NAME Solutions

Math 308E
Winter 2016

Midterm 1
January 29, 2016

Instructions

- Point totals for each problem are shown in parentheses.
- You must show all your work on the examination to receive credit. You must also use the techniques of this course on each problem; ask if you are not sure.
- Read each problem carefully. You will not receive credit if you misunderstand or misread a problem.
- Your work must be neat and organized.
- Be very careful with your arithmetic. None of the calculations or answers are too complicated.
- Make sure your test has 5 questions.

(6) 1. Find all solutions to the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 7 \\ -2x_1 - 4x_2 - x_3 &= -9 \\ -3x_1 - 6x_2 - x_3 &= -11 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ -2 & -4 & -1 & -9 \\ -3 & -6 & -1 & -11 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 2 & 10 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

equivalent to linear system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 7 \\ x_3 &= 5 \end{aligned}$$

$x_3 = 5$

$x_2 = s$ any real number

$x_1 = 2 - 2s$

(6) 2. Let $\mathbf{u}_1 = (1, 2, 3)$ and $\mathbf{u}_2 = (-1, 2, 5)$ be vectors in $\mathbf{R}^3$. For what value(s) of h is the vector $(h, 1, 4)$ in the span of $\{\mathbf{u}_1, \mathbf{u}_2\}$?

$(h,1,4)$ is in the span of $\{\vec{u}_1, \vec{u}_2\}$ if and only if the linear system represented by

$$\left[\begin{array}{cc|c} 1 & -1 & h \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{array}\right]$$

has a solution.

$$\left[\begin{array}{cc|c} 1 & -1 & h \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & h \\ 0 & 4 & 1-2h \\ 0 & 8 & 4-3h \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 0 & -1 & h \\ 0 & 4 & 1-2h \\ 0 & 0 & (4-3h)-2(1-2h) \end{array}\right]$$

Must have $(4-3h)-2(1-2h) = 0$

$$2 + h = 0$$

$$h = -2$$

(6) 3. Let $\mathbf{v}_1 = (2, 1, 2)$, $\mathbf{v}_2 = (3, -1, 1)$, and $\mathbf{v}_3 = (6, b, -2)$ be vectors in $\mathbf{R}^3$. For what value(s) of b is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ not linearly independent?

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent if and only if $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]\vec{x} = \vec{0}$ has only the trivial solution

$$\left[\begin{array}{ccc|c} 2 & 3 & 6 & 0 \\ 1 & -1 & b & 0 \\ 2 & 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & b & 0 \\ 2 & 3 & 6 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & b & 0 \\ 0 & 5 & 6-2b & 0 \\ 0 & 3 & -2-2b & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & b & 0 \\ 0 & 5 & 6-2b & 0 \\ 0 & 0 & -2-2b-\frac{3}{5}(6-2b) & 0 \end{array}\right]$$

nontrivial solutions $\Leftrightarrow$ $-2-2b-\frac{3}{5}(6-2b)=0$

$$-10-10b-18+6b=0$$

$$-4b-28=0$$

$$b=-7$$

(3) 4. Find a 3×3 matrix A such that the equation $A\mathbf{x} = \mathbf{0}$ has a unique solution but the equation

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

has no solution, or explain why such a matrix can't exist.

Such a matrix can't exist. If $A\vec{x} = \vec{0}$ has a unique solution, then the columns of A are linearly independent. But these are 3 vectors in $\mathbb{R}^3$, so they must span $\mathbb{R}^3$. This means that the equation $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$; in particular, for $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(3) 5. Find distinct non-zero vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ such that $\mathbf{u}_3$ is not in the span of $\{\mathbf{u}_1, \mathbf{u}_2\}$ but that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent, or explain why such vectors cannot exist.

Take $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.