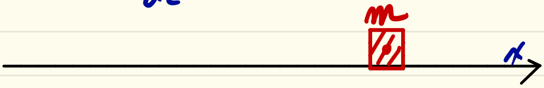
Lecture 02

(Modeling)

Newton's Second Law of Motion

Force = mass × acceleration

$$m \frac{d^2x}{dt} = F(t, x, dx/dt)$$



$v = \frac{dx}{dt}$: velocity, $a = \frac{dv}{dt}$

units:

mks:
- length : meters
- mass : kilograms
- time : seconds
- force : Newtons
- energy : Joules

British:
- length feet (0.3 m)
- mass slugs (14.6 kg)
- time seconds
- force pounds (4.45 N)
- energy foot-pounds (1.36 J)

Example (Free fall)

g : acceleration due to gravity $\approx 9.8\ m/sec^2$

$F = mg$ force due to gravity

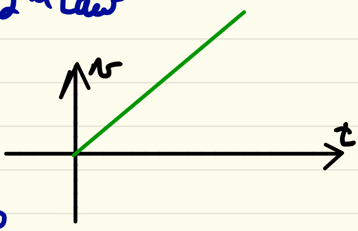
x = distance object has fallen (in meters)

m = mass (in kilograms)

$v = \frac{dx}{dt}$: (downward velocity)

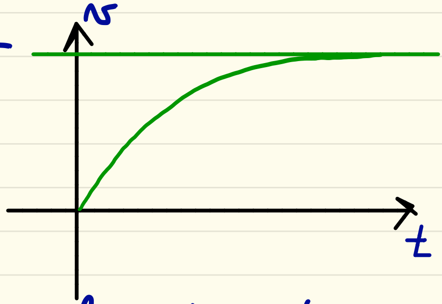
$$m \frac{dv}{dt} = mg \quad \text{(Newton's 2nd Law)}$$

$$\Rightarrow \begin{cases} \frac{dv}{dt} = g \\ v(0) = 0 \end{cases} \Rightarrow \begin{cases} v = gt + C \\ g \cdot (0) + C = 0 \end{cases}$$



Free fall in air:

$$m \frac{dv}{dt} = mg - \gamma v$$



γ = drag coefficient

units: kg/sec

$$\begin{cases} \dfrac{dv}{dt} + \dfrac{\gamma}{m} v = g \\ v(0) = 0 \end{cases}$$

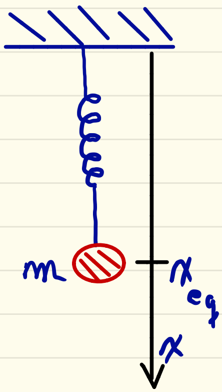
$$v_\infty = \lim_{t \to \infty} v(t) = \text{terminal velocity}$$

$$= mg/\gamma$$

(when $mg = \gamma v$)

$$v(t) = \frac{mg}{\gamma}\left(1 - e^{-\gamma/m\, t}\right)$$

Free fall with bungie cord (spring)



$$m \frac{d^2x}{dt^2} = mg - \gamma \frac{dx}{dt} - kx$$

$$\frac{d^2x}{dt^2} + \left(\frac{\gamma}{m}\right)\frac{dx}{dt} + \left(\frac{k}{m}\right) x = g$$

Units: $\frac{\text{meters}}{\text{sec}^2}$ $\quad \frac{1}{\text{sec}}$ $\quad \frac{\text{meters}}{\text{sec}}$ $\quad \frac{1}{\text{sec}^2}$ $\quad$ meters

Equilibrium position: $\frac{k}{m} x_{eq} = g$

Change variables:

$y = x - x_{eq}$

$$\boxed{\frac{d^2y}{dt^2} + \left(\frac{\gamma}{m}\right)\frac{dy}{dt} + \frac{k}{m} y = 0}$$

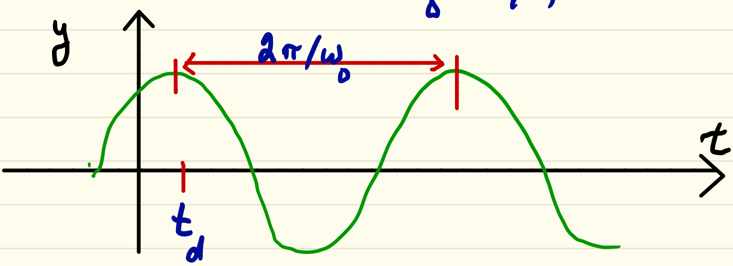
Harmonic Oscillator ($\gamma = 0$)

$$\frac{d^2y}{dt^2} + \left(\frac{k}{m}\right) y = 0$$

Let $\omega_0 = \sqrt{k/m}$ Units: 1/sec

$$\boxed{\frac{d^2y}{dt^2} + \omega_0^2\, y = 0}$$

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$
$$= A \cos(\omega_0 t - \phi)$$



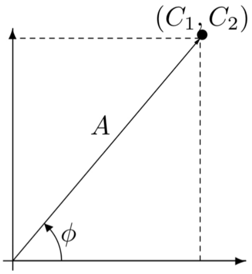
$t_d = \phi/\omega_0$ $\qquad T = \dfrac{2\pi}{\omega_0}$ period

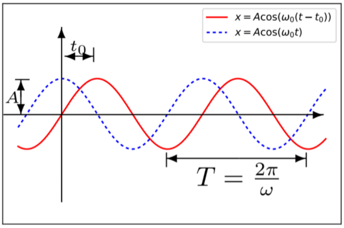
The phase-shift formula:

$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t) = A\cos(\omega t - \phi) = A\cos\left(\omega(t - t_0)\right)$,

where $t_0 = \dfrac{\phi}{\omega}$; $A = \sqrt{C_1^2 + C_2^2}$,

$\cos(\phi) = \dfrac{C_1}{\sqrt{C_1^2+C_2^2}}$. $\sin(\phi) = \dfrac{C_2}{\sqrt{C_1^2+C_2^2}}$, $\tan(\phi) = \dfrac{C_2}{C_1}$

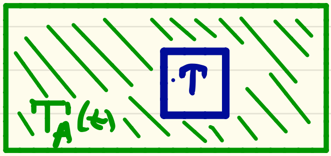




- **Newton's Law of Cooling**

$T_A(t)$: ambient temperature

$T = T(t)$: temperature of small object



$$\boxed{\frac{dT}{dt} = -k\left(T - T_A(t)\right)}$$

Can rewrite as

$$\boxed{\frac{dT}{dt} + kT = kT_A(t)}$$

Experiment:

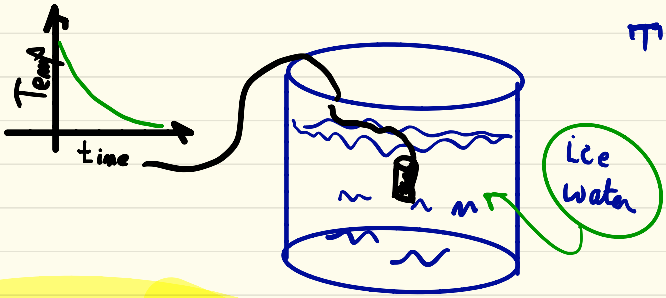
$T_A(t) = 0^\circ C$ (ice water)

T = temperature of sensor

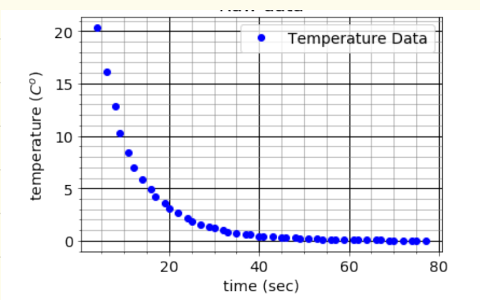
$$\begin{cases} \dfrac{dT}{dt} = -k(T - 0) \\ T(0) = T_0 \end{cases} \quad \text{(I.V.P)}$$

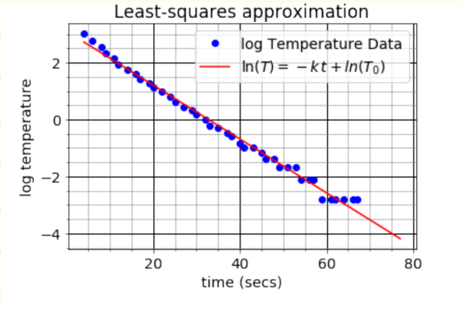
Solution:

$$T = T_0 e^{-kt}$$

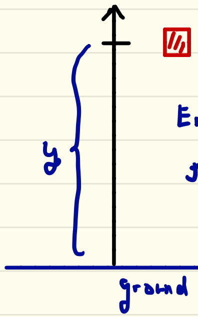


(Show demo.)





Gravitational Potential



m = mass

$$\text{Energy} = (g \cdot y)\cdot m = (mg) \times (y)$$

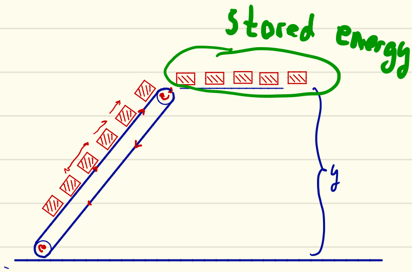
Force × distance

$$\text{Joules} = U \cdot m$$

$$\left(\frac{\text{Joules}}{\text{kg}}\right) \cdot \text{kg}$$

$U(y)$ = "gravitational potential"

$$= 9.8\, y \ \text{J/kg}$$

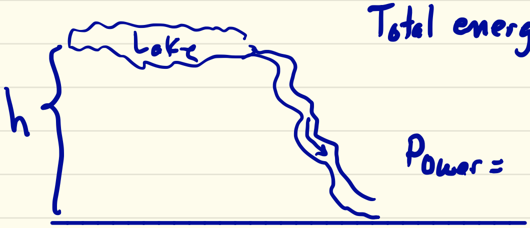


Power = rate that energy is used

units: Watt   1 Watt = 1 Joule/sec.

Example: Hydroelectric power

mass density of water $= 1 \ \text{kg/liter}$



$$\text{Total energy} = (9.8 \cdot h)\left(\text{volume of lake}\right)$$

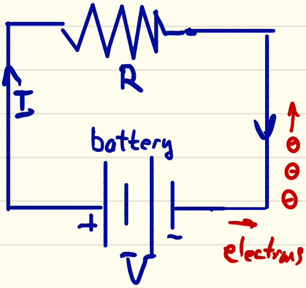
$$\text{Power} = (9.8 \cdot h) \cdot \frac{1\ \text{kg}}{\text{liter}} \cdot \frac{\#\text{liters}}{\text{sec}}$$

Intro to Electrical Circuits

Electrical Potential energy

Electric charge: coulomb ($\approx$ charge on 6.25×10^{18} protons)

Volt = Joule / coulomb (Electrical potential)



$V \approx 9$ volts electromotive force (emf) "pressure"

R = resistance (ohms Ω)

$V = RI$

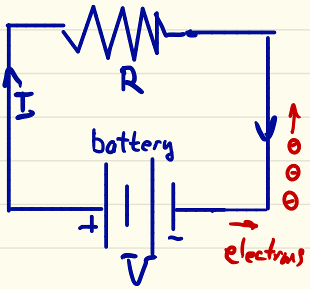
I = current (amperes) (coulombs / sec)

Power = $V \cdot I$

volts · ampers

$\frac{\text{Joules}}{\text{Coulomp}} \; \frac{\text{coulombs}}{\text{sec}}$ = watts

Volt = Joule / coulomb (Electrical potential)



$V \approx 9$ volts   electromotive force (emf) "pressure"

R = resistance (ohms Ω)

$$V = RI$$

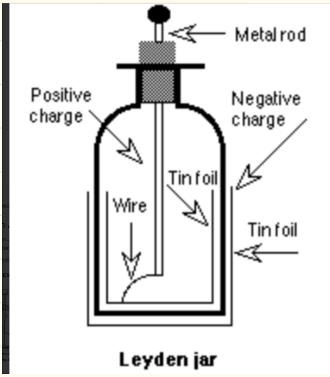
I = current (amperes (coulombs / sec))

$$\text{Power} = V \cdot I = V \cdot \left(\frac{V}{R}\right) = \frac{V^2}{R}$$

Example: Electric stove: $R \approx 25\,\Omega$ $V = 220$ Volts

$$\text{Power} \approx V \cdot I = (220)\left(\frac{220}{25}\right) \approx 2000 \text{ Watts}$$

Capacitors

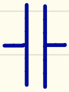


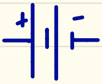
Leyden jar

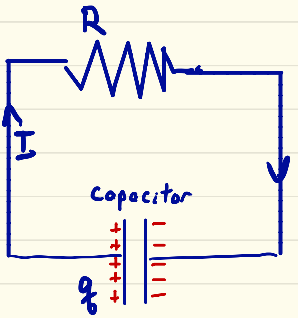
Capacitors

battery

Symbol:







$V_c = q/C$ $\quad C =$ capacitance

units: Farads $= \dfrac{\text{coulombs}}{\text{volt}}$

$q =$ charge $\quad I = -\dfrac{dq}{dt}$ (current)

$$q/C = R\left(-\frac{dq}{dt}\right) \Rightarrow \boxed{\frac{dq}{dt} + \frac{1}{RC}q = 0}$$

Demo here!

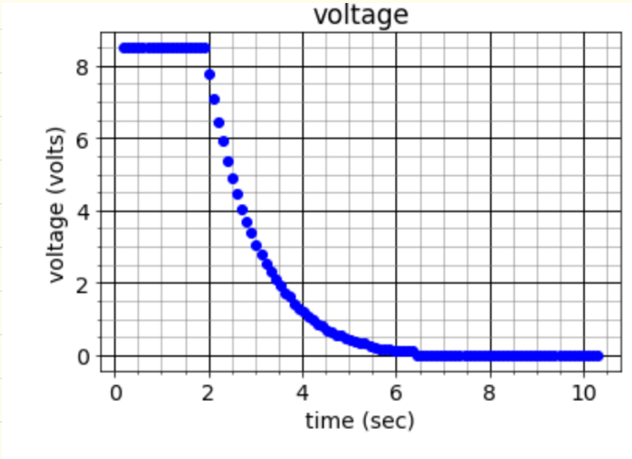
Experiment

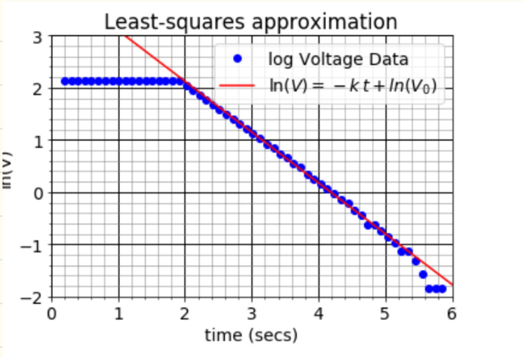
$$\frac{dq}{dt} + \frac{1}{RC} q = 0$$

$$\Rightarrow q = q_0 e^{-t/RC}$$

Note: $V = q/C$ so

$$V = V_0 e^{-t/RC}$$

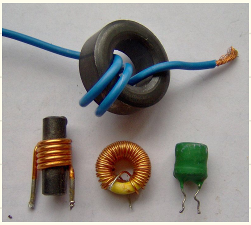




Components of a typical circuit

Resistors

Capicators



Inductors

- Resistors regulate, impede or set the flow of current through a particular path or impose a voltage reduction in an electric circuit as a result of this current flow. Resistance is denoted by R and is measured in *Ohms* (denoted by Ω).
- The capacitor is a component that has the ability or "capacity" to store energy in the form of an electric charge like a small battery. Capacitance is denoted by C and is measured in *Farads* (denoted by F) or micro[2] Farads (denoted by μF).
- An inductor is a coil of wire that induces a magnetic field within itself or within a central core as a direct result of current passing through the coil. Inductance[3] is denoted by L and is measured in *Henries*. (denoted by H) or in *micro Henries* (denoted by μH)

Units

q charge — coulomb (C)
(6.24×10^18 protons)

I current — ampere (A)
(coulomb/sec)

V Emf — volts (V)

R resistance — Ohms (Ω)
C capacitance — Farads (F)

Joule = Newtons · Meter
= Volts · coulomb

$\text{Volt} = \dfrac{\text{Joule}}{\text{coulomb}}$

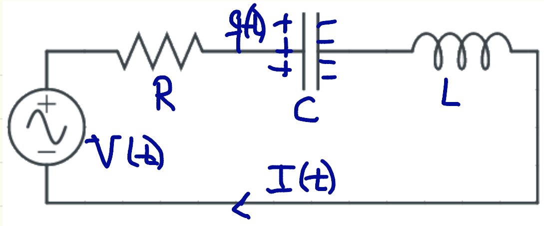
Watt = Joul/sec = Volt-Ampere

Benjamin Franklin (1706–1790)
James Watt (1736–1819)
Charles-Augustin de Coulomb 1737–1806
Alessandro Volta 1745–1827
Von Kleist 1745–1746 (Leyden Jar)
André-Marie Ampère 1775–1836
Michael Faraday 1791–1867
James Prescott Joule (1818–1889)
James Clerk Maxwell 1831–1879
Heinrich Hertz 1857–1894

Note:
$mF = 10^{-3} F$ milli-F
$\mu F = 10^{-6} F$ micro-F
$nF = 10^{-9} F$ nano-F
$pF = 10^{-12} F$ pico-F

(More about)

Electrical Circuits



Kirkhoff's (Voltage) Law

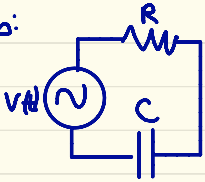
$$-V(t) + V_R(t) + V_C(t) + V_L(t) = 0$$

$$\begin{cases} q : \text{charge on capacitor} \\ I = \dfrac{dq}{dt} = \text{current} \end{cases} \qquad \begin{cases} V_R = R \cdot I \\ V_C = \dfrac{q}{C} \\ V_L = L\dfrac{dI}{dt} = L\dfrac{d^2q}{dt^2} \end{cases}$$

$$\boxed{L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V(t)}$$

Special Cases:

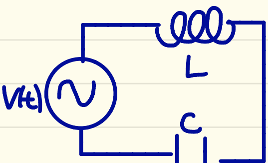
RC-circuit



$$R\frac{dq}{dt} + \frac{1}{C}q = V(t)$$

$$\boxed{\frac{dV_C}{dt} + \frac{1}{RC}V_C = \frac{V(t)}{RC}}$$

LC-circuit

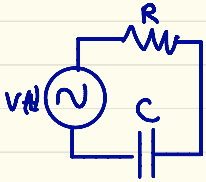


$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = V(t)$$

$$\boxed{\frac{d^2V_C}{dt^2} + \frac{1}{LC}V_C = \frac{V(t)}{LC}}$$

Special Cases:

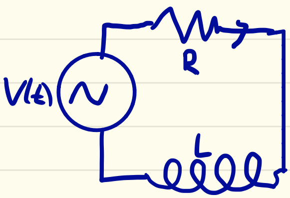
RC-circuit



$$R\frac{dq}{dt} + \frac{1}{C}q = V(t)$$

$$\boxed{\frac{dV_C}{dt} + \frac{1}{RC}V_C = \frac{V(t)}{RC}}$$

RL-circuit



$$L\frac{dI}{dt} + RI = V(t)$$

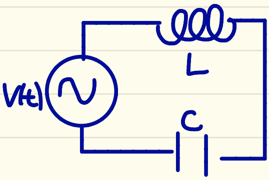
$$\frac{dI}{dt} + \frac{R}{L}I = \frac{V(t)}{L} \quad \text{or}$$

Since $I = V_R/R$

$$\frac{L}{R}\frac{dV_R}{dt} + V_R = V(t)$$

$$\boxed{\frac{dV_R}{dt} + \frac{R}{L}V_R = \frac{R}{L}V(t)}$$

LC-circuit



$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = V(t)$$

$$\boxed{\frac{d^2V_C}{dt^2} + \frac{1}{LC}V_C = \frac{V(t)}{LC}}$$