Math 564

Fall 2019

Homework 3

1. Let X be the space obtained from the standard 2-simplex with vertices v_0, v_1, v_2 by identifying the edges v_0v_1, v_1v_2, and v_2v_0 linearly with v_1v_2, v_2v_0, and v_0v_1 respectively. Compute $\widetilde{H}_*X$. (Hint: Do not try to triangulate X.)

2. Prove that for any space X and any $n \geq 0$ there are (natural) isomorphisms

$$H_q(X \times S^n, X \times e) \approx H_{q-n}(X).$$

(Hint: Use induction on n and the fact that if Y is contractible, then $H_*(X \times Y, X \times y_0) = 0$.) Next prove that there are natural isomorphisms

$$H_q(X \times S^n) \approx H_qX \oplus H_{q-n}X.$$

Use this to prove that if a space has the homotopy type of a finite product of spheres, then the set of spheres which are the factors is unique.

3. A simplicial complex is said to be *homogenously n-dimensional* if every simplex is a face of some n-simplex of the complex. An *n-dimensional pseudomanifold* is a simplicial complex K such that

a) K is homogenously n-dimensional

b) Every $(n-1)$-simplex of K is the face of at most two n simplexes of K

c) If s and s' are n-simplexes of K, there is a finite sequence $s = s_1, s_2, \ldots, s_m = s'$ of n-simplices of K such that s_i and s_{i+1} have an $(n-1)$-face in common for $1 \leq i < m$.

The *boundary* of an n-dimensional pseudomanifold K, denoted $\dot{K}$ is defined to be the subcomplex of K generated by the set of $(n-1)$-simplexes which are faces of exactly one n-simplex of K.

For example, if M is a smooth connected manifold with boundary ∂M then there exists an n-dimensional pseudomanifold K such that

$$(M, \partial M) = (|K|, |\dot{K}|).$$

Now let s be an n-simplex of K. An *orientation* $\sigma(s)$ of s is just a generator of $H_n(\bar{s}, \dot{s})$. A collection of orientations

$$\{\, \sigma(s) : \ s \text{ an } n\text{-simplex of } K \,\}$$

is said to be *compatible* if for any $(n-1)$-simplex $t \in K \setminus \dot{K}$ which is a face of the two n-simplexes s_1 and s_2 of K, $\sigma(s_1)$ and $-\sigma(s_2)$ correspond under the homomorphisms

$$H_n(\bar{s}_1, \dot{s}_1) \to H_{n-1}(\dot{s}_1) \to H_{n-1}(\dot{s}_1, \dot{s}_1 \setminus t) \xleftarrow{\approx} H_{n-1}(\bar{t}, \dot{t})$$

and

$$H_n(\bar{s}_2, \dot{s}_2) \to H_{n-1}(\dot{s}_2) \to H_{n-1}(\dot{s}_2, \dot{s}_2 \setminus t) \xleftarrow{\approx} H_{n-1}(\bar{t}, \dot{t}).$$

An *orientation* of K is a compatible collection of orientations.

Let K be a finite n-dimensional pseudomanifold. If K has an orientation, prove that $H_n(K, \dot{K}) \approx \mathbb{Z}$ and that there exists a (unique) $z \in H_n(K, \dot{K})$ such that $\sigma(s)$ is the image of z under the homomorphisms

$$H_n(K, \dot{K}) \to H_n(K, K \setminus s) \xleftarrow{\approx} H_n(\bar{s}, \dot{s}).$$

Prove that if K is not orientable, then $H_n(K, \dot{K}) = 0$.