(t)\,y = 0$.

Example: Solve $y' + t\,y = 0$

Use separation of variables:
$$\int \frac{dy}{y} = -\int t\,dt = -t^2/2 \Rightarrow \ln|y| = -t^2/2$$
So $y_h(t) = e^{-t^2/2}$ General Solution: $Ce^{-t^2/2}$

Works in general: $y' + p(t)\,y = 0$
$$\int \frac{dy}{y} = -\int p(t)\,dt = -P(t) \Rightarrow \ln|y| = -P(t)$$
$$\Rightarrow y_h(t) = e^{-P(t)}.$$
General Solution: $Ce^{-P(t)}$

Step 2: Solve $y' + p(t)\,y = f(t)$

$$\boxed{\text{Trick: Multiply by } e^{P(t)}.}$$

$P(t)$ = antiderivative of $p(t)$.

Example. $y' + t = t. \iff e^{t^2/2}(y' + ty) = e^{t^2/2} \cdot t$

Notice: $\frac{d}{dt}\left(e^{t^2/2} y\right) = e^{t^2/2} y' + t e^{t^2/2} y$

$$= e^{t^2/2}(y' + ty) = e^{t^2/2}(t)$$

So

$$\int \frac{d}{dt}\left(e^{t^2/2} y\right) dt = \int \underbrace{e^{t^2/2} t \, dt} + C$$

$u = t^2/2$

$du = t\,dt$

$\int e^u \, du$

$$\Rightarrow \quad e^{t^2/2} y = e^{t^2/2} + C$$

$$\Rightarrow \quad \boxed{y = 1 + C e^{-t^2/2}}$$

This works in general!

Method in general: To solve $y' + p(t)y = f(t)$.

① Let $P(t) = \int p(t)\,dt$ (no constant of integration!)

② Multiply by $e^{P(t)}$:

$$e^{P(t)}y' + p(t)e^{P(t)}y = e^{P(t)}f(t)$$

$$\Leftrightarrow \frac{d}{dt}\left(e^{P(t)}y\right) = e^{P(t)}f(t)$$

$$\Rightarrow e^{P(t)}y = \int e^{P(t)}f(t)\,dt + C$$

$$\Rightarrow y(t) = \underbrace{e^{-P(t)}\int e^{P(t)}f(t)\,dt}_{} + C\cdot e^{-P(t)}$$

$$= y_p(t) + C\,y_h(t)$$

Example Solve the initial value problem

$$y' + 3ty = te^{t^2}, \quad y(2) = 5.$$

Soln:

$$P(t) = \int 3t\,dt = \frac{3t^2}{2},$$

$$\left(e^{\frac{3}{2}t^2} y\right)' = e^{3t^2/2}\, t e^{t^2} = t e^{\frac{5t^2}{2}}$$

So

$$e^{\frac{3}{2}t^2} y = \int e^{\frac{5}{2}t^2}\, t\,dt = \frac{1}{5} e^{\frac{5}{2}t^2} + C$$

$$\Rightarrow \quad y(t) = \frac{1}{5} e^{t^2} + C e^{-\frac{3}{2}t^2}$$

(Gen solution)

Initial condition

$y(2) = 5 \Rightarrow$

$$5 = \frac{1}{5} e^4 + C e^{-\frac{3}{2}(2)^2}$$

$$= \frac{1}{5} e^4 + C e^{-6}$$

$$\Rightarrow \quad C = e^6\left(5 - \frac{1}{5} e^4\right)$$

$$= 5e^6 - \frac{1}{5} e^{10}$$

$$\approx -2388$$

So

$$(t) = \frac{1}{5} e^{t^2} + \left(5e^6 - e^{10}/5\right) e^{-\frac{3}{2}t^2}$$

$$\approx \frac{1}{5} e^{t^2} - 2388\, e^{-\frac{3}{2}t^2}$$

Example. Solve the following initial value problem:

$$(1+t^2)\frac{dy}{dt} + 2ty = t(1+t^2)\,, \quad y(1) = 1\,.$$

Solution.

Write in the form

$$\frac{dy}{dt} + \frac{2t}{1+t^2}y = t.$$

Then $P(t) = \int^t \frac{2s}{1+s^2}\,ds = \ln(1+t^2)$

So $e^{P(t)} = e^{\ln(1+t^2)} = (1+t^2)$

Hence, $\left((1+t^2)y(t)\right)' = (1+t^2)t$.

Integrate to get

$$(1+t^2)\,y(t) = \int (1+t^2)t\,dt$$

$$= \frac{(1+t^2)^2}{4} + c$$

or

$$y(t) = \frac{1+t^2}{4} + \frac{c}{1+t^2}$$

The initial condition

$$y(1) = 1$$

yields the equation

$$1 = \frac{1+(1)^2}{4} + \frac{c}{1+(1)^2}$$

$$\Leftrightarrow \quad 1 = \frac{1}{2} + \frac{c}{2}$$

$$\Rightarrow \quad c = 1$$

Hence,

$$\boxed{y(t) = \frac{(1+t^2)}{4} + \frac{1}{1+t^2}}$$

Important special case:

(where guessing is best approach)

$$\frac{dy}{dt} + ky = f(t) = \begin{cases} a e^{bt} \\ a_1 \cos(\omega t) + b_1 \sin(\omega t) \end{cases}$$

We know that $y(t) = y_p(t) + Ce^{-kt}$

So we only have to find $y_p(t)$!

Example 1. $\frac{dy}{dt} + 3y = e^{2t}$ $\quad \frac{dy}{dt} + 3y = 0 \quad y_h(t) = e^{-3t}$

Try $y_p(t) = Ae^{2t}$: "undetermined coefficients"

$y_p'(t) + 3y_p(t) = e^{2t}$. $\Rightarrow$

$2Ae^{2t} + 3Ae^{2t} = 5Ae^{2t} = e^{2t}$

So $A = 1/5$ and so $y(t) = \frac{1}{5}e^{2t} + Ce^{-3t}$.

Example 2. $y' + 3y = e^{-3t}$.

Try $y_p(t) = Ate^{-3t}$ (Ae^{-3t} won't work!) Why?

$$y_p' + 3y_p = (Ate^{-3t})' + 3(Ate^{-3t})$$

$$= Ae^{-3t} - 3tAe^{-3t} + 3Ate^{-3t}$$

$$= Ae^{-3t} = e^{-3t} \Rightarrow A = 1$$

So $y(t) = te^{-3t} + Ce^{-3t}$.