Lecture 7: Derivatives Continued

Ex: Find the slope of the tangent line to $y = \frac{1}{\sqrt{5-2x}}$ at $x=a$

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{5-2(a+h)}} - \frac{1}{\sqrt{5-2a}}}{h} =$$

$$= \lim_{h\to 0} \frac{\sqrt{5-2a} - \sqrt{5-2(a+h)}}{h\left(\sqrt{5-2(a+h)}\sqrt{5-2a}\right)} =$$

$$= \lim_{h\to 0} \frac{(5-2a)-(5-2(a+h))}{h\sqrt{5-2(a+h)}\sqrt{5-2a}\left(\sqrt{5-2a}+\sqrt{5-2(a+h)}\right)}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{5-2(a+h)}\sqrt{5-2a}\left(\sqrt{5-2a}+\sqrt{5-2(a+h)}\right)} =$$

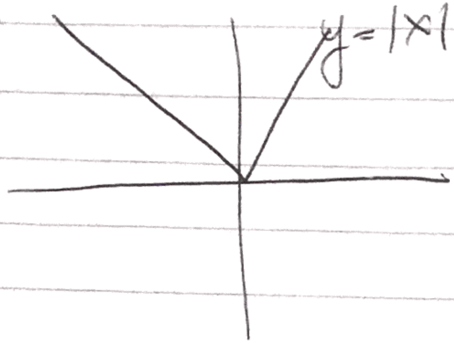
$$= \frac{1}{(5-2a)^{3/2}}$$

So The tangent line to $y = \frac{1}{\sqrt{5-2x}}$ at $x=a$ is

$$y - \frac{1}{\sqrt{5-2a}} = \frac{1}{(5-2a)^{3/2}}(x-a)$$

Defn: A function f is differentiable at $x=a$ if $f'(a)$ exists. The domain of the derivative f' consists of all points a such that $f'(a)$ exists and is finite

Ex:



$$\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\frac{|h|}{h}=1$$

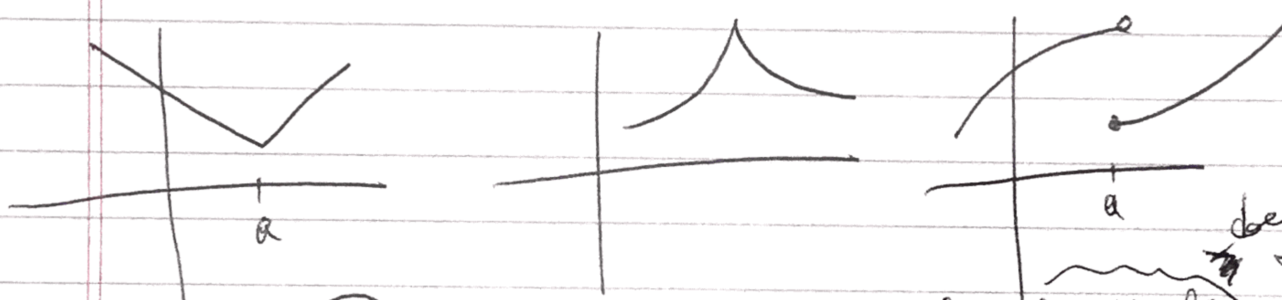
$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^-}\frac{|h|}{h}=-1$$

$\Rightarrow f'(0)$ does not exist

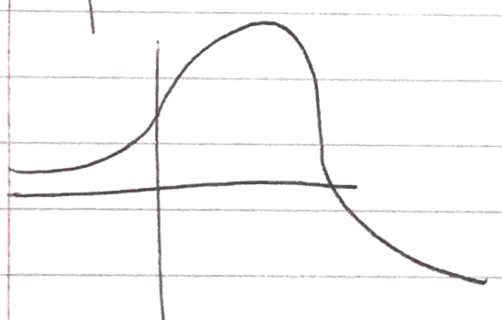
f is **not** differentiable at $a=0$.

Remark: In particular,

f continuous $\not\Rightarrow$ f is differentiable



$$\lim_{x\to a^-}\frac{\overbrace{f(x+h)-f(x)}^{\text{does not tend to zero}}}{h}$$



Thm: If f is differentiable at a, then f is continuous at a.

pf: $\displaystyle\lim_{x\to a} f(x)-f(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}(x-a)=\left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\lim_{x\to a}(x-a)$

$= f'(a)\cdot 0 = 0$ $\boxtimes$

Higher Derivatives

Defn: The second derivative of f at $x=a$ is

$$f''(a) = (f')'(a) = \lim_{x\to a} \frac{f'(x)-f'(a)}{x-a}$$

Ex: Find $f', f'', f''', f^{(4)}$ of $f(x) = 2x^2 - x^3$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[2(x+h)^2-(x+h)^3]-[2x^2-x^3]}{h} =$$

$$= \lim_{h\to 0}\frac{2(x^2+2xh+h^2)-(x+h)(x+h)^2-[2x^2-x^3]}{h} =$$

$$= \lim_{h\to 0}\frac{4xh+2h^2-x(x+h)^2-h(x+h)^2+x^3}{h} =$$

$$= \lim_{h\to 0}\frac{4xh+2h^2-x(x^2+2xh+h^2)-h(x+h)^2+x^3}{h} =$$

$$= \lim_{h\to 0} 4x+2h-2x^2-xh-(x+h)^2 = 4x-3x^2.$$

$$f''(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0}\frac{[4(x+h)-3(x+h)^2]-[4x-3x^2]}{h} =$$

$$= \lim_{h\to 0}\frac{4h-3(x^2+2xh+h^2)+3x^2}{h} = \lim_{h\to 0} 4-6x-3h = 4-6x.$$

$$f'''(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0}\frac{[4-6(x+h)]-(4-6x)}{h} = -6$$

$$f^{(4)}(x) = 0.$$

Application: If $f(x)$ is the position of a car at time t, then

$$f'(t) = \text{velocity}$$
$$f''(t) = \text{acceleration}.$$

Notation: We will sometimes use the notation $\frac{d}{dx} f(x) = f'(x)$.

Derivatives of polynomials and exponentials

Rule 1: $\frac{d}{dx} c = 0$

constant function

Rule 2: $\frac{d}{dx} x^n = n x^{n-1}$ where $n \geq 0$ is an integer.

~~pf.~~ Set $f(x) = x^n$.

Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} =$$

$$= \lim_{h \to 0} \frac{\left[x^n + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots + \binom{n}{n-1} x h^{n-1} + \quad\right] - x^n}{h}$$

$$= n x^{n-1}$$

Rule 3: $\frac{d}{dx} x^r = r x^{r-1}$ for any real number r.

Ex: $\frac{d}{dx} x^{1000} = 1000 x^{999}$, $\frac{d}{dx} x^{2/3} = \frac{2}{3} x^{-1/3}$.