Math 564

Fall 2019

Homework 1

1. Let R be a commutative ring and let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be an exact sequence of R-modules. (This is defined just as an exact sequence of abelian groups except that all maps are required to be R-module maps.) Prove that the following are equivalent: (All maps and isomorphisms are R-module homomorphisms.)

i. A is a direct summand of B. (This is the same as saying that there exists a submodule C' of B with $C \approx C'$ via $p|C'$ and $B = \operatorname{im} i \oplus C'$.)

ii. There exists a homomorphism $q : C \to B$ such that $p \circ q = \mathrm{id}_C$.

iii. There exists a homomorphism $j : B \to A$ such that $j \circ i = \mathrm{id}_A$.

A sequence satisfying the above conditions is called split exact. Now prove that if C is a free R-module, then any exact sequence $0 \to A \to B \to C \to 0$ is split exact.

2. Prove the Five Lemma (Exercise 4 on page 46 of Vick). (You don't have to read any of Vick to do this problem.) Note that the Five Lemma implies that if $f : (X, A) \to (Y, B)$, and if any two of the three induced maps $H_*X \to H_*Y$, $H_*A \to H_*B$, $H_*(X, A) \to H_*(Y, B)$ are isomorphisms, then so is the other one.