Lecture 04
Euler's Method
Review.

Direction Fields

General solution:
\[ y = y(t, c) \]

I.V.P.:
\[ \begin{cases} \frac{dy}{dt} = F(t, y) \\ y(t) = y_0 \end{cases} \]

Unique solutions \( \iff \) integral curves do not cross!

Unique solution of I.V.P.:
\[ \begin{cases} \frac{dy}{dt} = F(t, y) \text{ } \text{ } \text{ } \text{ } \text{ nice } \\ y(t_0) = y_0 \end{cases} \text{ initial value problem} \]

Theorem: If \( F \) is nice, I.V.P. has unique solution.
Special case:

\[ \frac{dy}{dt} = F(y) \quad \text{autonomous DE} \]

Example \( \frac{dy}{dt} = y(1-y) \)

![Graph showing the function \( F(y) = y(1-y) \) with equilibrium points at 0 and 1, indicating unstable and stable behavior.]
Example

Consider the ODE $y' = f(y)$ where the graph of $f(y)$ is displayed below (flipped along the diagonal and aligned with the direction field):

**Question:** Where are the equilibrium points? Which ones are stable and which ones are unstable?
Examples.

Free fall with air resistance:
\[
\frac{dN}{dt} = \frac{N}{m_0} - \frac{\gamma N}{m}
\]

\[ V_b = 9 \text{ volts} \]
\[
\frac{dV}{dt} + \frac{1}{RC} V = \frac{1}{RC} V_b
\]

\[
\frac{dT}{dt} = -k \left( T - T_a \right)
\]

Assume \( T \rightarrow T_a \) = const Newton's Law of Cooling.

General Form
\[
\begin{cases}
\frac{dy}{dt} + ky = ky_a \\
\frac{dy}{dt} = -k (y - y_a)
\end{cases}
\]

\( k > 0 \)

Stable fixed point.

\( y = y_a \)
Example. Classify the fixed points of the ODE
\[
\frac{dx}{dt} = (1-x^2) (4-x^2) x^2
\]

Solution.
Example. Classify the fixed points of the ODE
\[
\frac{dx}{dt} = (1-x^2)(4-x^2)x^2
\]
So let \( f(x) = (1-x^2)(4-x^2)x^2 \)
\[
= (1-x)(1+x)(2-x)(2+x)x^2
\]
\( f(0) = 0 \) for \( x = 1, -1, 2, -2, 0 \).

These are the fixed points.

- Graph \( f(x) \) (only the sign of \( f(x) \) is important!)

\[
\begin{array}{cccccccc}
+ & + & + & D & - & - & O & + & + & + & + & + & +
\end{array}
\]

\[-2 & -1 & 0 & 1 & 2\]

\[
\begin{array}{cccccccc}
\text{stable} & \text{unstable} & \text{semi stable} & \text{stable} & \text{unstable}
\end{array}
\]

\[-2 & -1 & 0 & 1 & 2\]

\[
\begin{array}{cccccccc}
\text{unstable} & \text{stable} & \text{semi stable} & \text{stable} & \text{unstable}
\end{array}
\]

\[-3 & -2 & -1 & 0 & 1 & 2 & 3 & 4\]
Numerical Solution of Differential Equations

Simplest method: Euler’s Method

Based on the tangent line approximation:

\[ y(t) \approx y_0 + y'(t_0)(t - t_0) \]

Example: \( y' = y, \ y(0) = 1 \).

<table>
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<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>( y_n )</td>
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<td>( y(t) = e^t )</td>
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Figure 2.5. Euler’s method applied to the initial value problem \( y' = y, \ y(0) = 1.0 \), with step size \( h = 0.1 \).
The method in general:

To solve the IVP

\[ \begin{align*}
\frac{dy}{dt} &= F(t, y) \\
y(t_0) &= y_0
\end{align*} \]

Proceed as follows:

1. Choose a step size \( h > 0 \).
2. Set \( n = 0 \).
3. Set \( y'_n = F(t_n, y_n) \).
4. Set \( t_{n+1} = t_n + h \).
5. Set \( y_{n+1} = y_n + y'_n \cdot h \).
6. Increase \( n \) by one and go to step (2).
Example

\[ y' + y = 10 \cos(\pi t), \quad y(0) = 0 \quad h = 0.1 \]
Caution: h can't be too big!

\[
\frac{dy}{dt} = (10-y)y, \quad y(0)=1 \quad h = 0.2
\]

<table>
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