Lecture 4: Continuity

Definition: A function \( f \) is continuous at \( a \) if
\[
\lim_{{x \to a}} f(x) = f(a)
\]

In other words:
1. \( \lim_{{x \to a}} f(x) \) exists
2. \( f(a) \) is defined
3. \( \lim_{{x \to a}} f(x) = f(a) \)

We say that \( f \) is discontinuous at \( a \) if \( f \) is not continuous at \( a \).

Example:

\( f(x) = \begin{cases} x^2 - x & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} \)

Continuous at \( x = 1 \)

\[
\lim_{{x \to 1}} f(x) = \lim_{{x \to 1}} \frac{x^2 - x}{x^2 - 1} = \lim_{{x \to 1}} \frac{x(x-1)}{(x-1)(x+1)} = \lim_{{x \to 1}} \frac{x}{x+1} = \frac{1}{2} \neq 1 = f(1)
\]

So \( f \) is discontinuous at \( x = 1 \).
Defn: \( f \) is continuous from left at \( a \)
\[
\text{if } \lim_{x \to a^-} f(x) = f(a).
\]
Continuity from right is similar.

Defn: \( f \) is continuous on \([a, b]\) if
\( f \) is continuous at any \( x \) in \((a, b)\)
right continuous at \( a \) and
left continuous at \( b \).

Thm: If \( f, g \) are continuous at \( a \) and \( c \) is a constant, then
\( f + g \), \( f - g \), \( c f \), \( f \cdot g \) (if \( g(x) \neq 0 \))
are continuous at \( a \).

- Polynomials are continuous everywhere.
- Polynomial is continuous at any point in its domain.
- All functions below are continuous at points in their domain.
  - Power functions, trig, inverse trig,
  - Exponential, logarithm.

Ex: Compute \( \lim_{x \to 1} \frac{2x - 3x^2}{1 + x^3} \)

\[
\begin{align*}
2x - 3x^2 - \frac{1}{1 + x^3} &\quad \text{continuous} \\
\Rightarrow &\quad \frac{2x - 3x^2}{1 + x^3} \text{ continuous} \Rightarrow \lim_{x \to 1} \frac{2x - 3x^2}{1 + x^3} = \frac{2(1) - 3(1)^2}{1 + (1)^3} = -\frac{1}{2}
\end{align*}
\]
**Theorem:** If \( f \) is continuous at \( b \) and \( \lim_{{x \to a}} g(x) = b \), then
\[
\lim_{{x \to a}} f(g(x)) = f(\lim_{{x \to a}} g(x)) = f(b).
\]

**Example:**
\[
\lim_{{x \to 1}} e^{x^2 - x} = e^{\lim_{{x \to 1}} x^2 - x} = e^0 = 1
\]

**Theorem:** If \( g \) is continuous at \( a \) and \( f \) is continuous at \( g(a) \), then \( f \circ g \) is continuous at \( a \).

**Intermediate Value Theorem (IVT):**
If \( f \) is continuous on \([a, b]\) and \( z \) is any number between \( f(a) \) and \( f(b) \), then there exists \( c \) in \([a, b]\) such that \( f(c) = z \).

**Example:** Show that \( e^x = 3 - 2x \) has a solution \( x \) in \((0, 1)\).

Define \( f(x) = e^x - 3 + 2x \).

Then \( f(0) = 1 - 3 = -2 < 0 \) and \( f(1) = e - 3 + 2 > 0 \).

**Apply** Intermediate Value Theorem (IVT).