Lecture 05

Separable ODE's

Friday Oct 5 (Separable DEs)

From Math 125:

$$\frac{dy}{dt} = g(t) \Rightarrow \int y'(t)\,dt = \int g(t)\,dt \Longrightarrow y(t) = \int g(t)\,dt$$
$$= G(t) + C$$

antiderivative

Example $\frac{dy}{dt} = \frac{1}{t+1} \Rightarrow$

$$y(t) = \int \frac{1}{t+1}\,dt = \ln|t+1| + C$$

$$\begin{cases} \frac{dy}{dt} = g(t) \\ y(t_0) = y_0 \end{cases} \Rightarrow \int_{t_0}^{t} y'(s)\,ds = \int_{t_0}^{t} g(s)\,ds$$

$$\Rightarrow y(t) - y(t_0) = G(t) - G(t_0)$$
$$y(t) - y_0 = G(t) - G(t_0)$$
$$\Longrightarrow y(t) = G(t) + y_0 - G(t_0)$$

Example $\frac{dy}{dt} = \frac{1}{t+1},\ y(1) = 7$

$$\int_1^t y'(s)\,ds = \int_1^t \frac{1}{s+1}\,ds$$

$$y(t) - y(1) = \ln(t+1) - \ln(1+1)$$

$$y(t) - 7 = \ln\left(\frac{t+1}{2}\right) \Rightarrow y(t) = \ln\left(\frac{t+1}{2}\right) + 7$$

Generalize:

Example: $\frac{dy}{dt} = 2ty^2$, $y(0) = 4$.

Suppose $y = y(t)$ is a solution.

Then $\frac{1}{y(t)^2}\frac{dy(t)}{dt} = 2t$. So $\int \frac{1}{y(t)^2}\frac{dy(t)}{dt}\,dt = \int 2t\,dt$

$$\Rightarrow \int \frac{1}{y^2}\,dy = \int 2t\,dt$$

$$\Rightarrow -\frac{1}{y(t)} = t^2 + C$$

Can solve for $y(t)$

$y = \frac{-1}{t^2+C}$ (General solution

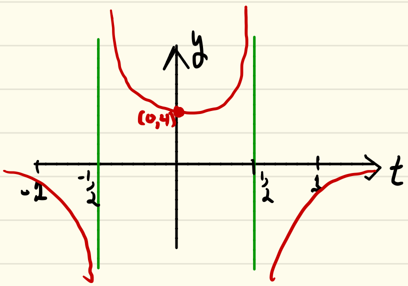
I.V.P. $-\frac{1}{y} = t^2 + C$, $y(0) = 4$

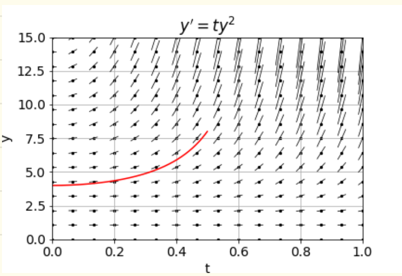
So $-\frac{1}{4} = (0)^2 + C$

$\Rightarrow C = -1/4$

Hence,

$$y = \frac{-1}{t^2 - 1/4} = \frac{1}{1/4 - t^2}$$





Separation of Variables in General:

Consider an ODE of the form

$$h(y)y' = g(t)$$

where $g(t)$ and $h(y)$ are continuous functions.
To solve let

$$\int h(y)\,dy = H(y) + C_1 \text{ and } \int g(t)\,dt = G(t) + C_2 .$$

Then

$$\frac{dG(t)}{dt} = g(t) \text{ and } \frac{dH(y)}{dt} = h(y)\frac{dy}{dt} .$$

So we get the **implicit** solution

$$H(y) + C_1 = G(t) + C_2 . \text{ or } H(y) = G(t) + C$$

where $C = C_2 - C_1$, again an arbitrary constant.
NOTE: Solve for y in terms of t if you can!

Example:

Solve the differential equation

$$\frac{dy}{dt} = (1 + y^2)e^t .$$

Step 1. Get into the form $h(y)y' = g(t)$:

$$\frac{1}{1+y^2}\frac{dy}{dt} = e^t .$$

Step 2. Integrate:

$$\int \frac{dy}{1+y^2} = \int e^t\,dt .$$

$$\tan^{-1}(y) = e^t + C \text{ or } y = \tan\left(e^t + C\right)$$

where C is an arbitrary constant.

To solve the initial value problem

$$h(y)y' = g(t) \quad y(t_0) = y_0$$

Step 1. Find the general solution:

$$H(y) = G(t) + C$$

where $H'(y) = h(y)$ and $G'(t) = g(t)$.

Step 2. Find C:
$H(y_0) = G(t_0) + C$ or $C = H(y_0) - G(t_0)$

Final solution:

$$H(y) - H(y_0) = G(t) - G(t_0)\,.$$

Example: Solve $y' = (1 + y^2)e^t\,, \quad y(0) = 1$

First solve the ODE :

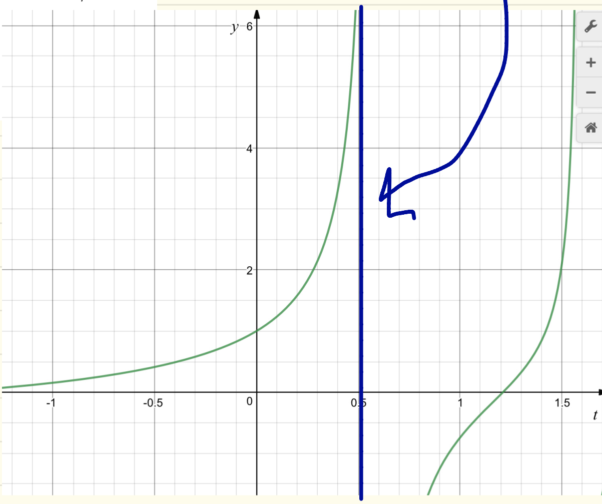
$$\tan^{-1}(y) = e^t + C$$

Set $t = 0$ and $y = 1$ to get

$$\tan^{-1}(1) = e^0 + C \implies \pi/4 = 1 + C \implies C = \pi/4 - 1$$

So $\tan^{-1}(y) - \tan^{1}(1) = e^t - e^0$
or $\tan^{-1}(y) - \pi/4 = e^t - 1$.
Solving for y gives $y = \tan\left(e^t - 1 + \pi/4\right)$.

$t = \ln(\pi/4 + 1) \approx 0.58$



Example: Find the general solution of the ODE

$$y' = 2(1-y)y$$

Solution: Separate variables: $\frac{1}{(1-y)y}\,y' = 2$

Then $$\int \frac{dy}{(1-y)y} = \int 2\,dt$$

Partial Fractions:

$$\frac{1}{(1-y)y} = \frac{A}{y} + \frac{B}{1-y}$$

$$\cancel{y}\,\frac{1}{(1-y)\cancel{y}} = y\,\frac{A}{y} + y\,\frac{B}{1-y} = A + \frac{yB}{1-y}$$

$$\cancel{(1-y)}\,\frac{1}{\cancel{(1-y)}y} = (1-y)\left(\frac{A}{y} + \frac{B}{(1-y)}\right) = B + \frac{(1-y)A}{y}$$

Set $y=0$: $\frac{1}{1-0} = A \Rightarrow A = 1$

Set $y=1$: $\frac{1}{(1)} = B \Rightarrow B = 1$

$$\int \frac{1}{y} + \frac{1}{1-y}\,dy = \int 2\,dt$$

$$\Rightarrow \ln|y| - \ln|1-y| = 2t + C$$

$$\Rightarrow \ln\left|\frac{y}{1-y}\right| = 2t + C$$

$$\Rightarrow \left|\frac{y}{1-y}\right| = e^{2t+C}$$

$$\Rightarrow \frac{y}{1-y} = Ke^{2t} \qquad K = \pm e^{C}$$

$$\Rightarrow y = Ke^{2t}(1-y)$$

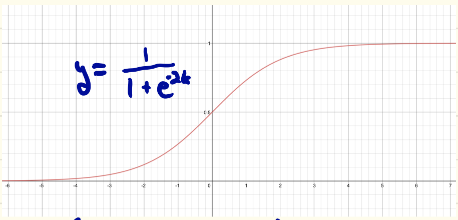
$$\Rightarrow (1+Ke^{2t})\,y = Ke^{2t}$$

$$\Rightarrow y = \frac{Ke^{2t}}{1+Ke^{2t}}$$

Better:

$$y = \frac{Ke^{2t}}{1+Ke^{2t}}\;\frac{(1/Ke^{2t})}{(1/Ke^{2t})}$$

$$\boxed{y = \frac{1}{1+Ae^{-2t}}} \qquad A = 1/K$$



(Logistic curve)