Lecture 8: Derivatives of polynomials and exponentials

Ex: Find $f'(x)$ for $f(x) = x\sqrt{x}$

Note $f(x) = x^{3/2}$

So $f'(x) = \frac{3}{2}x^{3/2-1} = \frac{3}{2}\sqrt{x}$

What is the equation of the tangent line to $y = f(x)$ at $x = 2$.

$$y - \sqrt{2}^3 = \frac{3}{2}\sqrt{2}(x-2)$$

$\boxed{\text{Rule 4:}}$ If c is a constant and f is differentiable, then $\frac{d}{dx} cf(x) = c\frac{d}{dx}f(x)$.

Ex: $\frac{d}{dx}\left(\frac{2}{3}x^{4/3}\right) = \frac{8}{9}\frac{d}{dx}x^{4/3} = \frac{8}{9}\left(\frac{4}{3}x^{1/3}\right) = \frac{32}{27}x^{1/3}$

$\boxed{\text{Rule 5:}}$ If f and g are differentiable, then

$$(f+g)'(x) = f'(x) + g'(x).$$

reason: Define $F(x) = f(x) + g(x)$.

Then

$$(f+g)'(x) = F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} =$$

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} =$$

$$= \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right) + \left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right) =$$

$$= f'(x) + g'(x)$$

More generally $(f+g+h)'(x) = f'(x) + g'(x) + h'(x)$

Also

$$(f-g)'(x) = (f+(-g))'(x) = f'(x) + (-g)'(x) =$$
$$= f'(x) - g'(x)$$

Ex: Find derivative $f'(x)$ of $f(x) = \frac{x^2+4x+3}{\sqrt{x}}$

$$\frac{d}{dx}\left[\frac{x^2+4x+3}{\sqrt{x}}\right] = \frac{d}{dx}\left(x^{3/2}+4x^{1/2}+3x^{-1/2}\right) =$$

$$= \frac{d}{dx}x^{3/2} + \frac{d}{dx}(4x^{1/2}) + \frac{d}{dx}(3x^{-1/2})$$

Rule 5

$$= \frac{d}{dx}x^{3/2} + 4\frac{d}{dx}x^{1/2} + 3\frac{d}{dx}x^{-1/2}$$

Rule 4

Rule 3. $= \frac{3}{2}x^{1/2} + 4\left(\frac{1}{2}\right)x^{-1/2} - \frac{3}{2}x^{-3/2}$

$$= \frac{3}{2}\sqrt{x} + \frac{2}{\sqrt{x}} - \frac{3}{2}x^{-3/2}$$

Ex: Evaluate $\lim_{x\to 1} \frac{x^{1000}-1}{x-1}$

Define $f(x) = x^{1000}$

Then

$1000(1)^{999} = f'(1) = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{x^{1000}-1}{x-1}$

$=$

1000

Exponential Function: Let $f(x) = a^x$.

Then $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} =$

$$= \lim_{h\to 0} a^x\left(\frac{a^h-1}{h}\right) = a^x \lim_{h\to 0}\frac{a^h-1}{h} = a^x \cdot f'(0)$$
$$= f(x)\cdot f'(0)$$

What is a such that ~~f'(0)~~ $\lim_{h\to 0} \frac{a^h - 1}{h} = 1$

Definition:
e is the number such that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$

$[e \approx 2.71828\ldots]$

Rule 6:

$$\frac{d}{dx} e^x = e^x.$$

[We will see how to compute $\frac{d}{dx} a^x$ later for general a]

aside.

Other characterizations of e:

$$e = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$$

$$= 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \ldots$$

Ex: Find equation of the tangent line to $y = f(x) = x^4 + 2e^x$ at $(0,2)$

$f'(x) = 4x^3 + 2e^x$

So $f'(0) = 2 \Rightarrow$ tangent line is

$$y - 2 = 2(x-0)$$
$$= 2x.$$

Ex: Show that $y = 2e^x + 3x + 5x^3$ has no tangent line with slope 2

Set $f(x) = 2e^x + 3x + 5x^3$

Then $f'(x) = 2e^x + 3 + 15x^2 = 2$

$\Rightarrow 2e^x + 15x^2 = -1 < 0$

No such x exists.