

Lecture 9: The product and quotient rules.

Last time

$$\frac{d}{dx} x^r = r x^{r-1} \quad \text{for any real } r.$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} [c \cdot f(x)] = c \frac{d}{dx} f(x)$$

$$\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

Goals: $\frac{d}{dx} [f(x)g(x)] = ?$ $\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = ?$

Thm (Product Rule)

If f and g are differentiable, then

$$\frac{d}{dx} [f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

Ex: Find the equation of the tangent line to $y = f(x) = 2xe^x$ at $(0,0)$

solution:

$$\begin{aligned} f'(x) &= 2 \frac{d}{dx} x e^x = 2 \left[x \frac{d}{dx} e^x + \left(\frac{d}{dx} x \right) e^x \right] = \\ &= 2 [x e^x + e^x] = 2(1+x) e^x \end{aligned}$$

$$\begin{aligned} f'(0) &= 2 \Rightarrow \text{Eqn of tangent line at } (0,0) \\ \text{is } y - f(0) &= f'(0)(x - 0) \\ y &= 2x. \end{aligned}$$

proof:

$$\frac{d}{dx} [f(x)g(x)] = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[f(x+h)g(x+h) - f(x)g(x+h)] + [f(x)g(x+h) - f(x)g(x)]}{h}$$

$$= \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right]$$

\nearrow $\frac{d}{dx}$ Differentiability
 $= \left[\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right] \left[\lim_{h \rightarrow 0} g(x+h) + f(x) \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \right]$

\nearrow continuity
 $= f'(x)g(x) + f(x)g'(x)$

Ex: Differentiate $f(x) = 2e^x(\sqrt{x} + x)$ \square

$$\begin{aligned} f'(x) &= 2 \left[e^x \frac{d}{dx} [\sqrt{x} + x] + \left(\frac{d}{dx} e^x \right) (\sqrt{x} + x) \right] = \\ &= 2 \left[e^x \left(\frac{1}{2} x^{-1/2} + 1 \right) + e^x (\sqrt{x} + x) \right] = \\ &= 2e^x \left[x + \sqrt{x} + \frac{1}{2\sqrt{x}} + 1 \right] \end{aligned}$$

Ex: Differentiate $f(x) = \sqrt{x}(x+4)$

[Trick Question]

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left[x^{3/2} + 4\sqrt{x} \right] = \frac{3}{2}\sqrt{x} + 2\frac{1}{\sqrt{x}}$$

Thm: [Quotient Rule]

If f and g are differentiable,
then

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x)f'(x) - g'(x)f(x)}{[g(x)]^2}$$

whenever $g(x) \neq 0$.

Ex: Differentiate $f(x) = \frac{x^2 - 2}{x^3 + x - 1}$

$$f'(x) = \frac{(x^3 + x - 1) \frac{d}{dx} [x^2 - 2] - \left[\frac{d}{dx} (x^3 + x - 1) \right] (x^2 - 2)}{(x^3 + x - 1)^2} =$$

$$= \frac{(x^3 + x - 1)(2x) - [3x^2 + 1](x^2 - 2)}{(x^3 + x - 1)^2}$$

$$= \frac{2x^4 + 2x^2 - 2x - 3x^4 + 6x^2 - x^2 + 2}{(x^3 + x - 1)^2}$$

multiply
out \rightarrow

$$= \frac{-x^4 + 7x^2 - 2x + 2}{(x^3 + x - 1)^2}$$

proof:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} =$$

$$= \lim_{h \rightarrow 0} \frac{g(x)f(x+h) - f(x)g(x+h)}{h g(x+h)g(x)} =$$

combine
fractions \rightarrow

$$\left[\lim_{h \rightarrow 0} \frac{1}{g(x+h)g(x)} \right] \lim_{h \rightarrow 0} \frac{[g(x)f(x+h) - g(x)f(x)] + [g(x)f(x) - f(x)g(x+h)]}{h}$$

$$\begin{aligned}
 &= \frac{1}{g(x)^2} \lim_{h \rightarrow 0} g(x) \left[\frac{f(x+h) - f(x)}{h} \right] \\
 &\quad - \lim_{h \rightarrow 0} f(x) \left[\frac{g(x+h) - g(x)}{h} \right] = \\
 &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \quad \square
 \end{aligned}$$

Ex: Differentiate $f(x) = \frac{1 - xe^x}{1 + e^x}$

$$f'(x) = \frac{(1+e^x) \frac{d}{dx}[1 - xe^x] - (1 - xe^x) \frac{d}{dx}(1+e^x)}{(1+e^x)^2} =$$

$$= \frac{-(1+e^x) \left[x \frac{d}{dx} e^x + e^x \frac{d}{dx} x \right] - (1 - xe^x) e^x}{(1+e^x)^2} =$$

$$= \frac{-(1+e^x)(xe^x + e^x) - (1 - xe^x)e^x}{(1+e^x)^2} =$$

$$= \frac{e^x \left(-(1+e^x)(1+x) - (1 - xe^x) \right)}{(1+e^x)^2} =$$

$$= \frac{e^x (-1 - x - e^x - xe^x - 1 + xe^x)}{(1+e^x)^2} =$$

$$= \frac{-(x + e^x + 2)e^x}{(1+e^x)^2}$$