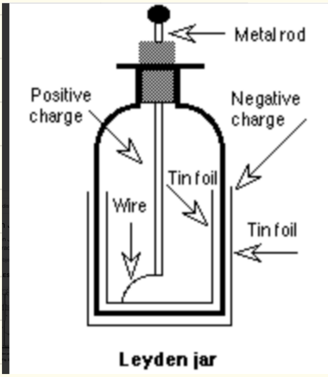
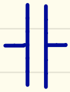
Lecture 03: direction fields

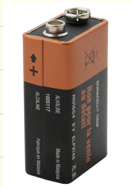
Capacitors



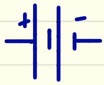
Leyden jar

Capacitors

Symbol: 

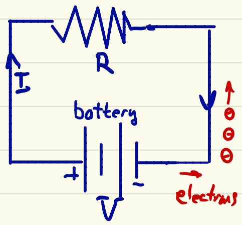


battery



Review:

Volt = Joule / coulomb (Electrical potential)



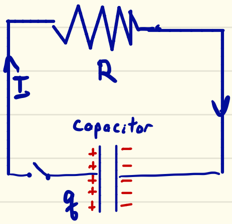
$V \approx 9$ volts electromotive force (emf) "pressure"

R = resistance (ohms Ω)

$V = RI$

I = current (amperes (coulombs / sec))

Replace battery with capacitor:



$V_c = q/C$

q = charge

$I = -\dfrac{dq}{dt}$ (current)

C = capacitance

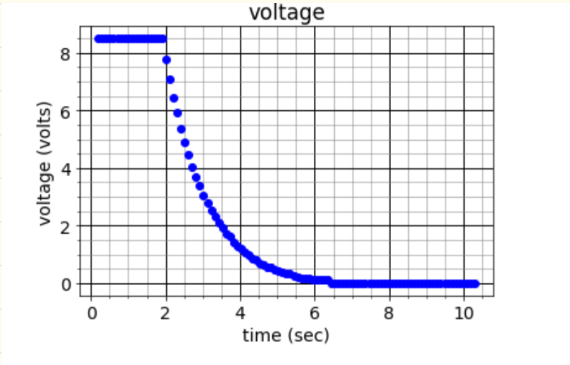
units: Farads = $\dfrac{\text{coulombs}}{\text{volt}}$

$V = RI$ becomes $q/C = R\left(-\dfrac{dq}{dt}\right)$

$$\Rightarrow \boxed{\frac{dq}{dt} + \frac{1}{RC}\, q = 0}$$

Experiment:

Discharge of capacitor

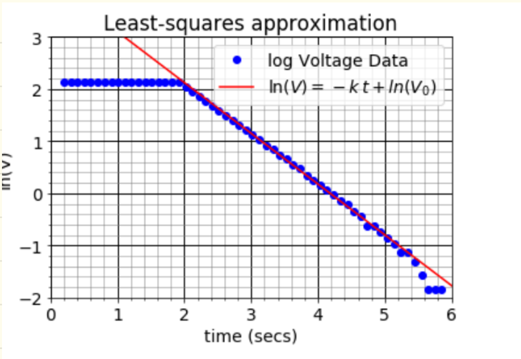


$$\frac{dq}{dt} + \frac{1}{RC}\, q = 0$$

$$\Rightarrow q = q_0\, e^{-t/RC}$$

Note: $V_c = q/C$ so can change variables:

$$\frac{d}{dt}(C \cdot V_c) + \frac{1}{RC}(C \cdot V_c) = 0$$



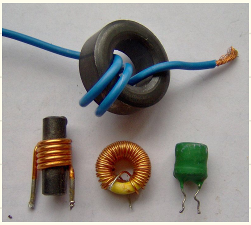
$$\Rightarrow \frac{dV_c}{dt} + \frac{1}{RC} V_c = 0$$

$$V = V_0\, e^{-t/RC}$$

Components of a typical circuit

Resistors

Capacitors



Inductors

- Resistors regulate, impede or set the flow of current through a particular path or impose a voltage reduction in an electric circuit as a result of this current flow. Resistance is denoted by R and is measured in *Ohms* (denoted by Ω).
- The capacitor is a component that has the ability or "capacity" to store energy in the form of an electric charge like a small battery. Capacitance is denoted by C and is measured in *Farads* (denoted by F) or micro[2] Farads (denoted by μF).
- An inductor is a coil of wire that induces a magnetic field within itself or within a central core as a direct result of current passing through the coil. Inductance[3] is denoted by L and is measured in *Henries.* (denoted by H) or in *micro Henries* (denoted by μH)

Units

q charge — coulomb (C)
(6.24×10^{18} protons)

I current — ampere (A)
(coulomb/sec)

V Emf — volts (V)

R resistance — Ohms (Ω)

C capacitance — Farads (F)

Joule = Newtons · Meter
= Volts · Coulomb

$$\text{Volt} = \frac{\text{Joule}}{\text{Coulomb}}$$

Watt = Joul/sec. = Volt-Ampere

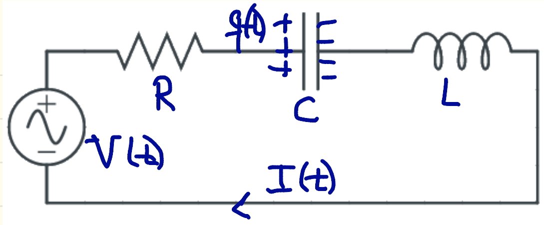
Benjamin Franklin (1706–1790)
James Watt (1736–1819)
Charles-Augustin de Coulomb 1737–1806
Alessandro Volta 1745–1827
Von Kleist 1745–1746 (Leyden Jar)
André-Marie Ampère 1775–1836
Michael Faraday 1791–1867
James Prescott Joule (1818–1889)
James Clerk Maxwell 1831–1879
Heinrich Hertz 1857–1894

Note:

$mF = 10^{-3}F$ mili-F
$\mu F = 10^{-6} F$ micro-F
$nF = 10^{-9}F$ nano-F
$pF = 10^{-12}F$ pico-F

(More about)

Electrical Circuits



$$I = \frac{dq}{dt}$$

Kirkhoff's (Voltage) Law

$$-V(t) + V_R(t) + V_C(t) + V_L(t) = 0$$

$$V_R = R\,I, \quad V_C = \frac{q}{C}, \quad V_L = L\frac{dI}{dt}$$

So

$$\boxed{-V(t) + RI + q/C + L\frac{dI}{dt} = 0}$$

$$\Rightarrow \quad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V(t)$$

Can rewrite in terms of other quantities.

E.g. $$LC\frac{d^2V_C}{dt^2} + RC\frac{dV_C}{dt} + V_C = V(t)$$

or $$\frac{d^2V_C}{dt^2} + \frac{R}{L}\frac{dV_C}{dt} + \frac{1}{LC}V_C = \frac{1}{LC}V(t)$$

Example.

$$RI + \frac{q}{C} + L\frac{dI}{dt} = V(t)$$

Differentiate:

$$R\frac{dI}{dt} + \frac{1}{C}I + L\frac{d^2I}{dt^2} = \frac{d}{dt}V(t)$$

$$\Leftrightarrow \quad L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \dot{V}(t)$$

$V_R = RI$ so can rewrite this to get

$$\frac{L}{R}\frac{d^2V_R}{dt^2} + \frac{dV_R}{dt} + \frac{1}{RC}V_R = \dot{V}(t)$$

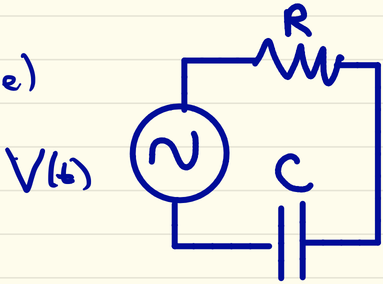
$$\Rightarrow \quad \frac{d^2V_R}{dt^2} + \frac{R}{L}\frac{dV_R}{dt} + \frac{1}{LC}V_R = \frac{R}{L}\dot{V}(t)$$

<u>Suggestion:</u> Try to find an ODE for V_L

Special Cases: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V(t)$

RC-circuit

(no inductance)

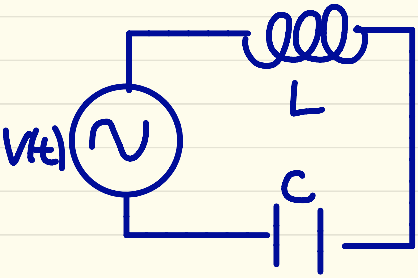


$$R\frac{dq}{dt} + \frac{1}{C}q = V(t)$$

$$\boxed{\frac{dV_C}{dt} + \frac{1}{RC}V_C = \frac{V(t)}{RC}}$$

LC-circuit

(no resistance)



$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = V(t)$$

$$\boxed{\frac{dV_C}{dt} + \frac{1}{LC}V_C = \frac{V(t)}{LC}}$$

Changing units

Example 

$R = 32\,\Omega$

$C = 33\,\mu F$ (μF: micro Farads)

$$\frac{dq}{dt} + \frac{1}{RC}q = 0 \qquad RC = 32 \times (33 \times 10^{-6})\ \text{sec}$$

$$= 1.056 \times 10^{-2}\ \text{sec.}$$

$$\frac{dq}{dt} + \frac{1}{1.056 \times 10^{-2}}q = 0$$

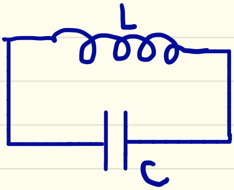
deci	d	$1000^{-1/3}$	10^{-1}	0.1
centi	c	$1000^{-2/3}$	10^{-2}	0.01
milli	m	1000^{-1}	10^{-3}	0.001
micro	μ	1000^{-2}	10^{-6}	0.000 001
nano	n	1000^{-3}	10^{-9}	0.000 000 001
pico	p	1000^{-4}	10^{-12}	0.000 000 000 001

Change to time in milliseconds:

$$\begin{cases} t: \text{sec} \\ \tau: \text{millisec} \end{cases} \qquad \tau = 10^3 \cdot t$$

$$\frac{dq}{dt} = \frac{d\tau}{dt}\cdot\frac{dq}{d\tau} = 10^3\frac{dq}{d\tau} \Rightarrow \frac{dq}{d\tau} + \frac{1}{10.56}q = 0$$

$$\frac{dq}{dt} = \frac{d\tau}{dt}\frac{dq}{d\tau}$$



$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$$

$$\frac{d^2q}{dt^2} = \left(\frac{d\tau}{dt}\right)^2\frac{d^2q}{d\tau^2} = 10^6\frac{d^2q}{d\tau^2}$$

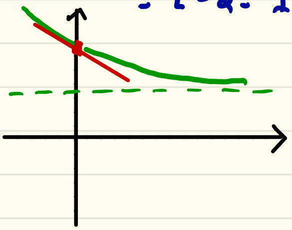
$$\begin{cases} C = 10\,\mu F = 10^{-5} F \\ L = 10\ mH = 10^{-2} H \end{cases} \qquad LC = 10^{-7}\ \text{sec}^2$$

$$\frac{d^2q}{dt^2} + 10^7 q = 0 \Rightarrow 10^6\frac{d^2q}{d\tau^2} + 10^7 q = 0 \Rightarrow \frac{d^2q}{d\tau^2} + 10q = 0$$

The Geometry of First Order ODE's

<u>Example</u> $\frac{dy}{dt} = 1 - y$

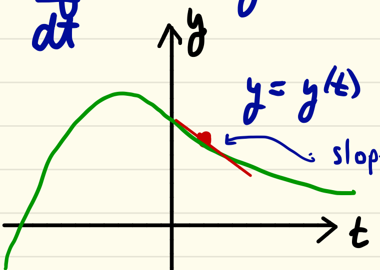
$y'(0) = 1 - y(0)$
$= 1 - 2 = -1$



$y(t) = e^{-t} + 1$

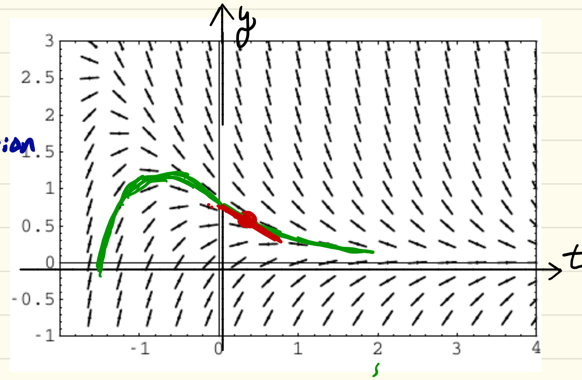
$y'(t) = -e^{-t}$

$\frac{dy}{dt} = F(t, y)$



$y = y(t)$ a solution

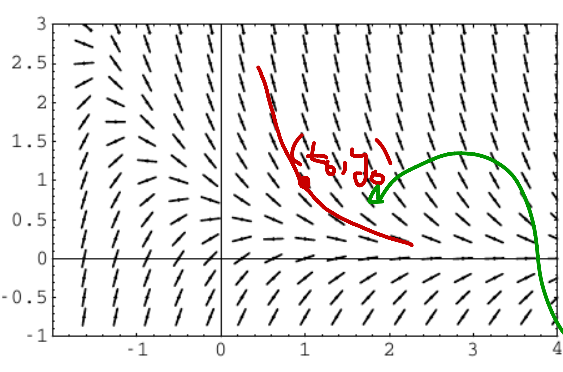
slope $= F(t, y)$
$= y'(t)$



<u>direction field</u>

(also called <u>slope field</u>
<u>field of line elements</u>)

Direction Fields

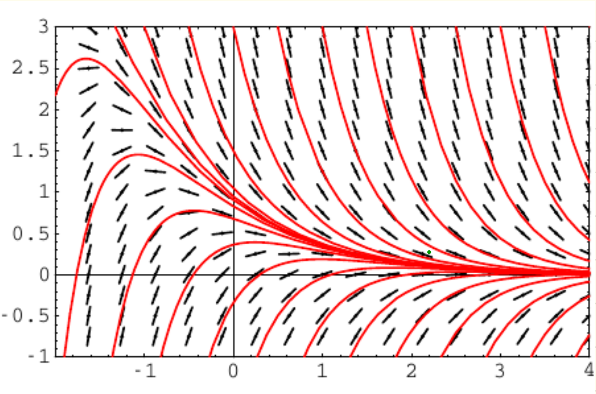


General soln:

$$y = y(t, C)$$

I.V.P.

$$\begin{cases} \dfrac{dy}{dt} = F(t,y) \\ y(t_0) = y_0 \end{cases}$$



Unique solns $\Longleftrightarrow$ integral curves do not cross!

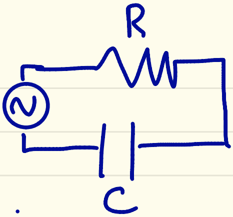
Unique soln of I.V.P.:

$$\begin{cases} \dfrac{dy}{dt} = F(t,y) \quad \text{"nice"} \\ y(t_0) = y_0 \end{cases}$$

initial value problem

Theorem If F is "nice" IVP has unique solution.

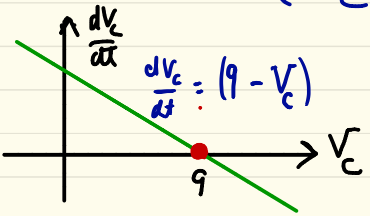
Example

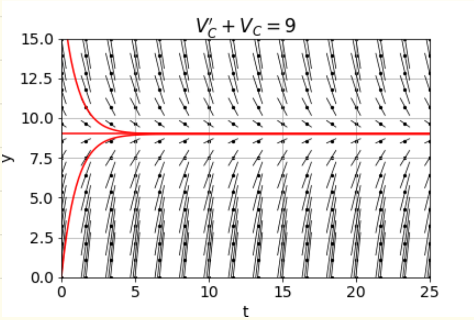


$$\frac{dV_C}{dt} + \frac{1}{RC} V_C = \frac{V(t)}{RC}$$

$R = 1\ k\Omega$.
$C = 1000\,\mu F$.
$RC = 1$ sec.
$V(t) = 9.0\ V$.

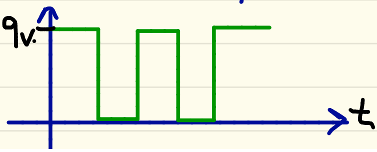
$$\begin{cases} \dfrac{dV_C}{dt} + V_C = 9 \\ V_C(0) = 0 \end{cases}$$

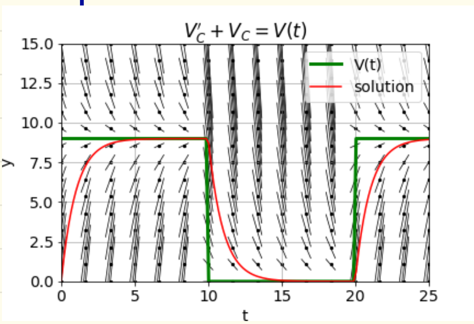


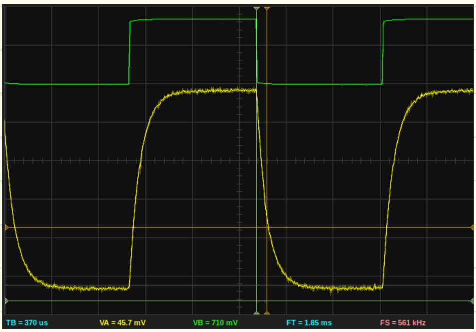


Let $V(t)$ be square wave:

$$\frac{dV_c}{dt} = V(t) - V_c$$



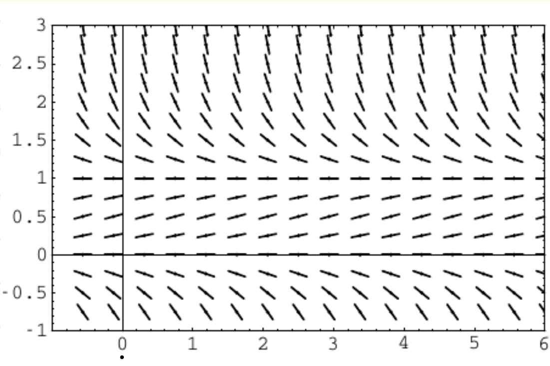


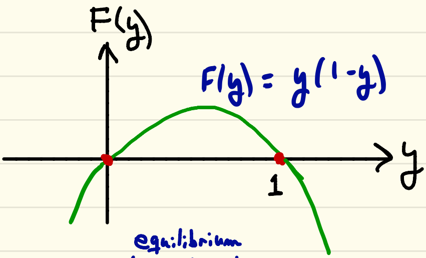


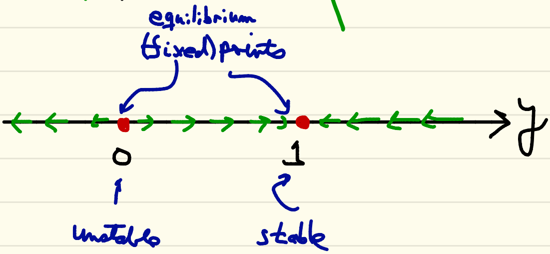
Special case:

$$\frac{dy}{dt} = F(y) \quad \text{autonomous DE}$$

Example $\frac{dy}{dt} = y(1-y)$

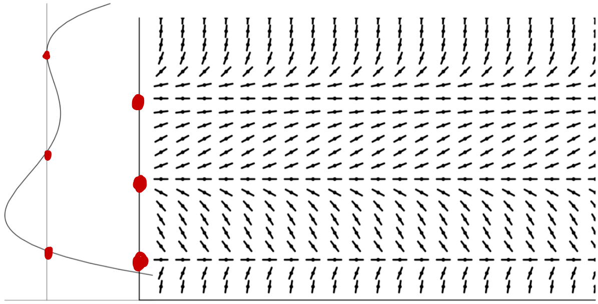






Example

Consider the ODE $y' = f(y)$ where the graph of $f(y)$ is displayed below (flipped along the diagonal and aligned with the direction field):



Question: Where are the equilibrium points? Which ones are stable and which ones are unstable?