Lecture 3: Calculating Limits

Thm: Suppose c is a constant, and
$P = \lim_{x \to a} f(x)$ and $Q = \lim_{x \to a} g(x)$ exist.

Then

① $\lim_{x \to a} (f(x) + g(x)) = P + Q$

② $\lim_{x \to a} (f(x) - g(x)) = P - Q$

③ $\lim_{x \to a} c \cdot f(x) = cP$

④ $\lim_{x \to a} f(x)g(x) = P \cdot Q$

⑤ $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{P}{Q}$ <u>if</u> $Q \neq 0$.

⑥ $\lim_{x \to a} (f(x))^n = P^n$

⑦ $\lim_{x \to a} c = c$

⑧ $\lim_{x \to a} x = a$

⑨ $\lim_{x \to a} x^n = a^n$

⑩ $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{P}$ (if n even, assume $P > 0$)

Similar results hold for one-sided limits.

Ex:

$$\lim_{x\to 2}\sqrt{\frac{2x^2+1}{3x-2}} = \sqrt{\lim_{x\to 2}\frac{2x^2+1}{3x-2}} = \sqrt{\frac{\lim_{x\to 2} 2x^2+1}{\lim_{x\to 2} 3x-2}}$$

because
$\lim_{x\to 2} 3x-2 = 3\lim_{x\to 2} x - 2 = 4$

$$= \sqrt{\frac{2\lim_{x\to 2} x^2+1}{3\lim_{x\to 2} x - 2}} = \sqrt{\frac{9}{4}} = \frac{3}{2}.$$

Observation (A)

If $f(x)$ is a polynomial or $\frac{\text{polynomial}}{\text{polynomial}}$ and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Ex: Find $\lim_{x\to 1}\frac{x^2-1}{x-1}$.

We can not use observation (A) because $x=1$ is not in the domain of the function

Instead observe

$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1}$ coincides with $x+1$ whenever $x \neq 1$

So $\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 2.$

Fact: If $f(x) = g(x)$ for all $x \neq a$, then
$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$

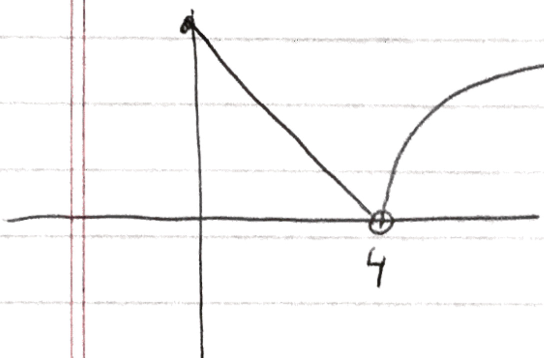
Ex:

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{t+1}}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{1-\sqrt{t+1}}{t\sqrt{t+1}}\cdot\left(\frac{1+\sqrt{t+1}}{1+\sqrt{t+1}}\right)$$

$$=\lim_{t\to 0}\frac{1-(t+1)}{t\sqrt{t+1}\,(1+\sqrt{t+1})}=\lim_{t\to 0}\frac{-t}{t\sqrt{t+1}\,(1+\sqrt{t+1})}$$

$$=\lim_{t\to 0}\frac{-1}{\sqrt{t+1}\,(1+\sqrt{t+1})}=\frac{-1}{\sqrt{1}\,(1+\sqrt{0+1})}=-\frac{1}{2}.$$

Ex: $f(x)=\begin{cases}\sqrt{x-4}, & \text{if } x>4\\ 8-2x, & \text{if } x<4\end{cases}$

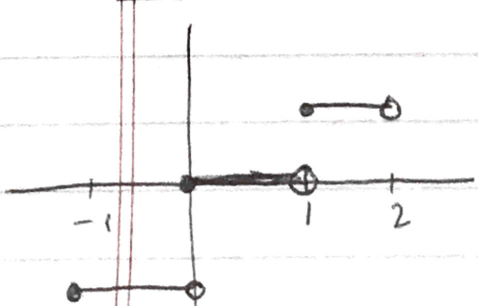


$$\lim_{x\to 4^+} f(x)=\lim_{x\to 4^+}\sqrt{x-4}=0$$

$$\lim_{x\to 4^-} f(x)=\lim_{x\to 4^-} 8-2x=0$$

So $\lim_{x\to 4} f(x)=0$.

Ex: Let $f(x)=\lfloor x\rfloor=$ largest integer $\le x$



$\lim_{x\to 1} f(x)$ does not exist

because

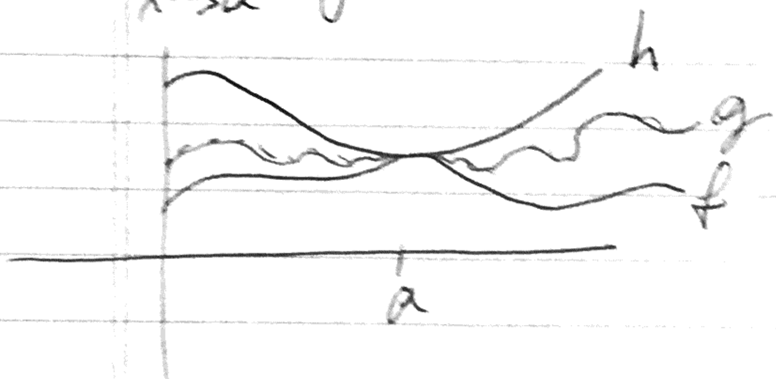
$\lim_{x\to 1^-} f(x)=0 \neq 1=\lim_{x\to 1^+} f(x)$.

Squeeze Theorem:

If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at $x=a$) and

$$L = \lim_{x\to a} f(x) = \lim_{x\to a} h(x) \text{ then}$$

$$\lim_{x\to a} g(x) = L$$



Ex: What is $\lim_{x\to 0^+} \sqrt{x^3+x^2}\,\sin\frac{\pi}{x}$?

Remember $-1 \le \sin\frac{\pi}{x} \le 1$

So

$$-\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\,\sin\frac{\pi}{x} \le \sqrt{x^3+x^2}$$

$$\lim_{x\to 0} -\sqrt{x^3+x^2} = 0 = \lim_{x\to 0} \sqrt{x^3+x^2}$$

So $\lim_{x\to 0^+} \sqrt{x^3+x^2}\,\sin\frac{\pi}{x} = 0$.

Ex: What is $\lim_{x\to 0^+} \sqrt{x}\, e^{\sin\frac{\pi}{x}}$

$$e^{-1} \le e^{\sin\frac{\pi}{x}} \le e$$

So $e^{-1}\sqrt{x} \le \sqrt{x}\, e^{\sin\frac{\pi}{x}} \le \sqrt{x}$

0 as $x \to 0^+$ 0 as $x \to 0^+$

$$\Rightarrow \lim_{x\to 0^+} \sqrt{x}\, e^{\sin\frac{\pi}{x}} = 0.$$