Lecture 6: Derivatives (Section 2.7)

Definition: Given a function \( f \) and a point \( x = a \), the derivative of \( f \) at \( a \) is
\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

The tangent line to the curve \( y = f(x) \) is
\[
y - f(a) = f'(a)(x - a)
\]

Interpretation: \( f'(a) \) is the rate of change of \( f \) with respect to \( x \) at \( x = a \).

Applications:
- \( f(t) \) = distance traveled by car at time \( t \).
- \( f'(a) \) = velocity of the car at \( t = a \).
- \( |f'(a)| \) = speed of the car at \( t = a \).

Remark: We can also write
\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]
Ex: A ball is thrown up in the air with a velocity of 40 ft/sec. Its height in feet after \( t \) seconds is

\[ f(t) = 40t - 16t^2 \]

Calculate its velocity as a function of \( t \).

\[ f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \]

\[ = \lim_{h \to 0} \frac{40(t+h) - 16(t+h)^2 - (40t - 16t^2)}{h} \]

\[ = \lim_{h \to 0} \frac{40t - 16(t^2 + 2th + h^2) + 16t^2}{h} \]

\[ = \lim_{h \to 0} \frac{40t - 32th - 16h^2}{h} \]

\[ = \lim_{h \to 0} \frac{40 - 32t - 16h}{1} = 40 - 32t \text{ ft/sec}. \]

What is the highest point?

Set \( f'(t) = 0 = 40 - 32t \)

\[ \Rightarrow t = \frac{40}{32} = \frac{5}{4} \text{ sec} \]

At \( t = \frac{5}{4} \) sec, the ball is at its highest point.

Graph \( f'(t) \)

- Speed ↑: 40 ft/sec
- Time the ball is going up:
- Speed ↓: \( \frac{40}{32} \) ft/sec
- Time ball is going down:
Ex: Find the slope of the tangent line to \( y = \frac{1}{\sqrt{5 - 2x}} \) at \( x = a \)

solution:
\[
\lim_{x \to a} \frac{\frac{1}{\sqrt{5 - 2x}} - \frac{1}{\sqrt{5 - 2a}}}{x - a} = \\
= \lim_{x \to a} \frac{\sqrt{5 - 2a} - \sqrt{5 - 2x}}{(5 - 2a) - (5 - 2x)} \cdot \frac{(5 - 2a)(5 - 2x)}{(5 - 2a + 5 - 2x)(\sqrt{5 - 2x} \cdot \sqrt{5 - 2a})} \cdot (x - a) \\
= \lim_{x \to a} \frac{2(x - a)}{(5 - 2a + 5 - 2x)(\sqrt{5 - 2x} \cdot \sqrt{5 - 2a})} \\
= \frac{2}{2(5 - 2a)^{3/2}} = \frac{1}{(5 - 2a)^{3/2}}
\]

Ex: If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom in 1 hour, Torricelli's law shows the volume \( V(t) \) remaining at time \( t \) (in minutes) is
\[ V(t) = 100,000 \left(1 - \frac{t}{60}\right)^2 \] gallons \( 0 \leq t \leq 60 \). What is the rate at which water is flowing out at time \( t \)?
Rate of change of volume at \( t = a \):

\[
\lim_{{h \to 0}} \frac{V(a+h) - V(a)}{h} = \lim_{{h \to 0}} \frac{100,000 \left(1 - \frac{a+h}{60}\right)^2 - 100,000 \left(1 - \frac{a}{60}\right)^2}{h}
\]

\[
= \lim_{{h \to 0}} 10^5 \left( \left(1 - \frac{a}{60}\right)^2 - \frac{h}{60} \left(1 - \frac{a}{60}\right) + \frac{h^2}{60^2} - \left(1 - \frac{a}{60}\right)^2 \right)
\]

\[
= \lim_{{h \to 0}} 10^5 \left( \frac{h^2}{60^2} - \frac{h}{30} \left(1 - \frac{a}{60}\right) \right)
\]

\[
= \frac{-10^5}{30} \left(1 - \frac{a}{60}\right)
\]

At \( A = 0 \), rate = \(-\frac{10^5}{30}\)

At \( a = 60 \), rate = 0