Variational analysis with smooth substructure

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Goals

- Intuitive notion of identifiable sets.
- Existence, calculus, properties.
- Connection to critical cones (Generalized Reduction Lemma).
- Illustration: Spectral functions.

Critical points

Definition (Generalized critical points)

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- For convex f, critical points are global minimizers.
- If f is $\mathbf{C}^1$-smooth, criticality reduces to the classical condition $\nabla f(x) = 0$.

Motivation (Sensitivity Analysis)

Consider the perturbed functions

$$f_v(x) = f(x) - \langle v, x \rangle. \qquad [\textit{For simplicity}],$$

and suppose $\bar{x}$ is critical for $f_{\bar{v}}$, that is $\bar{v} \in \partial f(\bar{x})$.

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Sensitivity question: How do critical points of f_v, near $\bar{x}$, behave as v varies near $\bar{v}$? or equivalently how do solutions x_v of

$$v \in \partial f(x),$$

vary, as we perturb v near $\bar{v}$.

Motivating example

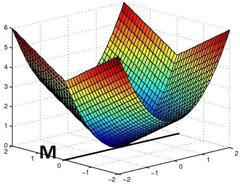

Figure: $f(x, y) = x^2 + |y|, \quad M = \{(t, 0) : -1 < t < 1\}$

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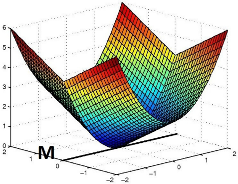

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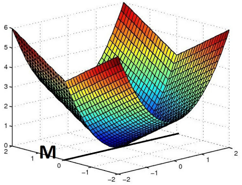

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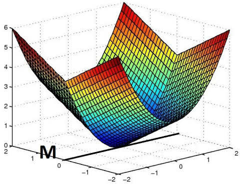

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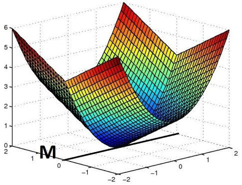

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- Only the restriction $f\big|_M$ matters!
- Goal: Look for small, well-behaved sets capturing only the essential information.

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Consider the system

$$v \in \partial f(x).$$

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Definition (Identifiable sets)

A set $M \subset \mathbf{R}^n$ is identifiable at $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$ if locally near $\bar{x}$ have

$$M = \pi_{\mathbf{R}^n}((U \times V) \cap \operatorname{gph} \partial f),$$

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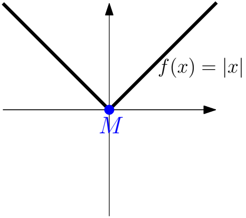

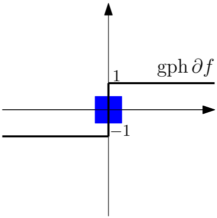

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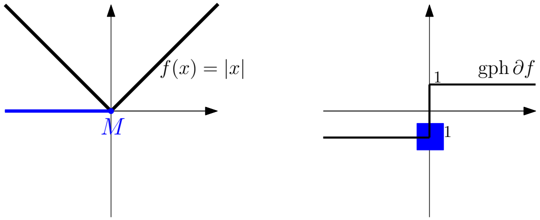

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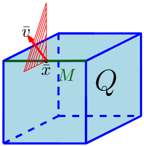

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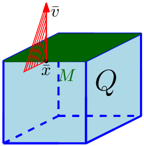

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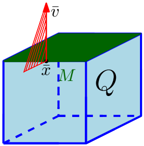

In this case $M = \bar{x} + K_Q(\bar{x}, \bar{v})$.

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- *$\bar{x}$ is a (strict) local minimizer of f $\iff$ $\bar{x}$ is a (strict) local minimizer of f on M.*
- *f grows quadratically near $\bar{x}$ $\iff$ f grows quadratically on M near $\bar{x}$.*

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Clearly all of $\mathbf{R}^n$ is identifiable at $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$ (not interesting).
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Definition

An identifiable set M at $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$ is locally minimal if

$$M' \text{ identifiable at } (\bar{x}, \bar{v}) \implies M \subset M', \text{ locally near } \bar{x}.$$

Existence and calculus

Locally minimal identifiable sets exist for

- fully amenable functions: $f(x) = g(F(x))$ where
  1. F is $\mathbf{C}^2$-smooth,
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A strong chain rule is available for composite functions

$$f(x) = g(F(x)).$$

Dimension Reduction

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Proposition (D, Lewis)

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May use this to study nonpolyhedral variational inequalities!

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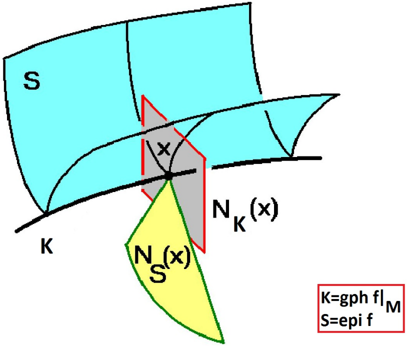

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- So can reduce to the classical setting.

Lifts of identifiable manifolds

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Identifiable manifolds "lift": (D, Lewis), (Daniilidis, Malick, Sendov)

$$M \text{ identifiable manifold at } (\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$$
$$\implies \lambda^{-1}(M) \text{ identifiable manifold at } (\bar{X}, \bar{V}) \in \operatorname{gph} \partial (f \circ \lambda).$$

Lifts of identifiable manifolds

Study "facial" structure of spectral sets (and functions). E.g.

$$\mathbf{S}^n_+ = \bigcup_{k=1}^{n} \{X \in \mathbf{S}^n_+ : \mathrm{rank}\, X = k\}.$$

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May lead to sensitivity analysis, provided can project onto M easily.

Summary

- Presented the intuitive notion of identifiable sets.
- Showed how identifiable sets capture the essence of previously developed concepts (dimension reduction, critical cones, optimality conditions).
- Application to spectral functions.

Thank you.