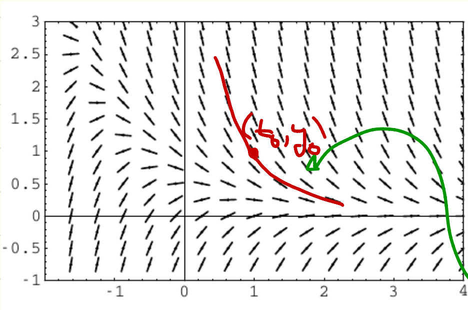
Lecture 04

Euler's Method

Review.

Direction Fields

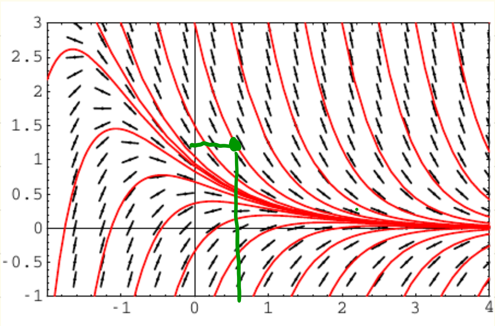


General soln:

$y = y(t, C)$

I.V.P.

$$\begin{cases} \frac{dy}{dt} = F(t,y) \\ y(t_0) = y_0 \end{cases}$$



Unique solns $\Longleftrightarrow$ integral curves do not cross!

Unique soln of I.V.P.

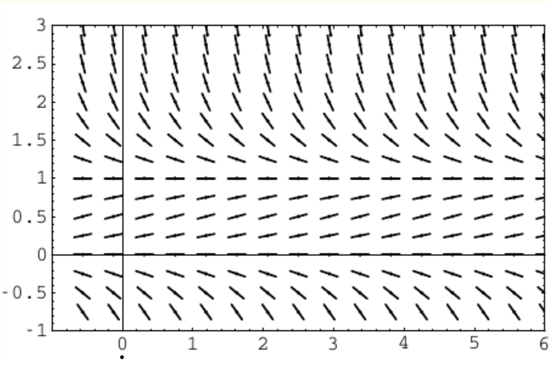
$$\begin{cases} \frac{dy}{dt} = F(t,y) \text{ "nice"} \\ y(t_0) = y_0 \end{cases}$$ initial value problem

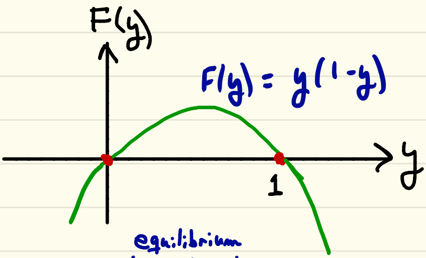
Theorem If F is "nice" IVP has unique solution.

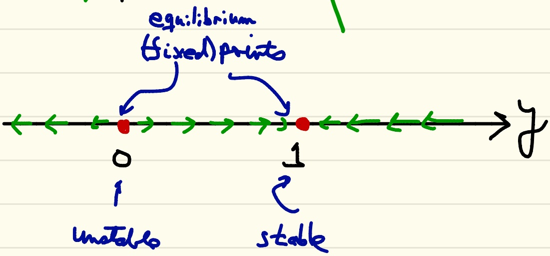
Special case:

$$\frac{dy}{dt} = F(y)$$ autonomous DE

Example $\frac{dy}{dt} = y(1-y)$

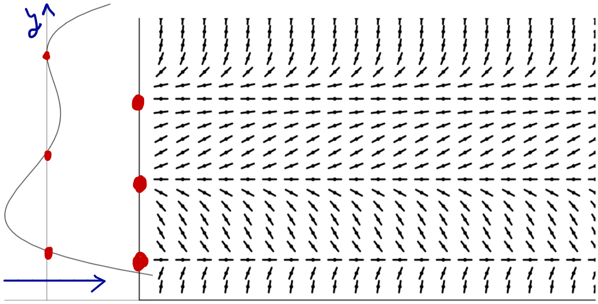




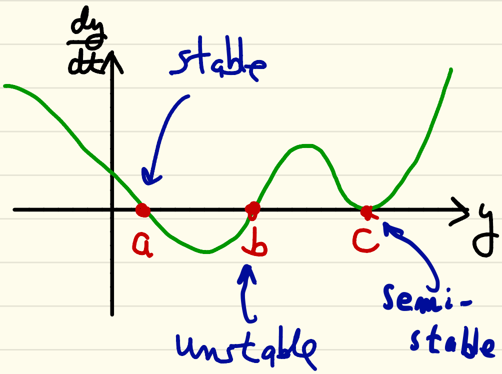


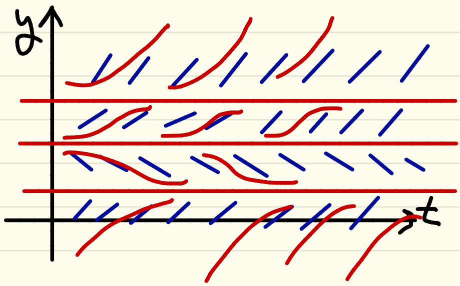
Example

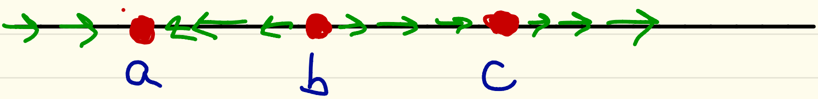
Consider the ODE $y' = f(y)$ where the graph of $f(y)$ is displayed below (flipped along the diagonal and aligned with the direction field):



Question: Where are the equilibrium points? Which ones are stable and which ones are unstable?

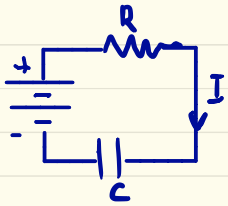






Examples.

$V_b = 9$ volts



$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{1}{RC} V_b$$

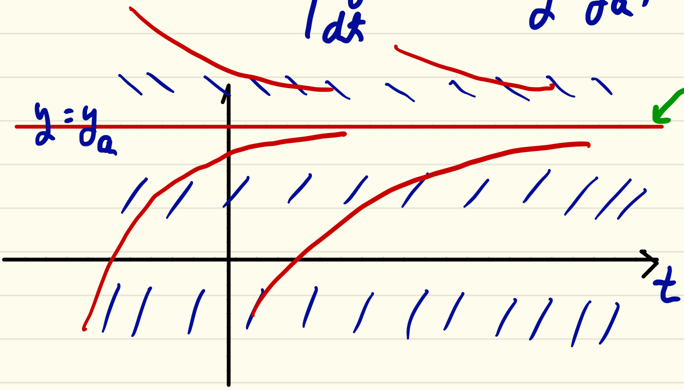
free fall with air resistance:

$$\frac{dv}{dt} + \frac{\gamma}{m} v = \frac{\gamma}{m} v_\infty$$

$$\frac{dT}{dt} = -k\,(T - T_A)$$

assume T_A = Const. Newton's Law of Cooling.

General Form
$$\begin{cases} \dfrac{dy}{dt} + k y = k y_a \\[2mm] \dfrac{dy}{dt} = -k(y - y_a) \end{cases} \qquad k > 0$$



stable fixed point

Example. Classify the fixed points of the ODE

$$\frac{dx}{dt} = (1-x^2)(4-x^2)x^2$$

Solution.

Example. Classify the fixed points of the ODE

$$\frac{dx}{dt} = (1-x^2)(4-x^2)x^2$$

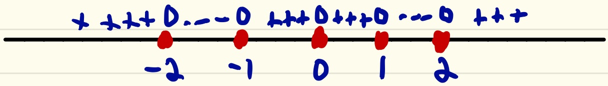
Solution. • Let $f(x) = (1-x^2)(4-x^2)x^2$

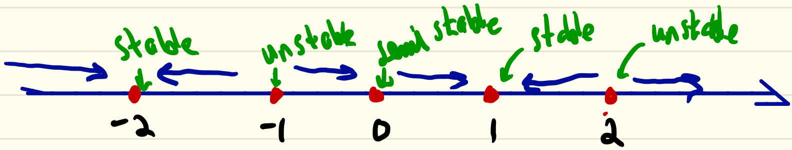
$$= (1-x)(1+x)(2-x)(2+x)x^2$$

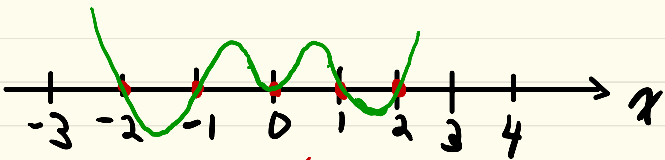
$f(x) = 0$ for $x = 1, -1, 2, -2, 0$.

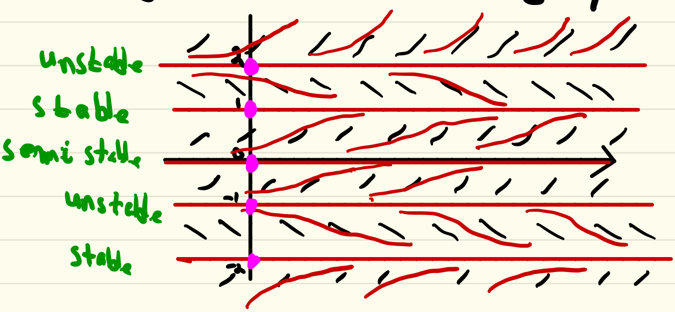
These are the fixed points.

• Graph $f(x)$ (Only the sign of $f(x)$ is important!)





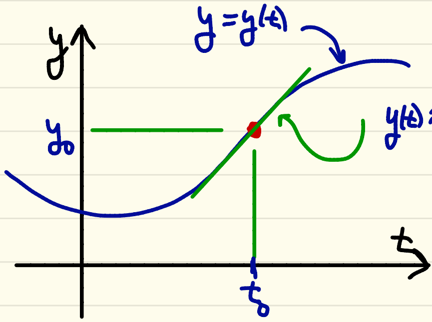




Numerical Solution of Differential Equations

Simplest method: Euler's Method

Based on The tangent line approximation:



$$y(t) \approx y_0 + y'(t_0)(t - t_0)$$

Example: $y' = y$, $y(0) = 1$.

$n =$	0	1	2	3	4	5	6	7	8	9	10
$t =$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$y_n =$	1.000	1.100	1.210	1.331	1.464	1.611	1.772	1.949	2.144	2.358	2.594
$y(t) = e^t =$	1.000	1.105	1.221	1.349	1.492	1.649	1.922	2.014	2.226	2.460	2.718

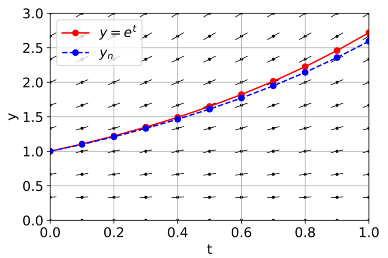
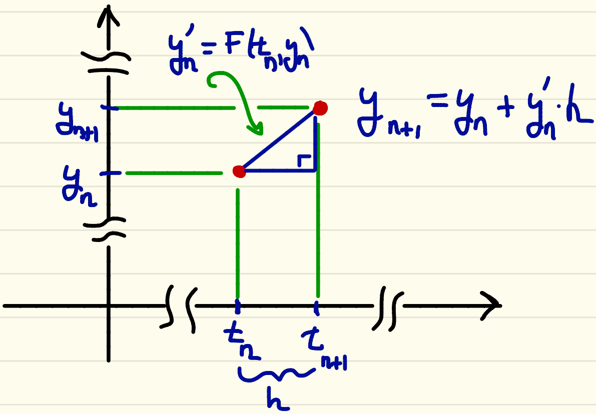


FIGURE 2.5. Euler's method applied to the initial value problem $y' = y$, $y(0) = 1.0$, with step size $h = 0.1$.

The method in general:

To solve the IVP

$$\begin{cases} \dfrac{dy}{dt} = F(t, y) \\ y(t_0) = y_0 \end{cases}$$

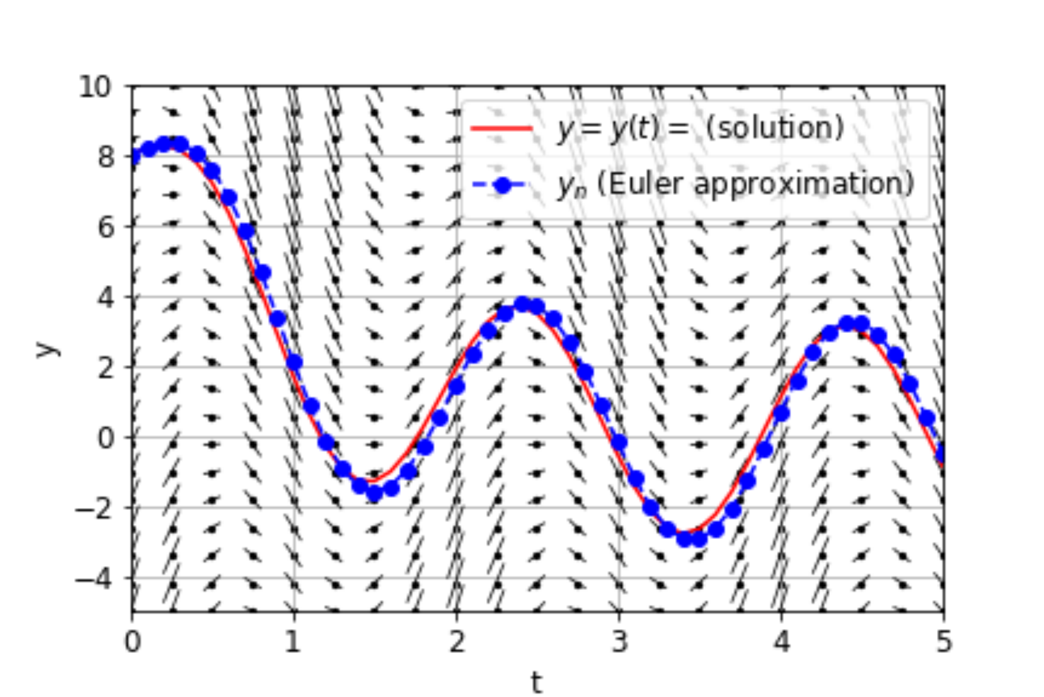


Proceed as follows:

(0) Choose a step size $h > 0$.
(1) Set $n = 0$.
(2) Set $y'_n = F(t_n, y_n)$.
(3) Set $t_{n+1} = t_n + h$.
(4) Set $y_{n+1} = y_n + y'_n \cdot h$
(5) Increase n by one and go to step (2).

Example

$$y' + y = 10\cos(\pi t), \quad y(0) = 8 \qquad h = 0.1$$



Caution: h can't be too big!

$$\frac{dy}{dt} = (10-y)y, \quad y(0)=1 \qquad h = 0.2$$

t	yapprox	yexact	error
0.	1.	1.	0.
0.2	2.8	4.50853	-1.70853
0.4	6.832	8.58486	-1.75286
0.6	11.1608	9.78178	1.37897
0.8	8.56977	9.9699	-1.40012
1.	11.0211	9.99592	1.0252
1.2	8.77035	9.99945	-1.2291
1.4	10.9272	9.99993	0.927318
1.6	8.9008	9.99999	-1.09919
1.8	10.8576	10.	0.857553
2.	8.99537	10.	-1.00463