Lecture 01

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- Class webpage:

https://sites.math.washington.edu/~duchamp/courses/307/

- WebWork webpage:

https://courses1.webwork.maa.org/webwork2/Washington-Math307C

User name: Your UW user name

Initial password: UW I.D.

<u>Note</u>: First homework due Tuesday!

What is an ODE (ordinary differential equation)?

1^{st} order ODE: $\frac{dy}{dt} = F(t, y)$: relation among t, y, and $\frac{dy}{dt}$

2^{nd} order ODE: $\frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right)$: relation among t, y, $\frac{dy}{dt}$, and $\frac{d^2y}{dt^2}$

Examples:

$\frac{dy}{dt} = 6t^2 + e^t$, $\quad \frac{dy}{dt} = 7y$, $\quad \frac{dy}{dt} + 6y = e^t$,

$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = \cos(t) \qquad t^2 y'' + 4t y' - 4y = 0$

What does it mean to "solve" an ODE?

1st order: A function $y = y(t)$ is a solution of $y' = F(t,y)$ if $y'(t) = F(t, y(t))$ for all t on some interval.

2nd order: A function $y = y(t)$ is a solution of $y'' = F(t,y,y')$ if $y''(t) = F(t, y(t), y'(t))$ for all t on some interval.

Example. $y = t^3 + t$ is a solution of $y' = 3t^2 + 1$.

Check: $(t^3+t)' = 3t^2+1$

Example. $y = e^{-4t}$ is a solution of $y' + 4y = 0$.

$(e^{-4t})' + 4\cdot(e^{-4t})$

$-4e^{-4t} + 4e^{-4t} = 0$

Example. $y = 3\cos(2t)$ is a solution of $y'' + 4y = 0$

$y' = -6\sin(2t)$

$y'' = -12\cos(2t)$

$(-12\cos(2t)) + 4(3\cos(2t))$

$= 0$

Example. Find all solutions of the ODE $t^2y'' + 4ty' - 4y = 0$ of the form $y(t) = t^r$.

Solution.

$y = t^r$

$y' = rt^{r-1}$

$y'' = r(r-1)t^r$

$t^2 y'' + 4ty - 4'y =$

$r(r-1)t^r + 4rt^r - 4t^r$

$= (r^2 - r + 4r - 4)t^r = 0$

$= (r^2 + 3r - 4)t^r$

$= (r+4)(r-1)t^r$

Equal to 0 if and only if $r = -4$ or $r = 1$

So solutions are t^{-4} and t

General Soln of ODEs

Example What is the most general solution of the ODE $y' = 6\cos(2t)$?

Soln.

$$y(t) = 3\sin(2t) + C$$

Example $\frac{dy}{dt} + 5y = 0$.

$y(t) = Ce^{-5t}$ is the "general solution"

That is every solution is of this form.

Initial Value Problems (I.V.P.)

First order initial value problems

$$\boxed{\frac{dy}{dt} = F(t,y), \quad \underbrace{y(t_0) = y_0}_{\text{initial condition}}}$$

Example. $\begin{cases} y' + 4y = 8 \\ y(0) = 7 \end{cases}$

General solution: $y(t) = 2 + Ce^{-4t}$

$y(0) = 7 \Rightarrow 2 + C = 7 \Rightarrow C = 5$

So $y(t) = 2 + 5e^{-4t}$ is the solution to the I.V.P.

Second Order Initial Value Problems:

$$\boxed{\frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right) \quad y(t_0) = y_0, \; y'(t_0) = y_0'}$$

Example. $y'' + 4y = 0, \quad y(0) = 3 \quad y'(0) = 5$

General Solution: $y(t) = c_1 \cos(2t) + c_2 \sin(2t)$

Initial Conditions:
$$\begin{cases} y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 = 3 \\ y'(0) = -4c_1 \sin(0) + 4c_2 \cos(0) = 4c_2 = 5 \end{cases}$$

So
$$y(t) = 3\cos(2t) + \frac{5}{4}\sin(2t)$$
is the solution to the I.V.P.

Can rewrite as $y(t) = A\cos(2t - \phi)$:

https://physics.stackexchange.com/questions/159021/why-is-the-harmonic-oscillator-so-important

$$y'' + \omega_0^2 \, y = f(t)$$

external force

Why are these mathematical models important?

In short, the significance of both the classical and quantum harmonic oscillator comes from their ubiquity -- they are absolutely everywhere in physics. We could spend an enormous amount of time trying to understand why this is so, but I think it's more productive to just see the pervasiveness of these models with some examples. I'd like to remark that although it's certainly true that the harmonic oscillator is a simple an elegant model, I think that answering your question by saying that it's important *because* of this fact is kind of begging the question. Simplicity is not a sufficient condition for usefulness, but in this case, we're fortunate that the universe seems to really "like" this system.

Where do we find the classical harmonic oscillator?

(this is by no means an exhaustive list, and suggestions for additions are more than welcome!)

1. **Mass on a Hooke's Law spring** (the classic!). In this case, the classical harmonic oscillator equation describes the *exact* equation of motion of the system.
2. Many (but not all) classical situations in which a particle is moving **near a local minimum of a potential** (as rob writes in his answer). In these cases, the classical harmonic oscillator equation describes the approximate dynamics of the system provided its motion doesn't appreciably deviate from the local minimum of the potential.
3. Classical systems of **coupled oscillators**. In this case, if the couplings are linear (like when a bunch of masses are connected by Hooke's Law springs) one can use linear algebra magic (eigenvalues and eigenvectors) to determine normal modes of the system, each of which acts like a single classical harmonic oscillator. These normal modes can then be used to solve the general dynamics of the system. If the couplings are non-linear, then the harmonic oscillator becomes an approximation for small deviations from equilibrium.
4. **Fourier analysis and PDEs**. Recall that Fourier Series, which represent either periodic functions on the entire real line, or functions on a finite interval, and Fourier transforms are constructed using sines and cosines, and the set $\{\sin, \cos\}$ forms a basis for the solution space of the classical harmonic oscillator equation. In this sense, any time you are using Fourier analysis for signal processing or to solve a PDE, you are just using the classical harmonic oscillator on massively powerful steroids.
5. **Classical electrodynamics**. This actually falls under the last point since electromagnetic waves come from solving Maxwell's equations which in certain cases yields the wave equation which can be solved using Fourier analysis.

Where do we find the quantum harmonic oscillator?