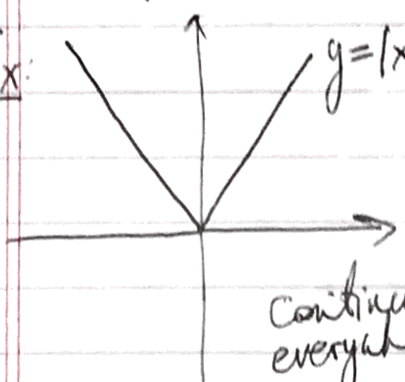
Lecture 4: Continuity

Defn: A function f is <u>continuous at a</u> if
$$\lim_{x\to a} f(x) = f(a)$$

(In other words
① $\lim_{x\to a} f(x)$ exists,
② $f(a)$ is defined
③ $\lim_{x\to a} f(x) = f(a)$)

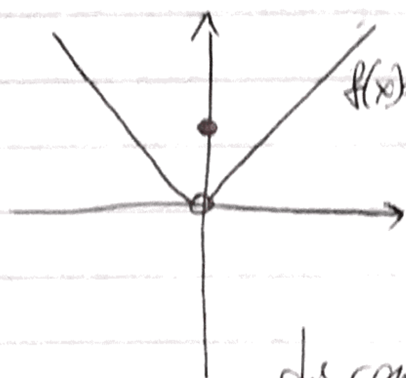
We say that f is <u>discontinuous at a</u>, if f is not continuous at a.

Ex:



$y=|x|$

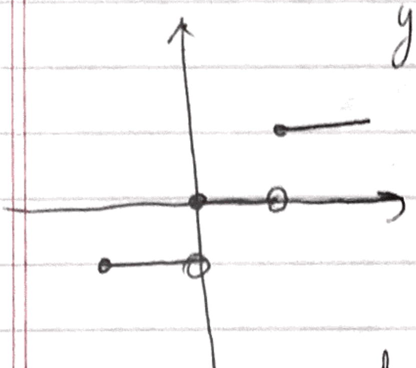
continuous everywhere



$f(x) = \begin{cases} |x|, & x \neq 0 \\ 1, & x = 0 \end{cases}$

<u>removable discontinuity</u> because redefining $f(a)$ makes function continuous

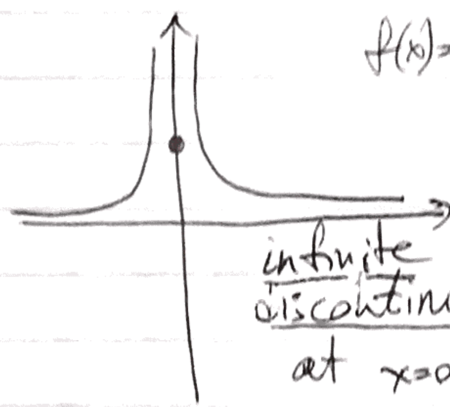
discontinuous at $x=0$



$y = \lfloor x \rfloor$

<u>jump discontinuity</u>

discontinuous at $x=$ integer



$f(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

<u>infinite discontinuity</u> at $x=0$.

Ex: Is $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1}, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases}$

continuous at $x = 1$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-x}{x^2-1} = \lim_{x\to 1} \frac{x(x-1)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2} \neq 1 = f(1)$$

So f is <u>discontinuous</u> at $x=1$

Defn: f is continuous from left at a

if $\lim_{x \to a^-} f(x) = f(a)$.

Continuity from right is similar.

Def: f is continuous on $[a,b]$ if

f is continuous at any x in (a,b)

right continuous at a and

left continuous at b.

Thm: If f, g are continuous at a and

c is a constant, then

$f+g,\ f-g,\ c\cdot f,\ f\cdot g,\ \frac{f}{g}$ (if $g(a) \neq 0$)

are continuous at a.

- Polynomials are continuous everywhere,

$\frac{\text{polynomial}}{\text{polynomial}}$ is continuous at any point in its domain

- All functions below are continuous at points in their domain:

root functions, trig, inverse trig,

exponential, logarithm.

Ex: Compute $\lim_{x \to 1} \frac{2x-3x^2}{1+x^3}$

continuous $2x-3x^2$

continuous at $\neq 0$ at $x=1$ $1+x^3$

$\Rightarrow \frac{2x-3x^2}{1+x^3}$ continuous at $x=1$ $\Rightarrow \lim_{x \to 1} \frac{2x-3x^2}{1+x^3} =$

$= \frac{2(1)-3(1)^2}{1+(1)^3} = -\frac{1}{2}$

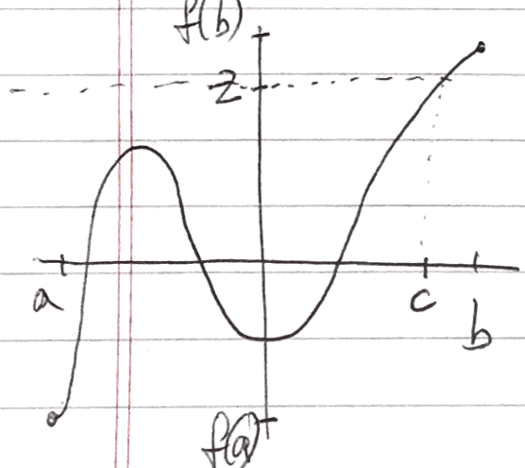
Thm: If f is continuous at b and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(b).$$

Ex: $$\lim_{x \to 1} e^{x^2 - x} = e^{\lim_{x \to 1} x^2 - x} = e^0 = 1$$

Thm: If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Intermediate Value Theorem (IVT):
If f is continuous on $[a,b]$ and z is any number between $f(a)$ and $f(b)$ then there exists c in $[a,b]$ such that $f(c) = z$.



Ex: Show that $e^x = 3 - 2x$ has a solution x in $(0,1)$

Define $f(x) = e^x - 3 + 2x$.

Then $f(0) = 1 - 3 = -2 < 0$

$f(1) = e - 3 + 2 > 0$

Apply ~~[illegible]~~. IVT