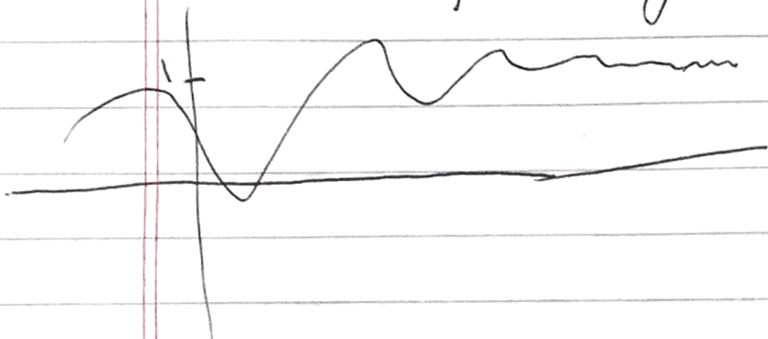
Lecture 5: Limits at infinity; Horizontal asymptotes

Defn: If $\lim_{x\to\infty} f(x) = c$ or $\lim_{x\to -\infty} f(x) = c$

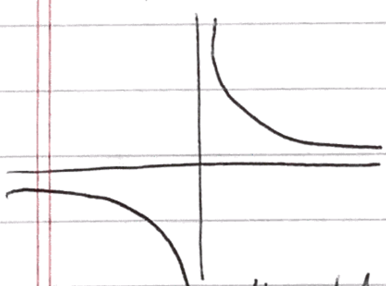
call line $y=c$ a horizontal asymptote

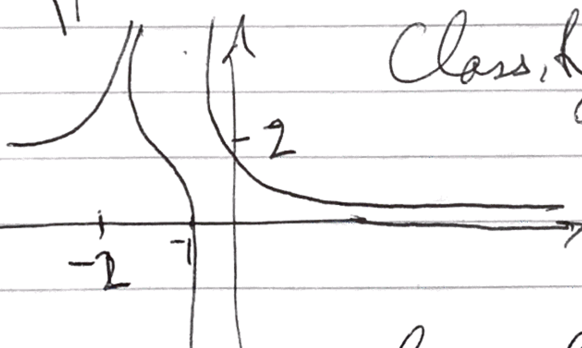


$\lim_{x\to\infty} f(x) = 1$

Ex: $\lim_{x\to -\infty} e^x = 0 \Rightarrow y=0$ is a horizontal asymptote

Ex: $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to -\infty} \frac{1}{x} = 0$.



Ex: 

Classify all asymptotes

$\lim_{x\to -\infty} f(x) = -2$ $\lim_{x\to -1^+} f(x) = +\infty$

$\lim_{x\to -2^+} f(x) = +\infty$ $\lim_{x\to +\infty} f(x) = 0$.

$\lim_{x\to -1^-} f(x) = -\infty$

So vertical asymptotes: $-2, -1$

horizontal " : $2, 0$.

Limit laws given in previous section hold for $\lim_{x\to\infty} f(x)$

Rule: $\lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to-\infty} \frac{1}{x^r} = 0$

for $r > 0$.

Pitfall: $\sqrt{x^2} = |x|$. $\Rightarrow$ if $x<0$, then $\sqrt{x^2+x} = \sqrt{x^2(1+\frac{1}{x})} = -x\sqrt{1+1/x}$.

Ex: Find all asymptotes of $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$

Horizontal: Rule of thumb: divide top and bottom by highest degree term in denominator

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{x\sqrt{2+\frac{1}{x^2}}}{x(3-\frac{5}{x})} = \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} =$$

$$= \frac{\sqrt{\lim_{x\to\infty} 2+\frac{1}{x^2}}}{\lim_{x\to\infty} 3-\frac{5}{x}} = \frac{\sqrt{2}}{3}.$$

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{-x\sqrt{2+\frac{1}{x}}}{x(3-\frac{5}{x})} = -\frac{\sqrt{2}}{3}$$

So $y = \frac{\sqrt{2}}{3}, \frac{-\sqrt{2}}{3}$ are horizontal asymptotes

Vertical:

function is continuous at all points except where it is undefined $x = \frac{5}{3}$.

$$\lim_{x\to\frac{5}{3}^+} \frac{\sqrt{2x^2+1}}{3x-5}$$

$3x-5 \to 0$ as $x \to \frac{5}{3}^+$

$\sqrt{2x^2+1} \to \sqrt{2(\frac{5}{3})+1} \neq 0$

as $x \to \frac{5}{3}^+$. So $\lim_{x\to\frac{5}{3}^+} \frac{\sqrt{2x^2+1}}{3x-5} = +\infty$

$\Rightarrow x = \frac{5}{3}$ is a v.a.

Ex: $\lim\limits_{x\to-\infty} x+\sqrt{x^2+2x} = \lim\limits_{x\to-\infty} x+\sqrt{x^2+2x}\left(\dfrac{x-\sqrt{x^2+2x}}{x-\sqrt{x^2+2x}}\right) =$

$$= \lim_{x\to-\infty} \frac{x^2-(x^2+2x)}{x-\sqrt{x^2+2x}} = \lim_{x\to-\infty} \frac{-2x}{x-\sqrt{x^2+2x}} =$$

$$= \lim_{x\to-\infty} \frac{-2}{1-\frac{1}{x}\sqrt{x^2+2x}} = \lim_{x\to-\infty} \frac{-2}{1+\sqrt{1+\frac{2}{x}}}$$

$$= -1.$$

Ex: $\lim\limits_{x\to\infty} x^2-x = \lim\limits_{x\to\infty} x(x-1) = \infty$

Ex: $\lim\limits_{x\to\infty} \dfrac{x^2+x}{3-x} = \lim\limits_{x\to\infty} \dfrac{x+1}{\frac{3}{x}-1} = -\infty$

Ex: $\lim\limits_{x\to\infty} \dfrac{1-e^x}{1+2e^x} = \lim\limits_{x\to\infty} \dfrac{e^{-x}-1}{e^{-x}+2} = -\dfrac{1}{2}.$

Ex: $\lim\limits_{x\to\infty} \dfrac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}} = \lim\limits_{x\to\infty} \dfrac{1-e^{-6x}}{1+e^{-6x}} = 1.$