NAME Solutions

Math 308E
Winter 2016

Midterm 1
January 29, 2016

Instructions

• Point totals for each problem are shown in parentheses.

• You must show all your work on the examination to receive credit. You must also use the techniques of this course on each problem; ask if you are not sure.

• Read each problem carefully. You will not receive credit if you misunderstand or misread a problem.

• Your work must be neat and organized.

• Be very careful with your arithmetic. None of the calculations or answers are too complicated.

• Make sure your test has 5 questions.
1. Find all solutions to the system of linear equations

\[
\begin{align*}
    x_1 + 2x_2 + x_3 &= 7 \\
    -2x_1 - 4x_2 - x_3 &= -9 \\
    -3x_1 - 6x_2 - x_3 &= -11
\end{align*}
\]

\[
\begin{bmatrix}
    1 & 2 & 1 & | & 7 \\
    -2 & -4 & -1 & | & -9 \\
    -3 & -6 & -1 & | & -11
\end{bmatrix} \rightarrow \begin{bmatrix}
    1 & 2 & 1 & | & 7 \\
    0 & 0 & 1 & | & 5 \\
    0 & 0 & 2 & | & 10
\end{bmatrix}
\rightarrow \begin{bmatrix}
    1 & 2 & 1 & | & 7 \\
    0 & 0 & 1 & | & 5 \\
    0 & 0 & 0 & | & 0
\end{bmatrix}
\]

equivalent to linear system

\[
\begin{align*}
    x_1 + 2x_2 + x_3 &= 7 \\
    x_3 &= 5
\end{align*}
\]

\[
\begin{align*}
    x_3 &= 5 \\
    x_2 &= s \text{ any real number} \\
    x_1 &= 2 - 2s
\end{align*}
\]
2. Let \( \mathbf{u}_1 = (1, 2, 3) \) and \( \mathbf{u}_2 = (-1, 2, 5) \) be vectors in \( \mathbb{R}^3 \). For what value(s) of \( h \) is the vector \((h, 1, 4)\) in the span of \( \{\mathbf{u}_1, \mathbf{u}_2\}\)?

\((h, 1, 4)\) is in the span of \( \{\mathbf{u}_1, \mathbf{u}_2\}\) if and only if the linear system represented by

\[
\begin{bmatrix}
1 & -1 & h \\
2 & 2 & 1 \\
3 & 5 & 4
\end{bmatrix}
\]

has a solution.

\[
\begin{bmatrix}
1 & -1 & h \\
2 & 2 & 1 \\
3 & 5 & 4
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & -1 & h \\
0 & 4 & 1-2h \\
0 & 8 & 4-3h
\end{bmatrix} 
\rightarrow
\begin{bmatrix}
0 & -1 & h \\
0 & 4 & 1-2h \\
0 & 0 & (4-3h)-2(1-2h)
\end{bmatrix}
\]

Must have

\[ (4-3h)-2(1-2h) = 0 \]

\[ 2+h = 0 \]

\[ h = -2 \]
3. Let \( \mathbf{v}_1 = (2, 1, 2) \), \( \mathbf{v}_2 = (3, -1, 1) \), and \( \mathbf{v}_3 = (6, b, -2) \) be vectors in \( \mathbb{R}^3 \). For what value(s) of \( b \) is the set \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) not linearly independent?

\[
\begin{align*}
\{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \text{ is linearly independent if and only if } \\
[\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3] \mathbf{x} = \mathbf{0} \text{ has only the trivial solution}
\end{align*}
\]

\[
\begin{bmatrix}
2 & 3 & 6 & 0 \\
1 & -1 & b & 0 \\
2 & 1 & -2 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1 & b & 0 \\
2 & 3 & 6 & 0 \\
2 & 1 & -2 & 0
\end{bmatrix} 
\rightarrow \begin{bmatrix}
1 & -1 & b & 0 \\
0 & 5 & 6-2b & 0 \\
0 & 3 & -2-2b & 0
\end{bmatrix}
\rightarrow \begin{bmatrix}
1 & -1 & b & 0 \\
0 & 5 & 6-2b & 0 \\
0 & 0 & -2-2b-\frac{3}{5}(6-2b) & 0
\end{bmatrix}
\]

Nontrivial solutions \( \Leftrightarrow \) 

\[
\begin{align*}
-2-2b-\frac{3}{5}(6-2b) &= 0 \\
-10-10b-18+6b &= 0 \\
-4b - 28 &= 0 \\
b &= -7
\end{align*}
\]
4. Find a $3 \times 3$ matrix $A$ such that the equation $Ax = 0$ has a unique solution but the equation

$$ Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} $$

has no solution, or explain why such a matrix can't exist.

Such a matrix can't exist. If $A\vec{x} = \vec{0}$ has a unique solution, then the columns of $A$ are linearly independent. But there are 3 vectors in $\mathbb{R}^3$, so they must span $\mathbb{R}^3$. This means that the equation $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$; in particular, for $\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. 

5. Find distinct non-zero vectors $u_1$, $u_2$, and $u_3$ such that $u_3$ is not in the span of $\{u_1, u_2\}$ but that $\{u_1, u_2, u_3\}$ is linearly dependent, or explain why such vectors cannot exist.

Take $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. 