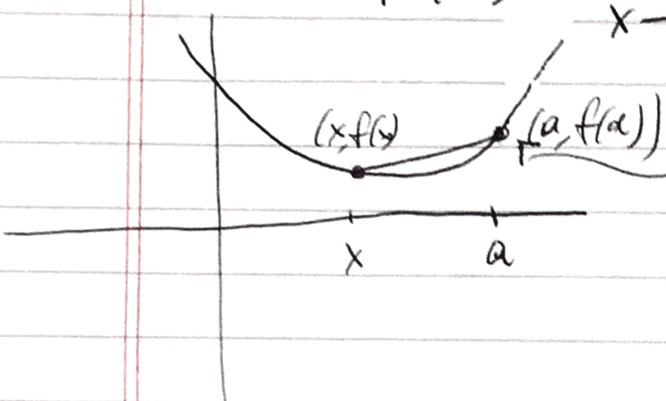
Lecture 6: Derivatives (Section 2.7)

Definition: Given a function f and a point $x=a$, the derivative of f at a is

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$



slope of secant line $= \dfrac{f(x)-f(a)}{x-a}$

The tangent line to the curve $y=f(x)$ is

$$y-f(a) = f'(a)(x-a).$$

Interpretation: $f'(a)$ is rate of change of f with respect to x at $x=a$.

Applications: $f(t)$ = distance traveled by car at time t.

$f'(a)$ = velocity of the car at $t=a$.

$|f'(a)|$ = speed at $t=a$.

Remark: We can also write

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Ex: A ball is thrown up in the air with a velocity of 40 ft/sec. It's height in feet after t seconds is

$$f(t) = 40t - 16t^2$$

Calculate its velocity as a function of t.

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} =$$

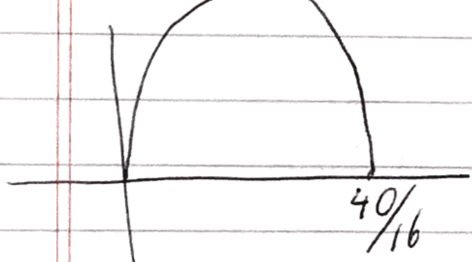
$$= \lim_{h \to 0} \frac{40(t+h) - 16(t+h)^2 - (40t - 16t^2)}{h} =$$

$$= \lim_{h \to 0} \frac{40h - 16(t^2 + 2th + h^2) + 16t^2}{h} =$$

$$= \lim_{h \to 0} \frac{40h - 32th - 16h^2}{h}$$

$$= \lim_{h \to 0} 40 - 32t - 16h = 40 - 32t \text{ ft/sec.}$$

What is the highest point?

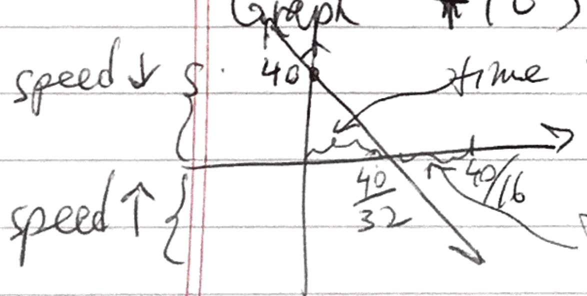


set $f'(t) = 0 = 40 - 32t$

$$\Rightarrow t = \frac{40}{32}$$

At $t = \frac{40}{32}$ sec, the ball is at its highest point.

Graph $f'(t)$



time the ball is going up.

time ball is going down.

Ex: Find the slope of the tangent line to $y = \frac{1}{\sqrt{5-2x}}$ at $x = a$

solution:

$$\lim_{x \to a} \frac{\frac{1}{\sqrt{5-2x}} - \frac{1}{\sqrt{5-2a}}}{x-a} =$$

$$= \lim_{x \to a} \frac{\sqrt{5-2a} - \sqrt{5-2x}}{\sqrt{5-2x}\sqrt{5-2a}(x-a)} =$$

$$= \lim_{x \to a} \frac{(5-2a)-(5-2x)}{(\sqrt{5-2a}+\sqrt{5-2x})(\sqrt{5-2x}\sqrt{5-2a})(x-a)} =$$

$$= \lim_{x \to a} \frac{2\cancel{(x-a)}}{(\sqrt{5-2a}+\sqrt{5-2x})(\sqrt{5-2x}\sqrt{5-2a})\cancel{(x-a)}}$$

$$= \frac{2}{2(5-2a)^{3/2}} = \frac{1}{(5-2a)^{3/2}}$$

Ex: If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom in 1 hour, Torricelli's law shows the volume $V(t)$ remaining at time t (in minutes) is $V(t) = 100{,}000\left(1 - \frac{t}{60}\right)^2$ gallons, $0 \le t \le 60$. What is the rate at which water is flowing out at time t?

Rate of change of volume at $t=a$:

$$\lim_{h\to 0} \frac{V(a+h)-V(a)}{h} = \lim_{h\to 0} \frac{100{,}000\left(1-\frac{a+h}{60}\right)^2 - 100{,}000\left(1-\frac{a}{60}\right)^2}{h}$$

$$= \lim_{h\to 0} \frac{10^5\left(\left(\left(1-\frac{a}{60}\right)-\frac{h}{60}\right)^2-\left(1-\frac{a}{60}\right)^2\right)}{h} =$$

$$= \lim_{h\to 0} \frac{10^5\left(\left(1-\frac{a}{60}\right)^2-\frac{h}{30}\left(1-\frac{a}{60}\right)+\frac{h^2}{60^2}-\left(1-\frac{a}{60}\right)^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{10^5\left(\frac{h^2}{60^2}-\frac{h}{30}\left(1-\frac{a}{60}\right)\right)}{h} =$$

$$= -\frac{10^5}{30}\left(1-\frac{a}{60}\right)$$

At $a=0$, rate $= -\frac{10^5}{30}$

At $a=60$, rate $= 0$